Fault Diagnosis and Fault Tolerant Control

Lecture 3

Fault Diagnosis of Deterministic Systems

Introduction

This lecture provides solutions to the
• fault detection,
• fault isolation and
• Fault estimation
problems for systems described by deterministic continuous-variable models.

The chapter considers faults that can be modelled as additive signals acting on the process. The methods presented lead to a diagnostic system which is separated in two parts: a
• residual generation module
• residual evaluation module.
Introduction

Continuous-variable models (or analytical models) consist of sets of differential or difference equations. They can be deduced by application of the laws of physics, chemistry, etc. to the supervised and/or controlled process.

The external variables entering these equations are called inputs:

- **Control inputs**: known and can be manipulated
- **Disturbances**: cannot be manipulated.
  Disturbances that are not measured are called unknown inputs.
- **Measurement noise** may be represented by stochastic processes (or sequences) appearing as additional inputs.

When such random input is used, one speaks about stochastic models, as opposed to deterministic models.

---

Introduction

This chapter and the next one will show the design of:

- Fault detection
- Fault isolation
- and/or Fault estimation

systems for processes described by deterministic continuous-variable models with unknown input. Such systems are made of two parts:

- Residual generation module
- Residual evaluation module (or decision system)
Introduction

The residuals are signals that, in the absence of faults, deviate from zero only due to modelling uncertainties, with nominal value being zero, or close to zero under actual working conditions.

If a fault should occur, the residuals deviate from zero with a magnitude such that the new condition can be distinguished from the fault free working mode.

The role of the decision system is to determine whether the residuals differ significantly from zero and, from the pattern of zero and non-zero residuals, to decide which is the most likely fault being present, if any, and in turn, isolate which component(s) could be the origin of a fault.
Introduction

The fundamental notion on which residual generation for continuous-variable systems rests is **analytical redundancy**. Analytical redundancy relations (ARR) are equations that are deduced from an analytical model, which solely use measured variables as input. Analytical redundancy relations (ARR) must be consistent in the absence of a fault, and can thus be used for residual generation.

Example 6.1 Residuals for the ship autopilot

Consider, the following part of the ship autopilot example. The turn rate $\omega_3$ and the heading angle $\psi$ are related through

$$\dot{\psi}(t) = \omega_3(t). \quad (6.1)$$

Let us neglect the effect of waves and assume that the measurements can only be affected by a bias. Hence, sensor faults are represented by additive signals and the measurement equations can be written:

$$\psi_m(t) = \psi(t) + f_\psi(t) \quad (6.2)$$
$$\omega_{3m}(t) = \omega_3(t) + f_\omega(t) \quad (6.3)$$

discrete model deduced from (6.1)

$$\psi(k+1) = \psi(k) + \omega_3(k)T_s, \quad (6.4)$$
### Introduction

By considering the equation error, \( r \), resulting from (6.4) when the variables are substituted by their measured value, the following expression is obtained:

\[
r(k) = \psi_m(k) - \psi_m(k - 1) - \omega_3m(k - 1)T_s. \tag{6.5}
\]

This quantity has the properties expected for a residual. Indeed, introducing (6.2), (6.3) into (6.5) yields

\[
r(k) = f\psi(k) - f\psi(k - 1) - f\omega(k - 1)T_s.
\]

This shows that, in the absence of a fault (namely when \( f\psi(k) = f\psi(k - 1) = f\omega(k - 1) = 0 \)), \( r(k) \) is zero. Upon occurrence of a bias in the measurement of \( \omega_3 \) say at time \( k_0 \), \( r(k) \) takes a constant non-zero value for all \( k \geq k_0 \). Finally, the appearance of a bias on the measurement of \( \psi \) at time instant \( k_0 \) shows up as a spike at time \( k_0 \), but has no permanent effect on \( r \). Both faults thus affect \( r \) and this signal is zero in the absence of fault. Hence it can be named a residual signal. For decision making, it suffices to compare the residual to a specified threshold. The latter should be chosen in such a way that biases that appear to be significant for the considered application are detected.
6.1. Introduction

When measurement noise is significant, comparison to a simple threshold might not be practicable, because the change in the mean of the residual due to the fault can be hidden by the effect of the noise on the residual. This noise needs to be taken into account as described in the following two discretised “noisy” versions of (6.2), (6.3):

\[
\begin{align*}
\psi_m(k) &= \psi(k) + f_\psi(k) + v_\psi(k) \\
\omega_3m(k) &= \omega_3(k) + f_\omega(k) + v_\omega(k),
\end{align*}
\]

where \(v_\psi(i), v_\omega(i), i = 1, 2, \ldots\) are mutually uncorrelated white noise sequences with variance \(E(v_\psi^2(k)) = Q_\psi\) and \(E(v_\omega^2(k)) = Q_\omega\), respectively.

Substituting (6.6) and (6.7) into (6.5) yields:

\[
r(k) = f_\psi(k) - f_\psi(k - 1) - f_\omega(k - 1)T_s + v_\psi(k) - v_\psi(k - 1) - v_\omega(k - 1)T_s.
\]

6.2. Analytical Redundancy in Nonlinear Deterministic Systems

Analytical redundancy can be seen as a tool for obtaining conditions, based on available measurements, that are necessarily fulfilled when the supervised system works in a specific operating mode.

In order to illustrate the principle of analytical redundancy, consider deterministic systems described in normal operation by state and measurement equations

\[
\begin{align*}
\dot{x}(t) &= g(x(t), u(t), d(t), \theta, t) \\
y(t) &= h(x(t), u(t), d(t), \theta, t),
\end{align*}
\]

Let \(H_0\) be the situation corresponding to normal operation, and \(H_1 = \neg H_0\) some faulty situation. The following logical statements are true:

\[
\begin{align*}
\mathcal{H}_0 \iff [\dot{x}(t) = g(x(t), u(t), d(t), \theta, t)] \land [y(t) = h(x(t), u(t), d(t), \theta, t)] \\
\mathcal{H}_1 \iff [\dot{x}(t) \neq g(x(t), u(t), d(t), \theta(t))] \lor [y(t) \neq h(x(t), u(t), d(t), \theta(t))].
\end{align*}
\]
6.2. Analytical Redundancy in Nonlinear Deterministic Systems

The violation of equality constraints that results from faults may be described in two ways:

- In the first option, faults are assumed to result from parametric variations, which is represented as
  \[ \theta_f(t) \neq \theta \iff \theta_f(t) = \theta + f(t), f(t) \neq 0, \]

- In the second option, no hypothesis is made about the origin of the discrepancy, which is just represented as an additive vector
  \[ \begin{align*}
  \dot{x}(t) &\neq g(x(t), u(t), d(t), \theta(t)) \lor y(t) \neq h(x(t), u(t), d(t), \theta(t)) \\
  \exists (f_x, f_y) \neq (0, 0) : \\
  \dot{x}(t) &= g(x(t), u(t), d(t), \theta(t)) + f_x(t) \\
  y(t) &= h(x(t), u(t), d(t), \theta(t)) + f_y(t). 
  \end{align*} \]

In both cases, the normal and the faulty system are represented using some “fault vector” \( f(t) \) where normal operation is associated with \( f(t) = 0 \).

Most often, the preliminary analysis of the system has identified a set of faults that are likely to occur, and that the FDI system to be designed should detect, isolate and estimate. When such knowledge is available, it results in the logical statement

\[ i \in I : H_i \iff f(t) = f_i(\eta_i, t) \neq 0, \]

where \( H_i \) denotes the \( i \)th fault situation, \( i = 1, 2, \ldots, n_f \) where \( n_f \) is the number of possible fault modes, and the knowledge available about each fault is modelled by the possible time evolution of the vector \( f \) which depends on some unknown parameters \( \eta_i \) (fault estimation therefore directly refers to the estimation of these parameters).
6.2.2. Analytical Redundancy Relations with No Unknown Inputs

Introducing the fault vector $f(t)$ in the state and measurement equations, and setting $d(t) = 0$, for all $t$ one gets

$$\dot{x}(t) = g(x(t), u(t), f(t)), \quad y(t) = h(x(t), u(t), f(t)), \quad (6.10)$$

It turns out that from (6.10), it is possible to construct residuals, i.e. quantities which can be computed in real time from the available data, and whose behaviour is different under the different situations $H_0$ and $H_1$.

6.2.2. Analytical Redundancy Relations with No Unknown Inputs

Such residuals are obtained from a two step construction:

**Step 1: Derivation of the outputs**

**Step 2: Elimination of the state**

**Step 1: Derivation of the outputs**
Assuming that all functions are differentiable with respect to their arguments, it is possible to construct the derivative $dy(t)/dt$ of the output signal $y(t)$:

$$\dot{y}(t) = \frac{\partial h}{\partial x}(\cdot)\dot{x}(t) + \frac{\partial h}{\partial u}(\cdot)\dot{u}(t) + \frac{\partial h}{\partial f}(\cdot)\dot{f}(t)$$
6.2.2. Analytical Redundancy Relations with No Unknown Inputs

Replacing $x'(t)$ by its value, one gets

$$\dot{y}(t) = \frac{\partial h}{\partial x}(\cdot) g(x(t), u(t), f(t)) + \frac{\partial h}{\partial u}(\cdot) \dot{u}(t) + \frac{\partial h}{\partial f}(\cdot) \dot{f}(t)$$

$$:= h_1(x(t), \tilde{u}^{(1)}(t), \tilde{f}^{(1)}(t)),$$

where $\tilde{u}^{(1)}(t)$ is a short notation for $(\mathbf{u}^T(t), \dot{\mathbf{u}}^T(t))^T$.

Iterating this process until some order of derivation $q$ (to be determined later), and assuming the existence of all required derivatives, one obtains.

$$\tilde{y}^{(q)}(t) = H^q \left( x(t), \tilde{u}^{(q)}(t), \tilde{f}^{(q)}(t) \right)$$

(6.11)

which is a set of $(q + 1) p$ equations—or constraints—that the dimension of $\tilde{y}^{(q)}(t))$, where the different variables have the following dimensions: $x \in \mathbb{R}^n$, $\tilde{u}^{(q)}(t) \in \mathbb{R}^{(q+1) \times m}$, $\tilde{f}^{(q)}(t) \in \mathbb{R}^{(q+1) \times n_f}$. The known variables are $\tilde{y}^{(q)}$ and $\tilde{u}^{(q)}$ while the unknown variables are $x$. $\tilde{f}^{(q)}(t)$ has a particular status, since it is known (equal to zero) when $\mathcal{H}_0$ is true, while it is unknown when $\mathcal{H}_1$ is true.

$p$: Number of output variables
$m$: Number of input variables
$n_f$: Number of faults
6.2.2. Analytical Redundancy Relations with No Unknown Inputs

Example 6.2 Redundancy in a nonlinear system

\[
\begin{pmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t)
\end{pmatrix} = \begin{pmatrix}
-x_1(t) + x_2^2(t) + u(t) + f_1(t) \\
-2x_2(t) + f_2(t)
\end{pmatrix}
\]
\[y(t) = x_1(t) + f_3(t)\]

gives

\[
\dot{y}(t) = -x_1(t) + x_2^2(t) + u(t) + f_1(t) + \dot{f}_3(t)
\]
\[\ddot{y}(t) = x_1(t) - 5x_2^2(t) - u(t) - f_1(t) + 2x_2(t)f_2(t) + \dot{u}(t) + \dot{f}_1(t) + \dot{f}_3(t).\]

(6.11) is thus a system of three equations.

\[
\begin{pmatrix}
\dot{y}(t) \\
\ddot{y}(t)
\end{pmatrix} = \begin{pmatrix}
y(t) \\
\dot{y}(t)
\end{pmatrix} \quad (6.12)
\]
\[
\begin{pmatrix}
\dot{y}(t) \\
\ddot{y}(t)
\end{pmatrix} = \begin{pmatrix}
-x_1(t) + x_2^2(t) + u(t) + f_1(t) + f_3(t) \\
x_1(t) - 5x_2^2(t) - u(t) - f_1(t) + 2x_2(t)f_2(t) + \dot{u}(t) + \dot{f}_1(t) + \dot{f}_3(t)
\end{pmatrix}.
\]
6.2.2. Analytical Redundancy Relations with No Unknown Inputs

Step 2: Elimination of the state.

Assume that:

- \((q + 1) p > n\)
  
  This condition gives a lower bound on the order of derivation that is necessary in establishing (6.11).

- the Jacobian \(\frac{\partial H^q}{\partial \mathbf{x}}\) is of rank \(n\).

It follows that (6.11) can be decomposed into

\[
\begin{pmatrix}
\dot{\mathbf{y}}_m^{(q)}(t) \\
\dot{\mathbf{y}}_m^{(q)}(t)
\end{pmatrix}
= \begin{pmatrix}
H_m^q \left( \mathbf{x}(t), \mathbf{u}^{(q)}(t), \mathbf{f}^{(q)}(t) \right) \\
H_m^q \left( \mathbf{x}(t), \mathbf{u}^{(q)}(t), \mathbf{f}^{(q)}(t) \right)
\end{pmatrix}
= \mathbf{0} \quad (6.13)
\]

where the first subsystem is of dimension \(n\) and allows to compute \(x(t)\) (at least locally) as a function of the other variables

\[
x(t) = \phi(\dot{\mathbf{y}}_m^{(q)}(t), \mathbf{u}^{(q)}(t), \mathbf{f}^{(q)}(t))
\]

(this results from the implicit function theorem).

Replacing \(x(t)\) by its value in the second subsystem, which is of dimension \((q + 1) p - n\), one obtains a system that is equivalent to (6.11)

\[
x(t) = \phi(\dot{\mathbf{y}}_m^{(q)}(t), \mathbf{u}^{(q)}(t), \mathbf{f}^{(q)}(t)) \quad (6.14)
\]

\[
0 = r(\dot{\mathbf{y}}^{(q)}(t), \mathbf{u}^{(q)}(t), \mathbf{f}^{(q)}(t)), \quad (6.15)
\]
6.2.2. Analytical Redundancy Relations with No Unknown Inputs

Step 2: Elimination of the state.

where

\[ r(\tilde{y}(q)(t), \tilde{u}(q)(t), \tilde{f}(q)(t)) \]
\[ = \tilde{y}_{\text{arr}}(q)(t) - H_{\text{arr}}^q(\phi(\tilde{y}_m(t)), \tilde{u}(q)(t), \tilde{f}(q)(t)), \tilde{u}(q)(t), \tilde{f}(q)(t)). \]

The set of constraints (6.15) is seen to contain only inputs, outputs and fault signals (along with their derivatives).

It is called an analytical redundancy relations (ARR) associated with the pair \((g, h)\).

\[ r(\tilde{y}(q), \tilde{u}(q), \tilde{f}(q)) \]

is called the residual vector.

---

6.2.2. Analytical Redundancy Relations with No Unknown Inputs

Example 6.2 (Cont.) Redundancy in a nonlinear system

Step 2 is now applied to (6.12). The variable \(t\) is omitted below. The state \((x_1, x_2)\) can be computed from the first two equations of (6.12) leading to the equivalent system

\[
\begin{equation}
\begin{pmatrix}
\dot{y}(t) \\
\dot{x}(t)
\end{pmatrix}
\begin{pmatrix}
y(t) \\
x(t)
\end{pmatrix}
= 
\begin{pmatrix}
x_1(t) + x_3(t) \\
x_1(t) - 5x_2(t)
\end{pmatrix}
\begin{pmatrix}
x_2(t) \\
x_2(t)
\end{pmatrix}
\end{equation}
\]

\(x_1 = y - f_3\)

\(x_2 = \pm \sqrt{\dot{y} + y - f_1 - u - f_3}\)

\(0 = \ddot{y} - y + f_3 + 5(\dot{y} + y - f_3 - u - f_1 - f_3) + u + \cdots\)

where the third equation is seen to depend only on the available inputs and outputs and on the faults. \(\Box\)
6.2.3 Unknown Inputs, Exact Decoupling

When unknown inputs are present, a state-space model of the system takes the form

\[ \dot{x}(t) = g(x(t), u(t), d(t), f(t)) \]
\[ y(t) = h(x(t), u(t), d(t), f(t)). \]  
(6.17)

Applying the same technique as above leads to

\[ \tilde{y}^{(q)}(t) = H^q \left( x(t), \tilde{u}^{(q)}(t), \tilde{d}^{(q)}(t), \tilde{j}^{(q)}(t) \right). \]  
(6.18)

Under the condition that \((q + 1) p > n + (q + 1) n_d\) and the Jacobian

\[ \begin{bmatrix} \frac{\partial H^q(\cdot)}{\partial x} & \frac{\partial H^q(\cdot)}{\partial \tilde{d}^{(q)}} \end{bmatrix} \]

is of rank \(n + (q + 1) n_d\), the analytical redundancy relations, which are independent of the unknown inputs, hence the name “exact decoupling” which is given to this approach.

* Note that exact decoupling is possible only if the structural graph of system (6.18) is overconstrained with respect to both the unknowns \(x\) and \(d^-(q)\).
6.2.4 How to Find Analytical Redundancy Relations

There are several procedures by which ARR can be found. They all rest on the elimination of \( x(t) \) (and \( \tilde{d}^{(q)}(t) \) when unknown inputs are present), either by starting with (6.8) and (6.9) or by establishing first (6.11).

Elimination procedures fit the nature of the functions \( g \) and \( h \).
- When all functions are linear, projection approaches are well suited: this is the parity space approach. (Section 6.3)
- Most often, nonlinear models involve polynomial functions (because polynomials can approximate any smooth nonlinear function). There are, basically, three elimination techniques for polynomial functions. All three require the components of the state to be eliminated according to some selected order.

*** Elimination theory rests on Euclidean division and derivation

6.2.5 ARR-based Diagnosis

Fault detection. In the absence of unknown inputs, or when exact decoupling is possible, the following logical statements hold

\[
\begin{align*}
(6.10) & \iff (6.14), (6.15) \Rightarrow r(\tilde{y}^{(q)}(t), \tilde{u}^{(q)}(t), \tilde{f}^{(q)}(t)) = 0 \\
(6.17) & \iff (6.19) \Rightarrow r(\tilde{y}^{(q)}(t), \tilde{u}^{(q)}(t), \tilde{f}^{(q)}(t)) = 0
\end{align*}
\]

(6.21)

From (6.21) it follows that in both cases necessary conditions for normal system operation are given by

\[
H_0 \Rightarrow r(\tilde{y}^{(q)}(t), \tilde{u}^{(q)}(t), 0) = 0 \Leftrightarrow H_0
\]

(Non detectable Faults)

Therefore, fault detection immediately follows from

\[
r(\tilde{y}^{(q)}(t), \tilde{u}^{(q)}(t), 0) \neq 0 \Rightarrow H_1.
\]
6.2.5 ARR-based Diagnosis

\[ r(\tilde{y}(q)(t), \tilde{u}(q)(t), 0) = 0 \implies "H_0 \text{ is not falsified by the observations}" \]

or

"It is not impossible that the system is healthy".

In fact, special fault values that are not detectable through analytical redundancy could exist.

They correspond to non-zero values of \( f(t) \) that yield

\[ r(\tilde{y}(q)(t), \tilde{u}(q)(t), \tilde{f}(q)(t)) = 0 \]

Example 6.2 (cont.) Redundancy in a nonlinear system

The redundancy relation in (6.16) writes

\[
\ddot{y} + 5\dot{y} + 4y - 4u - \dot{u} = f_1 - 2\left(\sqrt{\dot{y} + y - f_3 - u - f_1 - \dot{f}_3}\right) f_2 + 4f_3 + \dot{f}_1 + 5\dot{f}_3 + \ddot{f}_3.
\]

Therefore, the residual is

\[
r(\tilde{y}^{(2)}, \tilde{u}^{(2)}, 0) = \ddot{y} + 5\dot{y} + 4y - 4u - \dot{u}
\]

and the fault detection rule is

\[
\ddot{y} + 5\dot{y} + 4y - 4u - \dot{u} \neq 0 \implies \mathcal{H}_1. \square
\]
### 6.2.5 ARR-based Diagnosis

**Fault isolation.** Fault isolation is approached in a similar way, by the design of structured residuals. Assume it is possible to separate the set of faults \( I \) into two subsets \( I_1 \) and \( I_2 \) such that \( I = I_1 \cup I_2 \). Set

\[
f(t) = (f^T_{I_1}(t) \ f^T_{I_2}(t))^T
\]

where only \( f_{I_1}(t) \) \( (f_{I_2}(t)) \) is non-zero upon occurrence of a fault in \( I_1 \) \( (I_2) \). If the set of residuals can also be separated in two subsets:

\[
r(y^{(s)}(t), \bar{u}^{(s)}(t), \bar{f}^{(s)}(t)) = \begin{pmatrix} r_1(y^{(s)}(t), \bar{u}^{(s)}(t), \bar{f}^{(s)}(t)) \\ r_2(y^{(s)}(t), \bar{u}^{(s)}(t), \bar{f}^{(s)}(t)) \end{pmatrix}
\]

so that

(a) \( r_1 \) is insensitive to faults in \( I_2 \) but sensitive to faults in \( I_1 \) while

(b) \( r_2 \) is insensitive to faults in \( I_1 \) but sensitive to faults in \( I_2 \)

\[
\begin{align*}
(\forall i \in I_1 : f_{I_1}(t) = f_i (\eta_i, t) = 0) \ \& \ (\exists i \in I_2 : f_{I_2}(t) = f_i (\eta_i, t) \neq 0) & \Rightarrow r_1(y^{(s)}(t), \bar{u}^{(s)}(t), \bar{f}^{(s)}(t)) = 0 \\
(\exists i \in I_1 : f_{I_1}(t) = f_i (\eta_i, t) \neq 0) \ \& \ (\forall i \in I_2 : f_{I_2}(t) = f_i (\eta_i, t) = 0) & \Rightarrow r_2(y^{(s)}(t), \bar{u}^{(s)}(t), \bar{f}^{(s)}(t)) \neq 0
\end{align*}
\]
### 6.2.5 ARR-based Diagnosis

There are four possible situations (logical 0 means \( r = 0 \) while logical 1 means \( r = 0 \)) and the following conclusions are true.

<table>
<thead>
<tr>
<th>( r_1 )</th>
<th>( r_2 )</th>
<th>Conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>( \mathcal{H}_0 ) is not falsified (no fault is detected)</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>( \mathcal{H}_0 ) is falsified by a fault ( i \in I_2 )</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>( \mathcal{H}_0 ) is falsified by a fault ( i \in I_1 )</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>( \mathcal{H}_0 ) is falsified by a fault ( i \in I_1 ) and a fault ( j \in I_2 )</td>
</tr>
</tbody>
</table>

Therefore, under (6.22), it is possible to isolate a fault within the subset \( I_1 \) or within the subset \( I_2 \). By designing several partitions of the set of faults into two classes it is obviously possible to isolate faults within smaller subsets that result from the intersections of all these partitions.

---

#### Example 6.3 Two-tank system

Pump: \( q_P = u \cdot f(h_1) \)

Tank 1: \( \dot{h}_1 = \frac{1}{A} (q_P - q_{L_1} - q_{12}) \)

Tank 2: \( \dot{h}_2 = \frac{1}{A} (q_{12} - q_2) \)

Pipe between tanks \( (h_1 > h_2) \): \( q_{12} = k_1 \sqrt{h_1 - h_2} \)

Output pipe: \( q_2 = k_2 \sqrt{h_2} \)

Outflow measurement: \( q_m = k_m q_2 \).

The state-space and measurement equations are:

\[
\begin{pmatrix}
\dot{h}_1 \\
\dot{h}_2
\end{pmatrix} = \begin{pmatrix}
\frac{-k_1}{A} \sqrt{h_1 - h_2} + \frac{f(h_1)}{A} u - \frac{1}{A} q_L \\
\frac{k_2}{A} \sqrt{h_1 - h_2} - \frac{k_2}{A} \sqrt{h_2}
\end{pmatrix} \quad (6.23)
\]

\[
q_m = k_m k_2 \sqrt{h_2}. \quad (6.24)
\]
6.2.5 ARR-based Diagnosis

Example 6.3 Two-tank system

Derivating once the output gives

\[
\dot{q}_m = k_m k_2 (h_2)^{-1/2} \dot{h}_2
\]
\[
\ddot{q}_m = k_m k_2 (h_2)^{-1/2} \left( \frac{k_1}{A} \sqrt{h_1 - h_2} - \frac{k_2}{A} \sqrt{h_2} \right).
\]  \hspace{1cm} (6.25)

From (6.24) and (6.25) the two states \( h_1 \) and \( h_2 \) can be computed

\[
h_2 = \left( \frac{q_m}{k_m k_2} \right)^2
\]
\[
h_1 = q_m^2 \left( 1 + (1 + \dot{q}_m)^2 \right).
\]  \hspace{1cm} (6.26)

Derivating once again gives

\[
\dddot{q}_m = \frac{(h_1 - h_2)^{-1/2} \sqrt{h_2} (\dot{h}_1 - \dot{h}_2) - (h_2)^{-1/2} \dot{h}_2 (h_1 - h_2)^{1/2}}{\dot{h}_2}.
\]

where replacing \( h_1, h_2, \dot{h}_1, \dot{h}_2 \) by their values taken from (6.26), (6.23) and (6.24)–(6.25) gives the redundancy relation

\[
r(q_m, \dot{q}_m, \ddot{q}_m, u, q_L)
= \sqrt{h_2 (h_1 - h_2)^{1/2}} \ddot{q}_m - \dot{h}_1 + \dot{h}_2 + (h_2)^{-1} \dot{h}_2 (h_1 - h_2) = 0
\]  \hspace{1cm} (6.27)

and the leakage detection rule

\[
r(q_m, \dot{q}_m, \ddot{q}_m, 0) \neq 0 \Rightarrow q_L \neq 0. \]
6.3 Analytical Redundancy Relations for Linear Deterministic Systems - Time Domain

Let us consider the following continuous-time state-space model

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) + Ex_d(t) + F_x f(t), \quad x(0) = x_0, \\
y(t) &= Cx(t) + Du(t) + Ey_d(t) + F_y f(t),
\end{align*}
\]

(6.28)

where \( x \in \mathcal{R}^n \) denotes the state vector, \( u \in \mathcal{R}^m \) is the vector of measured input signals, \( y \in \mathcal{R}^p \) is the vector of measured plant output signals, \( d \in \mathcal{R}^{nd} \) and \( f \in \mathcal{R}^{nf} \) are vectors of unknown input signals, \( f \) represents the faults one wishes to detect, while \( d \) are unknown disturbances that should not be detected.

The aim is to solve the following problem.

Problem 6.1 (Design of linear analytical redundancy relations)

Given a model of the supervised process of the form (6.28), determine, if possible, a set of linear relations between the measured inputs and outputs and their derivatives up to a certain order, say \( q \), such that,

• in the absence of fault,

\[
\sum_{i=1}^{q} W_{y,i} z^{(i)}(t) + \sum_{i=1}^{q} W_{u,i} u^{(i)}(t) = 0,
\]

where \( z^{(i)}(t) \) denotes the \( i \)th derivative of \( z(t) \) and \( W_{y,i}, W_{u,i} \) are \( n_r \times p \) and \( n_r \times m \) matrices of real elements, \( n_r \) being the number of relations (to be determined),
6.3 Analytical Redundancy Relations for Linear Deterministic Systems - Time Domain

**Problem 6.1** *(Design of linear analytical redundancy relations)*

Given a model of the supervised process of the form (6.28), determine, if possible, a set of linear relations between the measured inputs and outputs and their derivatives up to a certain order, say \( q \), such that,

- **in the absence of fault**

\[
\sum_{i=1}^{q} W_{y,i} y^{(i)}(t) + \sum_{i=1}^{q} W_{u,i} u^{(i)}(t) = 0,
\]

where \( y^{(i)}(t) \) denotes the \( i \)th derivative of \( y(t) \) and \( W_{y,i}, W_{u,i} \) are \( n_y \times p \) and \( n_y \times m \) matrices of real elements, \( n_y \) being the number of relations (to be determined),

- **in the presence of fault**

\[
\sum_{i=1}^{q} W_{y,i} y^{(i)}(t) + \sum_{i=1}^{q} W_{u,i} u^{(i)}(t) \neq 0.
\]

Such relations are a particular kind of analytical redundancy relations called *parity relations*. 
6.3 Analytical Redundancy Relations for Linear Deterministic Systems - Time Domain

In order to solve this problem, let us consider the successive time derivatives of $y$ up to order $q$

\[ y(t) = Cx(t) + Du(t) + E_y d(t) + F_y f(t) \]

\[ \dot{y}(t) = C \dot{x}(t) + D \dot{u}(t) + E_y \dot{d}(t) + F_y \dot{f}(t) \]
\[ = CAx(t) + CBu(t) + D\dot{u}(t) + CE_x d(t) + E_y \dot{d}(t) + F_y \dot{f}(t) \]

where the last equality is deduced by substitution of (6.28) for $\dot{x}(t)$. By iterating this process, the following expression for the $q_{th}$ derivative of $y$ is obtained:

\[ y^{(q)}(t) = CA^q x(t) + CA^{(q-1)} Bu(t) + \cdots + CBu^{(q-1)}(t) + Du^{(q)}(t) + \]
\[ + CA^{(q-1)} E_x d(t) + \cdots + CE_x d^{(q-1)}(t) + E_y d^{(q)}(t) + \]
\[ + CA^{(q-1)} F_x f(t) + \cdots + CF_x f^{(q-1)}(t) + F_y f^{(q)}(t). \]

(6.30)

by: Dr B. Moaveni

---

6.3 Analytical Redundancy Relations for Linear Deterministic Systems - Time Domain

The above set of equations can be concatenated into the expression

\[ \ddot{y}^{(q)}(t) = \mathcal{O} x(t) + T_{u,q} \ddot{u}^{(q)}(t) + T_{d,q} \ddot{d}^{(q)}(t) + T_{f,q} \ddot{f}^{(q)}(t), \]

(6.31)

where $\ddot{y}^{(q)}(t) = \begin{bmatrix} y(t)^T & \dot{y}(t)^T & \cdots & y^{(q)}(t)^T \end{bmatrix}^T$, and $\ddot{u}^{(q)}(t), \ddot{d}^{(q)}(t), \ddot{f}^{(q)}(t)$ have a similar form with $u(t), d(t)$ and $f(t)$ substituted for $y(t)$,

\[ \mathcal{O} = \begin{pmatrix} C & \vdots & CA^q \end{pmatrix}, \quad T_{u,q} = \begin{pmatrix} D & 0 & \cdots & 0 \\ CB & D & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{q-1} B & \cdots & CB D \end{pmatrix} \]

and a similar definition holds for the block Toeplitz matrices $T_{d,q}, T_{f,q}$ with respectively $E_y$ and $E_x$ or $F_y$ and $F_x$ substituted for $D$ and $B$. 

Fault D. and FTCS
by: Dr B. Moaveni
6.3 Analytical Redundancy Relations for Linear Deterministic Systems - Time Domain

set of equations:

\[
(6.31) \rightarrow \overline{y}^{(q)}(t) = \left( O \ T_{d,q} \right) \left[ \frac{x(t)}{d^{(q)}} \right] + T_{u,q} \tilde{u}^{(q)}(t) + T_{f,q} \tilde{f}^{(q)}(t)
\]

If there exists a value of \( q \) such that

\[
\text{rank} \left( O \ T_{d,q} \right) < (q + 1)p,
\]

the left nullspace of \( (O \ T_{d,q}) \) is not empty. The dimension of this subspace, say \( n_r \), is given as \( n_r = (q + 1)p - \text{rank} \left( O \ T_{d,q} \right) \). Let \( W \) be a \( n_r \times (q + 1)p \) matrix of which each row is a basis vector for this subspace. Multiplying \( (6.31) \) on the left by \( W \) results in the following equality

\[
W \tilde{y}^{(q)}(t) - WT_{u,q} \tilde{u}^{(q)}(t) = WT_{f,q} \tilde{f}^{(q)}(t),
\]

Equation \((6.32)\) describes \( n_r \) analytical redundancy relations. Indeed, in the absence of fault, the right-hand side is equal to zero, and it is normally different from zero in the presence of a fault.

\[
W \tilde{y}^{(q)}(t) - WT_{u,q} \tilde{u}^{(q)}(t) = WT_{f,q} \tilde{f}^{(q)}(t),
\]

In order to implement such relations, and thus to compute the quantity

\[
r(t) = W \tilde{y}^{(q)}(t) - WT_{u,q} \tilde{u}^{(q)}(t),
\]
6.3 Analytical Redundancy Relations for Linear Deterministic Systems - Time Domain

It is necessary to evaluate the derivatives that appear in the above relation. Such signals are highly sensitive to noise, so that filtered estimates of the derivatives have to be used. One approach is to resort to a state variable filter, which amounts to implementing the scheme of Fig. 6.2.

\[
\begin{align*}
\frac{z_f(s)}{s^q + a_1 s^{q-1} + \cdots + a_q} &= z(s)
\end{align*}
\]

Fig. 6.2 Block diagram of a third-order state variable filter

6.3 Analytical Redundancy Relations for Linear Deterministic Systems - Time Domain

Letting \( z(t) \) denote the input of such a filter, the \( i \)th integrator output provides the \( i \)th filtered derivative of \( z \), \( z_f^{(i)} \). This filter corresponds to the analog simulation of the observability canonical state-space representation for the relation.

\[
z_f(s) = \frac{1}{s^q + a_1 s^{q-1} + \cdots + a_q} z(s)
\]

By taking this filter into account, (6.33) can be rewritten in the frequency domain as

\[
r_f(s) = (W_y(s)y(s) + W_u(s)u(s))/p_f(s), \quad (6.34)
\]

where

\[
W_y(s) = \sum_{i=0}^{q} W_i s^i \quad p_f(s) = s^q + a_1 s^{q-1} + \cdots + a_q
\]
Vector $r$ is called a **parity vector**. It has generally different directions and magnitudes in the presence of the different fault modes. The $n_r$-dimensional space of all such vectors is called the **parity space**, and any linear combination of the rows of (6.33) is called a **parity relation**.

---

**Algorithm 6.1** Parity relations for deterministic linear systems

Given: A linear state-space model of the form (6.28) and a suitable order of derivation $q$

Compute off-line:
1. Matrices $O$, $T_{d,q}$, $T_{u,q}$
2. A basis $W$ for the left null space of $(O \ T_{d,q})$
3. State space filters for the estimation of the derivatives of $y$ and $u$ up to order $q$.

At each time instant:
1. Acquire the new data $y(t)$, $u(t)$.
2. Compute $r_f(t)$ from (6.34).

Result: A residual vector $r_f(t)$ for an increasing time horizon.
6.3 Analytical Redundancy Relations for Linear Deterministic Systems - Time Domain

Example 6.4 Parity relations for the ship

A linearized model of the ship example can be written as

\[
\begin{pmatrix}
\dot{\omega}_3 \\
\dot{\psi}
\end{pmatrix} = \begin{pmatrix}
b_1 & 0 \\
1 & 0
\end{pmatrix} \begin{pmatrix}
\omega_3 \\
\psi
\end{pmatrix} + \begin{pmatrix}
b \\
0
\end{pmatrix} \delta + \begin{pmatrix}
0 \\
1
\end{pmatrix} \omega_w
\]

(6.35)

\[
\begin{pmatrix}
\omega_{3m} \\
\psi_m
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix} \begin{pmatrix}
\omega_3 \\
\psi
\end{pmatrix} + \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix} \begin{pmatrix}
f_\omega \\
f_\psi
\end{pmatrix} + \begin{pmatrix}
1 \\
0
\end{pmatrix} \omega_w
\]

(6.36)

when linearisation around \( \omega_3 = 0 \) is considered. Here \( \delta \), the rudder angle, is a known input, while \( \omega_w \), the wave disturbance, is an unknown input.

6.3 Analytical Redundancy Relations for Linear Deterministic Systems - Time Domain

Example 6.4 Parity relations for the ship

Straightforward computations yield the following expression for (6.31) with \( q = 1 \):

\[
\begin{pmatrix}
\omega_{3m} \\
\psi_m
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix} \begin{pmatrix}
\omega_3 \\
\psi
\end{pmatrix} + \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix} \begin{pmatrix}
\delta
\end{pmatrix}
\]

\[
+ \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix} \begin{pmatrix}
\omega_w \\
\dot{\omega}_w
\end{pmatrix} + \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
f_\omega \\
f_\dot{\omega} \\
f_\dot{\psi} \\
f_\dot{\psi}
\end{pmatrix}
\]

(6.37)

The block matrix \( (\mathcal{O} \quad T_{d,1}) \) takes the form:

\[
(\mathcal{O} \quad T_{d,1}) = \begin{pmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
b_{\text{pr}} & 0 & 0 & 1 \\
1 & 0 & 1 & 0
\end{pmatrix}
\]
6.3 Analytical Redundancy Relations for Linear Deterministic Systems - Time Domain

Example 6.4 Parity relations for the ship

A basis vector for the one-dimensional left nullspace of this matrix can be written

\[ W = \begin{pmatrix} 1 & 0 & 0 & -1 \end{pmatrix} \]

Expression (6.32) then yields

\[ \omega_{3m} - \dot{\psi}_m = f_\omega - \dot{f}_\psi \]

Hence a residual can be computed according as:

\[ r_f(s) = (\omega_{3m}(s) - s\psi_m(s))/(s + a). \]  \hfill (6.38)
6.4 Analytical Redundancy Relations for Linear Deterministic Systems - FREQUENCY DOMAIN

In this section, the problems of fault detection, fault isolation and fault estimation are solved using the parity space approach to residual generation, from an input-output model of the supervised system.

An alternative method would be to design observer based residual generators, which yields similar filters, as indicated in the bibliographical notes.

6.4.1 Fault Detection

Consider again a system described by a linear continuous-time state-space model of the form

\[\dot{x}(t) = Ax(t) + Bu(t) + E_x d(t) + F_x f(t), \quad x(0) = x_0\]
\[y(t) =Cx(t) + Du(t) + E_y d(t) + F_y f(t),\]  \hspace{1cm} (6.39)

where \(x \in \mathbb{R}^n\) denotes the state vector, \(u \in \mathbb{R}^m\) is the vector of measured input signals, \(y \in \mathbb{R}^p\) is the vector of measured plant output signals, \(d \in \mathbb{R}^d\) and \(f \in \mathbb{R}^f\) are vectors of unknown input signals. \(f\) represents the faults one wishes to detect, while \(d\) are unknown disturbances that should not be detected.
6.4 Analytical Redundancy Relations for Linear Deterministic Systems - FREQUENCY DOMAIN

Such a model can also be written in terms of transfer functions:

\[ y(s) = H_{yu}(s)u(s) + H_{yd}(s)d(s) + H_{yx}(s)x(0) + H_{yf}(s)f(s), \]  
\hspace{1cm} (6.40)

where

\[ H_{yu}(s) = C(sI - A)^{-1}B + D \]
\[ H_{yx}(s) = C(sI - A)^{-1} \]
\[ H_{yd}(s) = C(sI - A)^{-1}E_x + E_y \]
\[ H_{yf}(s) = C(sI - A)^{-1}F_x + F_y. \]

As indicated in Fig. 6.1, a residual generator is a filter with input \( u \) and \( y \).

As supervision of linear time-invariant systems is addressed here, the class of considered filters will be restricted to linear time-invariant systems of the following form:

\[ \dot{z}(t) = A\cdot z(t) + B_{ru} u(t) + B_{zy} y(t), \quad z(0) = z_0 \]
\[ r(t) = C_{rz} z(t) + D_{ru} u(t) + D_{ry} y(t) \]  
\hspace{1cm} (6.41)

or, in transfer function form, assuming zero initial conditions:

\[ r(s) = V_{ru}(s)u(s) + V_{ry}(s)y(s) = \begin{pmatrix} V_{ru}(s) & V_{ry}(s) \end{pmatrix} \begin{pmatrix} u(s) \\ y(s) \end{pmatrix} \]  
\hspace{1cm} (6.42)
6.4 Analytical Redundancy Relations for Linear Deterministic Systems - FREQUENCY DOMAIN

**Problem 6.2 (Residual generator design for fault detection based on a deterministic model)**

Given, a model of the supervised process of the form (6.39) or (6.40) determine a stable linear time-invariant system (6.41) or (6.42) such that:

- In the absence of fault ($f(t) = 0$ for all $t$), the output signal $r(t)$, $t > 0$ asymptotically decays to zero for any input $u(t)$, $d(t)$, $t > 0$ and any initial conditions $x(0)$, $z(0)$.

- $r(t)$ is affected by $f(t)$ (Fault detectability issue)

---

**Definition 6.1 (Weak detectability)**

The $i^{th}$ fault ($f_i(t)=0$ for all $t \geq t_0$) is weakly detectable if there exists a stable residual generator such that $r(t)$ is affected by $f_i(t)$. $r(t)$ reaches to zero steady state value.

In the literature, weak detectability is also referred to as detectability.

**Definition 6.2 (Strong detectability)**

A fault $f_i$ is strongly detectable if there exists a stable residual generator such that $r(t)$ reaches a non-zero steady-state value for a fault signal that has a bounded final value different from zero.
6.4 Analytical Redundancy Relations for Linear Deterministic Systems - FREQUENCY DOMAIN

6.4.2 Solution by the Parity Space Approach

In order to determine the conditions to be fulfilled by $V_{ru}(s)$ and $V_{ry}(s)$ for (6.42) to be a residual generator, (6.40) is substituted for $y(s)$ in (6.42):

$$ r(s) = V_{ru}(s)u(s) + V_{ry}(s) \left( H_{yu}(s)u(s) + H_{yx}(s)x(0) \right) 
+ H_{yd}(s)d(s) + H_{yf}(s)f(s) 
= (V_{ru}(s) + V_{ry}(s)H_{yu}(s) \quad V_{ry}(s)H_{yd}(s)) \begin{pmatrix} u(s) \\ d(s) \end{pmatrix} 
+ V_{ry}(s)H_{yx}(s)x(0) + V_{ry}(s)H_{yf}(s)f(s) $$

(6.43)

Figure 6.3 illustrates this residual generator.
6.4 Analytical Redundancy Relations for Linear Deterministic Systems - FREQUENCY DOMAIN

Fulfillment of the first condition in Problem 6.2 requires:

\[
(V_{ru}(s) + V_{ry}(s)H_{yu}(s) \quad V_{ry}(s)H_{yd}(s)) = O
\]  \hspace{1cm} (6.44)

together with the asymptotic stability of \( V_{ry}(s)H_{yx}(s) \).

Since, in healthy working mode, the plant is normally stabilized by an appropriate controller, the latter condition amounts to requiring the stability of the filter. This can be guaranteed by an appropriate choice of the denominator of \( V_{ru}(s) \) and \( V_{ry}(s) \).

Therefore, we concentrate now on the way to achieve (6.44).

Note that (6.44) can be rewritten:

\[
(V_{ry}(s) \quad V_{ru}(s)) \begin{pmatrix} H_{yu}(s) & H_{yd}(s) \\ I & O \end{pmatrix} = 0.
\]  \hspace{1cm} (6.45)

For any filter, the least common multiple of the denominators of the entries of \( V_{ry}(s) \) and \( V_{ru}(s) \), \( p(s) \) can be determined. Using \( p(s) \), the left most matrix in (6.45) can be written:

\[
(V_{ry}(s) \quad V_{ru}(s)) = \frac{(\tilde{V}_{ry}(s) \quad \tilde{V}_{ru}(s))}{p(s)},
\]  \hspace{1cm} (6.46)
6.4 Analytical Redundancy Relations for Linear Deterministic Systems - FREQUENCY DOMAIN

where \( \bar{V}_{ry}(s) \) and \( \bar{V}_{ru}(s) \) are suitable polynomial matrices. Hence, the whole class of filters that meet (6.45) can be obtained by characterising the set of polynomial matrices \( \begin{pmatrix} \bar{V}_{ry}(s) & \bar{V}_{ru}(s) \end{pmatrix} \) that fulfil:

\[
\begin{pmatrix} \bar{V}_{ry}(s) & \bar{V}_{ru}(s) \end{pmatrix} \begin{pmatrix} H_{yu}(s) & H_{yd}(s) \\ I & O \end{pmatrix} = 0.
\]  

(6.47)

This is the set of polynomial matrices that lie in the left nullspace of

\[
H(s) = \begin{pmatrix} H_{yu}(s) & H_{yd}(s) \\ I & O \end{pmatrix}.
\]

(6.48)

This space is denoted \( N_L(H(s)) \). Its dimension is equal to the difference between the number of rows of \( H(s) \) and its rank, namely

\[
\dim(N_L(H(s))) = m + p - \text{rank } H(s) = m + p - (m + n_d) = p - n_d,
\]

where
\[
m: \text{the number of inputs},
\]
\[
p: \text{the number of outputs},
\]
\[
n_d: \text{the number of unknown inputs (disturbances)}.
\]

It has been assumed that \( H_{yu}(s) \) and \( H_{yd}(s) \) have full column rank.

- Note that the number of plant output signals must be larger than the number of disturbances for the left null space to be non-zero.

Fault D. and FTCS
by: Dr B. Moaveni
6.4 Analytical Redundancy Relations for Linear Deterministic Systems - FREQUENCY DOMAIN

One way to characterise the set of polynomial matrices \( \begin{bmatrix} V_{ry}(s) & V_{ru}(s) \end{bmatrix} \) that meet (6.47) is to determine an irreducible polynomial basis, for the rational vector space \( \mathcal{N}_L(H(s)) \). Further, let \( F(s) \) be a matrix of which the rows make such an irreducible polynomial basis, then any suitable matrix \( \begin{bmatrix} \tilde{V}_{ry}(s) & \tilde{V}_{ru}(s) \end{bmatrix} \) can be obtained by combinations of the rows of \( F(s) \), namely

\[
\begin{bmatrix} \tilde{V}_{ry}(s) & \tilde{V}_{ru}(s) \end{bmatrix} = Q(s)F(s), \tag{6.49}
\]

where \( Q(s) \) is an arbitrary polynomial matrix with appropriate number of columns.

A general parametrisation of the family of residual generators is obtained from (6.49). Substitution of (6.49) into (6.46) yields

\[
\begin{bmatrix} V_{ry}(s) & V_{ru}(s) \end{bmatrix} = \frac{Q(s)F(s)}{p(s)}. \tag{6.49}
\]

Introducing this expression into (6.42) finally results in

\[
r(s) = \frac{Q(s)F(s)}{p(s)} \begin{bmatrix} y(s) \\ u(s) \end{bmatrix}. \tag{6.50}
\]

The choice of the matrix \( Q(s) \) and the polynomial \( p(s) \) depends on the specification of the diagnosis problem. Typically, the residual generator should ensure filtering of high frequency disturbances which always exist, even though they were not considered in the model, and adequate properties at low frequencies. Sometimes, precise information on the frequency range of the fault is available, and \( Q(s)/p(s) \) can be designed to perform appropriate filtering.
### 6.4 Analytical Redundancy Relations for Linear Deterministic Systems - FREQUENCY DOMAIN

#### Modelling uncertainty

Although modelling uncertainties have not been introduced here, they can be accounted for a posteriori when \( F(s) \) has several rows. \( Q(s) \) is then used to select appropriate rows in \( F(s) \). To explain the idea, let \( F_i(s), i = 1, \ldots, n_r \) denote the \( i \)th row of \( F(s) \) and consider the scalar residuals

\[
r_i(s) = \frac{F_i(s)}{p(s)} \begin{pmatrix} y(s) \\ u(s) \end{pmatrix} \quad i = 1, \ldots, n_r.
\]

By performing a simulation of all these filters with actual plant measurements as input, one may compare how significantly the actual residuals \( r_i(t), i = 1, \ldots, n_r \) deviate from zero in the absence of fault, once the transient due to initial conditions has vanished. This reflects the effect of modelling errors on the residuals.

---

### 6.4 Analytical Redundancy Relations for Linear Deterministic Systems - FREQUENCY DOMAIN

Besides, by using faulty data obtained with a simulation or corresponding to actual plant measurements it is also possible to compare the actual sensitivities to faults. A kind of “signal to noise ratio” could be defined for each residual as

\[
SNR_i = \frac{\int_{t_0}^{t_0+T} r_i^{FF}(t) dt}{\int_{t_1}^{t_1+T} r_i^{FF2}(t) dt}, \quad (6.51)
\]

where \( r_i^{FF}(t) \) denotes the residual obtained with the measurement associated to the faulty mode, and \( r_i^{FF} \) corresponds to the fault free situation. \( T \) is a user defined horizon, \( t_0 \) and \( t_1 \) are time instants associated to faulty and fault free data sequences. Matrix \( Q(s) \) should then be chosen to select the components of \( r(s) \) for which the “signal-to-noise ratio” is significantly larger than 1.
6.4 Analytical Redundancy Relations for Linear Deterministic Systems - FREQUENCY DOMAIN

Computational aspects. The problem of finding an irreducible polynomial basis for $N_f(H(s))$ can be transformed into the determination of a similar basis for a polynomial matrix instead of the rational matrix $H(s)$. It suffices to notice that

$$H(s) = \bar{H}(s)/h(s)$$

where $h(s)$ is the least common multiple of all denominators. An irreducible polynomial basis for $\bar{H}(s)$ is also an irreducible polynomial basis for $H(s)$, and vice-versa. Numerically stable algorithms for the computation of an irreducible polynomial basis are available in the literature, and they have been programmed in the polynomial toolbox of MATLAB.

Example 6.4 (cont.) Parity relations for the ship

A model of the form (6.40) can be easily deduced from the linear state-space model for the ship example. The following transfer matrices are obtained when sensor faults are considered, and when state and sensor noise are neglected:

$$y(s) = H_{yu}(s)u(s) + H_{yd}(s)d(s) + H_{yx}(s)x(0) + H_{yf}(s)f(s),$$

(6.40)

$$H_{yu}(s) = \begin{pmatrix} \frac{b}{s-b\eta_1} \\ \frac{b}{(s-b\eta_1)s} \end{pmatrix}, \quad H_{yd}(s) = \begin{pmatrix} 1 \\ s \end{pmatrix}, \quad H_{yx}(s) = \begin{pmatrix} 1 \\ \frac{1}{s-b\eta_1} \\ 1 \end{pmatrix}, \quad H_{yf}(s) = \frac{1}{s(s-b\eta_1)} \begin{pmatrix} 0 \\ 1 \\ s \end{pmatrix}$$

$$H_{yf} = I_2$$
6.4 Analytical Redundancy Relations for Linear Deterministic Systems - FREQUENCY DOMAIN

Example 6.4 (cont.) Parity relations for the ship

where

\[ x(t) = (\omega_3(t) \quad \psi(t))^T \]
\[ y(t) = (\omega_3m(t) \quad \psi_m(t))^T \]
\[ d(t) = \omega_m(t) \]
\[ u(t) = \delta(t) \]
\[ f(t) = (f_\omega(t) \quad f_\psi(t))^T \]

It is assumed that \( \eta_1 \) is negative, so that the ship is stable. \( H_{yx}(s) \) is not asymptotically stable however, due to the integrator linking speed and position. We shall see below what slight modification must be introduced in the theory to handle the pole at the origin.

Fault D. and FTCS by: Dr B. Moaveni

---

6.4 Analytical Redundancy Relations for Linear Deterministic Systems - FREQUENCY DOMAIN

Example 6.4 (cont.) Parity relations for the ship

The matrix \( H(s) \) takes the form:

\[
H(s) = \begin{pmatrix}
\frac{b}{s - b\eta_1} & 1 & 0 \\
\frac{b}{s(s - b\eta_1)} & \frac{1}{s} & 0 \\
s(s - b\eta_1) & 0 & 0
\end{pmatrix} = \frac{1}{s(s - b\eta_1)} \begin{pmatrix}
brs & s(s - b\eta_1) \\
bs & (s - b\eta_1) \\
s(s - b\eta_1) & 0
\end{pmatrix}
\]

The last matrix corresponds to \( \bar{H}(s) \). An irreducible basis for its left nullspace can be calculated, or found by inspection, to be

\[
F(s) = (1 \quad -s \quad 0).
\]
6.4 Analytical Redundancy Relations for Linear Deterministic Systems - FREQUENCY DOMAIN

Example 6.4 (cont.) Parity relations for the ship

Thus, any vector of rational functions of the form

\[
\begin{pmatrix}
\frac{q(s)}{p(s)} & -\frac{s q(s)}{p(s)} & 0
\end{pmatrix}
\]

where \( p(s) \) is an arbitrary polynomial with roots in the left-half plane and \( q(s) \) is an arbitrary polynomial with degree less than \( p(s) \), fulfils condition (6.45). Candidate residual generators have the form:

\[
 r(s) = \frac{q(s)}{p(s)} \omega_3m(s) - \frac{s q(s)}{p(s)} \psi_m(s). \tag{6.52}
\]

Note that, by setting \( q(s) = 1 \), one recovers (6.38) with \( p(s) = s + a \).

Substituting the model equations for \( \omega_3(s) \) and \( \psi(s) \) yields

\[
 r(s) = \frac{q(s)}{p(s)} \left( \frac{b}{s - b\eta_1} \delta(s) + \omega_{\psi}(s) + \frac{1}{s - b\eta_1} \omega_3(0) + f_\omega(s) \right) \\
- \frac{s q(s)}{p(s)} \left( \frac{b}{s(s - b\eta_1)} \delta(s) + \frac{\omega_{\psi}(s)}{s} + \frac{1}{s(s - b\eta_1)} \omega_3(0) + \frac{1}{s} \psi(0) + f_\psi(s) \right) \\
= \frac{q(s)}{p(s)} \omega_3(0) + \frac{q(s)}{p(s)} f_\omega(s) - \frac{s q(s)}{p(s)} f_\psi(s).
\]
6.4 Analytical Redundancy Relations for Linear Deterministic Systems - FREQUENCY DOMAIN

Example 6.4 (cont.) Parity relations for the ship

In order to assure that the residual asymptotically vanishes, two solutions are possible:

- Introduction of a derivative action in \( q(s) \), so that \( q(s) = s\tilde{q}(s) \) and the term involving \( \psi(0) \) in the above equation is null at steady state.

- Modification of (6.52) by adding a correction term associated with the initial position (supposed to be measured correctly). This yields

\[
 r(s) = q(s) \frac{\psi'(0)}{p(s)} + \frac{q(s)}{p(s)} \omega_{3m}(s) - \frac{sq(s)}{p(s)} \psi_m(s)
\]

Fault D. and FTCS
by: Dr B. Moaveni

75

6.4 Analytical Redundancy Relations for Linear Deterministic Systems - FREQUENCY DOMAIN

Example 6.4 (cont.) Parity relations for the ship

or, after substitution of \( \omega_{3m}(s) \) and \( \psi_m(s) \) in terms of the model equations:

\[
 r(s) = \frac{q(s)}{p(s)} f_{\omega}(s) - \frac{sq(s)}{p(s)} f_{\psi}(s).
\]  

(6.53)

The first solution also introduces a derivative action in the transfer functions between \( f_{\omega}(s) \) and \( r(s) \), and between \( f_{\psi}(s) \) and \( r(s) \). Hence step like faults do not have any steady-state effect on the residual. On the other hand, in (6.53), \( q(s) \) can be chosen so that a step-like fault \( f_{\omega} \) has a steady-state effect on \( r \), but a step-like fault in \( f_{\psi} \) can only influence temporarily \( r \) due to the zero at the origin in \( \frac{sq(s)}{p(s)} \).

Application of the theory below will indicate that, indeed, fault \( f_{\omega} \) is strongly detectable, but \( f_{\psi} \) is only weakly detectable.

Fault D. and FTCS
by: Dr B. Moaveni

76
6.4 Analytical Redundancy Relations for Linear Deterministic Systems - FREQUENCY DOMAIN

Example 6.5 Parity relations - ship with three output measurements

Some useful observations can be made later from the above example but using an additional instrument to measure the ship heading. This third instrument is taken to be independent of the other two. This is a realistic case since redundant heading instruments are required for most merchant ships.

With two independent heading angle measurements

\[ y_2(s) = \psi_m^{(1)}(s) \]

and

\[ y_3(s) = \psi_m^{(2)}(s), \]

the matrix \( H(s) \) takes the form:

\[
H(s) = \frac{1}{s(s - b\eta_1)} \begin{pmatrix}
bs & s(s - b\eta_1) \\
b & (s - b\eta_1) \\
b & (s - b\eta_1) \\
(s(s - b\eta_1)) & 0
\end{pmatrix}
\]

The left nullspace basis for \( H(s) \) is computed to be

\[
\begin{pmatrix}
-\frac{1}{s} \\
1 \\
0 \\
0
\end{pmatrix}
\]
6.4 Analytical Redundancy Relations for Linear Deterministic Systems - FREQUENCY DOMAIN

Example 6.5 Parity relations - ship with three output measurements

This means a family of candidate residual generators exist, which have the form

\[ r(s) = \frac{q(s)}{p(s)} \begin{pmatrix} -\frac{1}{s} & 1 & 0 & 0 \\ -\frac{1}{s} & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \omega_m(s) \\ \psi_{m1}(s) \\ \psi_{m2}(s) \\ \delta(s) \end{pmatrix} \]

The relation between components of the residual vector \( r(s) \) to faults \( f(s) \) and wave disturbance \( \omega_w \) is

\[ r_1(s) = \frac{q(s)}{p(s)} \left( -\frac{1}{s} f_\omega(s) + f_{\psi1}(s) \right) \]
\[ r_2(s) = \frac{q(s)}{p(s)} \left( -\frac{1}{s} f_\omega(s) + f_{\psi2}(s) \right) \]

Fault D. and FTCS
by: Dr B. Moaveni

---

6.4 Analytical Redundancy Relations for Linear Deterministic Systems - FREQUENCY DOMAIN

Example 6.5 Parity relations - ship with three output measurements

It is obvious that all elements of the residual are decoupled from the wave disturbance, which was the intention.

Forming a third residual using the plain difference between heading angle measurements

\[ r_3(s) = \psi_{m1}(s) - \psi_{m2}(s) \]

which would be a straightforward choice as an output parity equation, is indeed possible, but since this would be a linear relation between the two residuals already defined, this would not add to the information contained in the residual vector.

Fault D. and FTCS
by: Dr B. Moaveni
6.4 Analytical Redundancy Relations for Linear Deterministic Systems - FREQUENCY DOMAIN

Fault detectability.
To deduce theoretical results on fault detectability, the expression of the residual in the presence of faults must be determined. Substituting (6.45) into (6.43) yields

\[
\begin{pmatrix}
V_{ry}(s) & V_{ru}(s)
\end{pmatrix}
\begin{pmatrix}
H_{yu}(s) & H_{yd}(s)
\end{pmatrix}
= 0. \tag{6.45}
\]

\[
\begin{align*}
 r(s) &= V_{ru}(s)u(s) + V_{ry}(s)\left[H_{yu}(s)u(s) + H_{yx}(s)x(0)ight. \\
 &\quad + H_{yd}(s)d(s) + H_{yf}(s)f(s) \bigg] \\
 &= \left(V_{ru}(s) + V_{ry}(s)H_{yu}(s) \right) \begin{pmatrix} u(s) \\ d(s) \end{pmatrix} \\
 &\quad + V_{ry}(s)H_{yx}(s)x(0) + V_{ry}(s)H_{yf}(s)f(s) \tag{6.43}
\end{align*}
\]

Fault D. and FTCS
by: Dr. B. Moaveni

---

6.4 Analytical Redundancy Relations for Linear Deterministic Systems - FREQUENCY DOMAIN

Fault detectability.
Substituting (6.45) into (6.43) yields

\[
r(s) = V_{ry}(s)H_{yx}(s)x(0) + V_{ry}(s)H_{yf}(s)f(s) \tag{6.54}
\]

\[
= V_{ry}(s)H_{yx}(s)x(0) + \sum_{i=1}^{n_f} V_{ry}(s)H_{yf}^i(s)f_i(s),
\]

where \(H_{yf}^i(s)\) denotes the \(i\)th column of \(H_{yf}(s)\). It can be shown that a necessary and sufficient condition for detectability of the \(i\)th fault is:

\[
V_{ry}(s)H_{yf}^i(s) \neq 0, \tag{6.55}
\]

where \(V_{ry}(s)\) also fulfills

\[
V_{ry}(s)H_{yd}(s) = 0. \tag{6.56}
\]

Fault D. and FTCS
by: Dr. B. Moaveni
6.4 Analytical Redundancy Relations for Linear Deterministic Systems - **FREQUENCY DOMAIN**

**Fault detectability.**
For (6.55) and (6.56) to be simultaneously verified, one should not be able to express \( H^i_{yf}(s) \) as a linear combination of the columns of \( H_{yd}(s) \). In other words, there cannot exist any nonzero polynomial set \( \alpha_0(s), \alpha_1(s), \ldots, \alpha_{nd}(s) \) such that:

\[
\alpha_0(s)H^i_{yf}(s) + \alpha_1(s)H^1_{yd}(s) + \cdots + \alpha_{nd}(s)H^{nd}_{yd}(s) = 0
\]

This condition is fulfilled when

\[
\text{rank } \begin{pmatrix} H_{yd}(s) & H^i_{yf}(s) \end{pmatrix} > \text{rank } H_{yd}(s), \quad (6.57)
\]
where

\[
\text{rank } A(s) = \max_s \text{rank } A(s)
\]

**Fault detectability.**

**Necessary and sufficient condition for the \( i^{th} \) fault to be weakly detectable:**

\[
\text{rank } \begin{pmatrix} H_{yd}(s) & H^i_{yf}(s) \end{pmatrix} > \text{rank } H_{yd}(s), \quad (6.57)
\]
### 6.4 Analytical Redundancy Relations for Linear Deterministic Systems - FREQUENCY DOMAIN

**Fault detectability.**

To determine a test for strong fault detectability, substitute the model (6.40) for \( y(s) \) in the parametrisation of the class of residual generators (6.50)

\[
    r(s) = \frac{Q(s)F(s)}{p(s)} \left( \begin{array}{ccc}
        H_{yf}(s) & H_{yd}(s) & H_{yf}(s) \\
        I & O & O
    \end{array} \right) \begin{pmatrix} u(s) \\ d(s) \\ f(s) \end{pmatrix}
\]

\[
    = \frac{Q(s)F(s)}{p(s)} \begin{pmatrix} H_{yf}(s) \\ O \end{pmatrix} f(s),
\]

where the second equality accounts for the fact that \( F(s) \) is a basis for the left nullspace of \( H(s) \).

---

**6.4 Analytical Redundancy Relations for Linear Deterministic Systems - FREQUENCY DOMAIN**

**Fault detectability.**

The transient term due to \( x(0) \) was not considered as its effect vanishes when \( t \) tends to infinity.

**Strong detectability** of the fault \( f_i \) is thus achieved if there exists some polynomial \( p(s) \) and polynomial matrix \( Q(s) \) such that

\[
    \left. \frac{Q(s)F(s)}{p(s)} \begin{pmatrix} H_{yf}^i(s) \\ O \end{pmatrix} \right|_{s=0} \neq 0.
\]  

(6.59)

As \( p(0) \) is necessarily chosen non-zero to assure asymptotic stability of the filter, and \( Q(s) \) can be chosen arbitrarily.

**Necessary and sufficient condition for strong fault detectability is**

\[
    F(s) \left( \begin{pmatrix} H_{yf}(s) \\ O \end{pmatrix} \right) \bigg|_{s=0} \neq 0.
\]  

(6.60)
6.4 Analytical Redundancy Relations for Linear Deterministic Systems - FREQUENCY DOMAIN

Fault detectability.

Note that this expression may be different from $F(0) \left( \frac{H^T_{W(0)}}{O} \right)^T$ and, hence, substitution by $s = 0$ must be performed after computation of the matrix product.

---

Example 6.6 Detectability - ship with two output measurements

To check that fault $f_{\omega}$ is detectable, (6.57) is applied as follows

$$\text{rank} \begin{pmatrix} 1 & 1 \\ \frac{1}{s} & 0 \end{pmatrix} > \text{rank} \begin{pmatrix} 1 \\ \frac{1}{s} \end{pmatrix}$$

Similarly, the inequality

$$\text{rank} \begin{pmatrix} 1 & 0 \\ \frac{1}{s} & 1 \end{pmatrix} > \text{rank} \begin{pmatrix} 1 \\ \frac{1}{s} \end{pmatrix}$$

ensures that $f_{\psi}$ is detectable.
6.4 Analytical Redundancy Relations for Linear Deterministic Systems - FREQUENCY DOMAIN

Example 6.6 Detectability - ship with two output measurements

Condition (6.60) is now used to check strong fault detectability. For fault $f_\omega$ it yields

$$(1 - s \ 0) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}_{s=0} = 1$$

Thus fault $f_\omega$ is strongly detectable. For fault $f_\psi$ one gets

$$(1 - s \ 0) \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}_{s=0} = 0$$

which indicates that $f_\psi$ is not strongly detectable, as was expected.

---

**Algorithm 6.2** Residual generator design with the parity space method

**Given:** A model of the supervised system in the form (6.40).

**Computation:**

1. Compute matrix $H(s)$ as defined by (6.48).
2. Determine an irreducible polynomial basis for $\mathcal{N}_I(H(s))$, and let $F(s)$ be the matrix whose rows make such a basis. If $F(s) = O$, the problem has no solution.
3. Design the filter $\frac{Q(s)}{p(s)}$ as a low-pass or a band-pass filter which possibly selects appropriate rows in $F(s)$ according to $SNR_i, (i = 1, \ldots, \beta)$ in Eq. (6.51).
4. Check for weak or strong fault detectability as needed.

**Result:** A residual generator in the form (6.50).
6.4 ARR for Linear Deterministic Systems - FREQUENCY DOMAIN

6.4.3 Fault Isolation
For fault-tolerant control, faults should not only be detected, but also be isolated, namely the faulty components should be determined.

Consider a system described by a continuous-time linear state-space

\[ \dot{x}(t) = Ax(t) + Bu(t) + \sum_{j=1}^{n_f} F_j^x f_j(t), \quad x(0) = x_0 \]

\[ y(t) = Cx(t) + Du(t) + \sum_{j=1}^{n_f} F_j^y f_j(t). \]  

(6.61)

where \( f_j \in \mathbb{R}^{n_j}, j = 1, \ldots, n_f \) represent the faults that must be detected and isolated. In terms of transfer functions, (6.61) can be written as

\[ y(s) = H_{yu}(s)u(s) + H_{yx}(s)x(0) + \sum_{j=1}^{n_f} H_{yf_j}(s)f_j(s), \]

(6.62)
6.4 ARR for Linear Deterministic Systems - FREQUENCY DOMAIN

6.4.3 Fault Isolation

Problem 6.3 (Residual generator design for fault detection and isolation based on a deterministic model) Given a model of the supervised process of the form (6.61) or (6.62), determine a set of $n_f$ stable linear time-invariant filters described by

$$\dot{z}_\ell(t) = A_{zz,\ell}z_\ell(t) + B_{zu,\ell}u(t) + B_{zy,\ell}y(t), \quad z_\ell(0) = z_{0,\ell} \quad (6.63)$$

$$r_\ell(t) = C_{zz,\ell}z_\ell(t) + D_{ru,\ell}u(t) + D_{ry,\ell}y(t), \quad \ell = 1, \ldots, n_f$$

or, in transfer function form, assuming zero initial conditions,

$$r_\ell(s) = V_{ru,\ell}(s)u(s) + V_{ry,\ell}(s)y(s), \quad \ell = 1, \ldots, n_f, \quad (6.64)$$

such that the following conditions are met.

- $r_\ell(t)$ asymptotically decays to zero for any $u(t)$ and any $f_j(t), \ j = 1, \ldots, n_f, \ j \neq \ell, \ t > 0$.
- $r_\ell(t)$ is affected by $f_\ell(t)$.

---

6.4 ARR for Linear Deterministic Systems - FREQUENCY DOMAIN

6.4.3 Fault Isolation

In this problem statement, the $\ell^{th}$ residual can only be affected by the $\ell^{th}$ fault, and not by the others. The table below represents this situation when $n_f = 3$. A symbol $\times$ in Table 6.1 indicates that the fault in the corresponding column affects the residual of the corresponding row.

<table>
<thead>
<tr>
<th></th>
<th>$f_1$</th>
<th>$f_2$</th>
<th>$f_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_1$</td>
<td>$\times$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$r_2$</td>
<td>0</td>
<td>$\times$</td>
<td>0</td>
</tr>
<tr>
<td>$r_3$</td>
<td>0</td>
<td>0</td>
<td>$\times$</td>
</tr>
</tbody>
</table>

Table 6.1 Effects of the faults on the residuals
6.4 ARR for Linear Deterministic Systems - FREQUENCY DOMAIN

6.4.3 Fault Isolation
The faults that do not affect the $l$th residual can be seen as unknown inputs to which this residual should not be sensitive.

Hence, to design a residual generator that output $r_l$, it suffices to use the solution of the problem of residual generation for fault detection in which vector $d$ is replaced by $(f_1^T \ldots f_{\ell-1}^T f_{\ell+1}^T \ldots f_{n_f}^T)^T$.

Such problems should be solved for $\ell = 1, \ldots, n_f$ in order to obtain the $n_f$ filters that make a solution to the fault isolation problem.

From the conditions for fault detectability, the following conditions can be deduced for the above scheme to work:

$$\text{rank} \begin{pmatrix} H_{y,f_l}(s) & H_{y,f_j}(s) \end{pmatrix} > \text{rank} H_{y,f_j}(s) \quad (6.65)$$

for all $\ell, j = 1, \ldots, n_f, \ell \neq j$.

A necessary condition for (6.65) to hold is

$$\sum_{\substack{j = 1, n_f \\ j \neq \ell}} n_{f_j} < p, \quad (6.66)$$

where $p$ is the number of measured output signals (dimension of $y$).

Note: When condition, (6.65) is not met, the diagonal structure of Table 6.1 cannot be obtained, and one should attempt to group the fault vectors in different classes and to generate residuals that are affected by a specific fault class and not by the others.
6.4 ARR for Linear Deterministic Systems - FREQUENCY DOMAIN

6.4.3 Fault Isolation
The table below illustrates one way to perform such a grouping, in a situation where \( n_f = 3 \) and two residual generators are designed. In the situation of Table 6.2, all three faults can be distinguished as the combination of \( r_1 \) and \( r_2 \) reacts differently to each fault. However, simultaneous faults cannot be isolated because they affect both residuals in all cases.

<table>
<thead>
<tr>
<th>( f_1 )</th>
<th>( f_2 )</th>
<th>( f_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r_1 )</td>
<td>✗</td>
<td>✗</td>
</tr>
<tr>
<td>( r_2 )</td>
<td>✗</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 6.2 Effects of the faults on the residuals—non-diagonal structure

Example 6.7 Isolability - ship with three output measurements
For the ship with one rate measurement and two heading measurements (Example 6.5), a residual generator is achieved, which was decoupled from the disturbance,

\[
\begin{pmatrix}
  r_1(s) \\
  r_2(s)
\end{pmatrix} =
\begin{pmatrix}
  \frac{1}{s} & 1 & 0 \\
  \frac{1}{s} & 0 & 1
\end{pmatrix}
\begin{pmatrix}
  f_{\omega 3}(s) \\
  f_{(1)}(s) \\
  f_{(2)}(s)
\end{pmatrix}.
\] (6.67)

This residual generator has the properties shown in Table 6.2.
6.4 ARR for Linear Deterministic Systems - FREQUENCY DOMAIN

Sensor fault isolation in a fault-tolerant control setting. If it has been detected that one out of a set of faults is present, but it has not been possible to isolate which fault is actually present, and this was due to the design of the residual generator specification, alternatives are available on the fault-tolerant setting because the supervisory system has control of the input signals to the plant. Similar to system identification, where a dedicated test signal is applied to obtain the optimal information about a particular parameter, a dedicated test signal can be applied on the control input to help confirm particular hypotheses. This procedure can help to reduce the time to diagnose and, hence, the time to reconfigure a controller.

Example 6.8 Dedicated test signal for isolation - ship steering

If two identical rate sensors are available in the ship steering example, and the residual generator was designed to be insensitive to the wave disturbance, it is not possible to isolate faults $f_{\delta}^1$ and $f_{\delta}^2$. In a fault-tolerant control setting, we employ active test signal generation to isolate the fault once it has been detected that one of the rate sensor units is defect. Let us define a dedicated test signal

$$\delta(t) = \tilde{\delta}(t), \ t \in [0, \ T],$$

which is applied immediately after the hypothesis of

$$\{ \hat{f}_{\delta}^1(t) \lor \hat{f}_{\delta}^2(t) \} \neq 0$$

is confirmed.
6.4 ARR for Linear Deterministic Systems - FREQUENCY DOMAIN

Example 6.8 Dedicated test signal for isolation - ship steering

Observe a-priori the response in the non-faulty condition

\[ \omega_3^{ref}(t) = g_{\omega_3}(\delta(t), U(t)), \ t \subset [0, T] \]

note that the function \( g_{\omega_3} \) is not calculated, the angular rate is merely recorded and stored. Calculate the correlations

\[
\text{cor}_{31}(t) = \frac{1}{t} \int_0^t \omega_3^{ref}(\tau) \omega_3^1(\tau) \, d\tau \\
\text{cor}_{21}(t) = \frac{1}{t} \int_0^t \omega_3^{ref}(\tau) \omega_3^2(\tau) \, d\tau \\
\text{cor}_{32}(t) = \frac{1}{t} \int_0^t \omega_3^{ref}(\tau) \omega_3^3(\tau) \, d\tau.
\]

These correlation signals with appropriate normalisation make it straightforward to determine which hypothesis is the most likely.

---

6.4 ARR for Linear Deterministic Systems - FREQUENCY DOMAIN

6.4.4 Fault Estimation

The isolation schemes signify which fault is present but do not assess the magnitude of the fault. Fault estimates are needed in certain fault accommodation approaches as was indicated in Sect. 6.1.

**Definition 6.3 (Fault estimation)** Fault estimation is the ability to estimate the magnitude of a fault \( f(t) \) and its time history.
### 6.4 ARR for Linear Deterministic Systems - FREQUENCY DOMAIN

#### 6.4.4 Fault Estimation

Combining (6.42) and (6.58), the link between the fault vector \( f(s) \) and the residual \( r(s) \) is seen to be

\[
r(s) = V_{ru}(s)u(s) + V_{ry}(s)y(s) = Q(s)F(s) \left( \begin{array}{c} H_yf(s) \\ 0 \end{array} \right)f(s),
\]

where it is assumed that initial conditions have vanished. Letting

\[
F(s) = \left( \begin{array}{c} F_1(s) \\ F_2(s) \end{array} \right),
\]

where \( F_1(s) \) has \( p \) columns and \( F_2(s) \) has \( m \) columns, Eq. (6.68) can be written

\[
r(s) = V_{ru}(s)u(s) + V_{ry}(s)y(s) = \frac{Q(s)F_1(s)}{p(s)}H_yf(s)f(s).
\]

Fault D. and FTCS
by: Dr B. Moaveni

---

#### 6.4 ARR for Linear Deterministic Systems - FREQUENCY DOMAIN

#### 6.4.4 Fault Estimation

On the other hand, Eq. (6.54) yields the following relation when the transient due to the initial conditions is neglected

\[
r(s) = V_{ry}(s)H_yf(s)f(s),
\]

hence

\[
V_{ry}(s) := \frac{Q(s)F_1(s)}{p(s)}.
\]

As a compact notation, introduce \( H_f(s) \) by

\[
H_f(s) := V_{ry}(s)H_yf(s) = \frac{Q(s)F_1(s)}{p(s)}H_yf(s).
\]

If it is possible to determine a suitable left inverse to \( H_f(s) \), say \( G(s) \), an estimate of \( f(s) \) would be

\[
\hat{f}(s) = G(s)r(s) = G(s)(V_{ru}(s)u(s) + V_{ry}(s)y(s)).
\]
6.4 ARR for Linear Deterministic Systems - FREQUENCY DOMAIN

**Left inverse transformation.** If the polynomial matrix $H_{gf}(s)$ is square, then the estimate $\hat{f}(s) = H_{gf}^{-1}(s)$ where the $i$th element of $H_{gf}^{-1}(s)$, call it $h_{ij}$ is the usual inverse

$$h_{ij}(s) = \frac{1}{\det(H_{gf}(s))}(-1)^{i+j}(M_{ji}(s)),$$

where $M_{ji}(s)$ is the determinant of the matrix formed by $H_{gf}(s)$ after deleting row $j$ and column $i$.

If $H_{gf}(s)$ is non-square, with $l$ rows and $n_f$ columns, then, there exists a left pseudo-inverse $G(s)$ of $H_{gf}(s)$ if and only if

$$\text{rank } (H_{gf}(s)) = n_f,$$

where the normal rank is considered. $G(s)$ is given as

$$G(s) = (H_{gf}^T(s)H_{gf}(s))^{-1}H_{gf}^T(s).$$  \hspace{1cm} (6.72)

---

6.4 ARR for Linear Deterministic Systems - FREQUENCY DOMAIN

**Remark 6.5 (Causality of solution)** To be able to implement the filter Eq. (6.71), $G(s) V_{r_0}(s)$ and $G(s) V_{r_3}(s)$ must be proper and stable transfer functions. This may not be true when $G(s)$ is computed as above. A modified procedure can be found in the literature (see the bibliographical notes for this chapter). □
6.4 ARR for Linear Deterministic Systems - FREQUENCY DOMAIN

Fault estimation after isolation. A necessary condition to be able to compute the above rational estimate, based on a pseudo inverse transformation, is that the rank of the $H_{rf}$ matrix is equal to the number of faults to be estimated. As the number of faults is often larger than the number of independent residuals, it is necessary to take advantage of the results of the fault isolation to limit estimation of faults to those that the isolation algorithm found to be present in the system.

Assume the subset of the fault vector $f_i$, $i \in [j, \ldots, k]$ has been determined necessary to estimate by the isolation algorithm. The above general expressions then hold for the entries of the transfer function matrices that relate to $f_i$, $i \in [j, \ldots, k]$.

Assume a single fault has been determined present. Then, a single column in $H_{rf}(s)$ needs to be considered. The result for this simplest case can be formulated as follows.

\[ r(s) = V_{r0}(s)u(s) + V_{ry}(s)y(s) \]

and a transfer function model relating this residual to faults

\[ r(s) = H_{rf}(s)f(s). \]

Assume that the isolation procedure indicates that fault number $i$ is present, and let the $i$th column of $H_{rf}(s)$ be

\[ h_i(s) = \frac{\tilde{h}_i(s)}{\eta(s)}. \]

where $\tilde{h}_i(s)$ is a polynomial vector with entries $\tilde{h}_{ji}(s)$ and $\eta(s)$ is the least common denomination of the entries of $h_i(s)$.
6.4 ARR for Linear Deterministic Systems - FREQUENCY DOMAIN

Theorem 6.1  (Single fault estimation) On the condition that \( \eta(s) \) and \( \hat{h}_i^T(s)\hat{h}_i(s) = \sum_{j=1}^{l} h_{ji}^2(s) \) are stable polynomials, an estimate of \( f_i \), \( \hat{f}_i \) is given by:

\[
\hat{f}_i(s) = \left( h_i^T(s) h_i(s) \right)^{-1} h_i(s)^T r(s).
\]  \hfill (6.73)

This estimator is causal when \( \deg \eta(s) = \max \deg h_{ji}(s) \). This is easily proved by direct computation of the pseudo inverse in Eq. (6.73):

\[
\left( h_i(s)^T h_i(s) \right)^{-1} h_i(s)^T = \frac{1}{\sum_{i=1}^{l} h_{ji}^2(s)} (\hat{h}_{i1}(s), \ldots, \hat{h}_{il}(s)).
\]  \hfill (6.74)

If the above condition on the degree is not met, a low-pass approximation for the fault estimate can be obtained by multiplying the denominator of Eq. (6.74) by \( (s+\alpha)\beta \), where \( \alpha \in \mathbb{R}^n \) and \( \beta \) is chosen so that all entries in Eq. (6.74) are causal.
Example 6.9 Fault estimation - ship with three output measurements

Fault estimation following isolation for the ship with three output measurements results from the residual generator obtained in Example 6.7

\[
H_{rf}(s) = \begin{pmatrix} -\frac{1}{s} & 1 & 0 \\ -\frac{1}{s} & 0 & 1 \end{pmatrix} = \frac{1}{s} \begin{pmatrix} -1 & s & 0 \\ -1 & 0 & s \end{pmatrix}
\]

**Fault 1 isolated:** The estimate of fault number 1 is

\[
\hat{f}_1 = \frac{s}{2} (r_1(s) + r_2(s)).
\]

Since this filter is not causal, a low-pass filtered approximation for the rate gyro fault is needed, where \( \alpha \in \mathbb{R}^+ \)

\[
\hat{f}_1 = \frac{s}{2(s + \alpha)} (r_1(s) + r_2(s)). \quad (6.75)
\]

Fault 2 isolated: The estimate of fault number 2 is

\[
\hat{f}_2 = r_1(s)
\]

Fault 3 isolated: The estimate of fault number 3 is

\[
\hat{f}_3 = r_3(s)
\]

It should be noted that an erroneous isolation will give gross errors in the fault estimate. In an implementation, the above fault estimators would run in parallel. Once a particular fault is isolated, the estimate can be rapidly provided.
6.4 ARR for Linear Deterministic Systems - FREQUENCY DOMAIN

Alternative methods to fault estimation.

In cases, where the above algebraic approach to fault estimation fails, asymptotic estimation of faults may be achievable using an observer on an augmented system, where the state is augmented by the fault(s) to be estimated (modelling faults to be constant):

\[
\frac{d}{dt} \begin{pmatrix} x \\ f \end{pmatrix} = \begin{pmatrix} A & F_x \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ f \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u(t)
\]

\[y(t) = \begin{pmatrix} C & F_y \end{pmatrix} \begin{pmatrix} x \\ f \end{pmatrix} .\]

A necessary condition for an asymptotically stable estimator to exist is that the pair is observable.

\[
\left( \begin{pmatrix} A & F_x \\ 0 & 0 \end{pmatrix} , \begin{pmatrix} C & F_y \end{pmatrix} \right)
\]

Observer-based methods are covered extensively in the literature (see the bibliographical notes for references).
6.4 ARR for Linear Deterministic Systems - FREQUENCY DOMAIN

Algorithm 6.3 *Fault estimation*

**Given:** A model of the supervised process of the form (6.40) and a residual generator of the form (6.42)

**Compute:**
1. The transfer matrix $H_{rf}(s)$ relating residuals to faults
2. A left inverse to $H_{rf}(s)$
3. An estimator of the form (6.71), possibly after appropriate filtering of the left inverse in order to obtain a causal and stable estimator for all faults.

**Result:** A causal and stable fault estimator based on the measurements of $u(s)$ and $y(s)$.

6.5 Optimization-Based Approach to Diagnosis

The above methods were based on algebraic or polynomial manipulations, and relied on the ability to achieve exact decoupling from disturbances and from input to the residual. *When this is not possible*, the influence $d(t)$ and $u(t)$ have on the residual competes with that generated by faults $f(t)$. If the effects of input and disturbance on the residual are non-zero, we do not obtain

$$
(V_{ru}(s) + V_{ry}(s)H_{yu}(s) \quad V_{ry}(s)H_{yd}(s)) \begin{pmatrix} u(s) \\ d(s) \end{pmatrix} = 0
$$

(6.76)

for all $u(s)$ and $d(s)$ and

$$
r(s) = (V_{ru}(s) + V_{ry}(s)H_{yu}(s) \quad V_{ry}(s)H_{yd}(s)) \begin{pmatrix} u(s) \\ d(s) \end{pmatrix} + V_{ry}(s)H_{yf}(s)f(s) + V_{ry}(s)H_{yx}(s)x(0)
$$

(6.77)

is strictly speaking not a residual generator according to the definition.
6.5 Optimization-Based Approach to Diagnosis

The purpose of this section is to find ways to relax the requirement on exact decoupling for the residual generator. Instead, some optimal approximation should be obtained in the sense that the design shall satisfy certain criteria. The design objectives should be to

1. Provide a sufficient suppression of disturbances $d$ seen from the residual,
   \[ V_{ry}(s)H_{yd}(s) = 0 \]

2. Maximise the sensitivity $r$ of the residual with respect to all or a to a selected set of faults in $f$.
   \[ V_{ry}(s)H_{yf}(s) \neq 0 \]

3. Make the residual signal sufficiently insensitive to variations in the input signal $u$.
   \[ V_{ru}(s) + V_{ry}(s)H_{yu}(s) = 0 \]

4. Provide the designer with tools to enter a specification of the desired performance.

---

6.5 Optimization-Based Approach to Diagnosis

**Norms**

Let $x \in \mathbb{R}^n$

\[ |x|_p = \left( \sum_{i=1}^{n} |x_i|^p \right)^{1/p} \]

\[ |x|_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2} \]

\[ |x|_\infty = \max_{1 \leq i \leq n} |x_i| \]
6.5 Optimization-Based Approach to Diagnosis

Norms
Let $A \in \mathbb{R}^{m \times n}$

$$|A|_p = \sup_{x \neq 0} \frac{|Ax|_p}{|x|_p}$$

$$|A|_2 = \sqrt{\lambda_{\text{max}}(A^T A)} = \bar{\sigma}(A)$$

$$|A|_\infty = \max_{1 \leq i \leq m} \sum_{j=1}^{n} |a_{ij}| \quad \text{(maximum absolute row sum)},$$

---

6.5 Optimization-Based Approach to Diagnosis

Norms
Let $H(j\omega) \in \mathbb{C}^{m \times n}$ be a stable transfer function

$$|H|_2 = \text{trace} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} H(j\omega) H^T(-j\omega) \, d\omega \right)^{\frac{1}{2}}$$

$$|H|_\infty = \max_{\omega} \bar{\sigma}(H(j\omega)).$$

$$|H f|_2 \leq |H|_\infty^2 |f|_2 = \max_{\omega} \bar{\sigma}(H(j\omega))^2 |f|_2$$

which shows $|H|_\infty^2$ is the upper bound for the signal power transmitted from input to output of the transfer function $H(s)$.
6.5 Optimization-Based Approach to Diagnosis

Formulation as an optimisation problem.

The first property of a relaxed residual generator should be minimisation of the effect of disturbances in the residual.

A direct minimisation of the effect the disturbance has on the residual is expressed in the induced norm

$$
\min_{V_{ry}} J_{id} = \min_{V_{ry}} \frac{\left| V_{ry}(s) H_{yd}(s) d(s) \right|_2^2}{\left| d(s) \right|_2^2} = \min_{V_{ry}} \left| V_{ry}(s) H_{yd}(s) \right|_\infty^2
$$

subject to

$$V_{ry}(s) H_{yf}(s) \neq 0.$$  

The constraint prevents the trivial solution $V_{ry}(s)=0$.

The signal power comprised in the residual caused by the disturbance over the power generated by faults should be minimised, hence a feasible index could be

$$
\max_{V_{ry}} J_2 = \max_{V_{ry}} \left( \frac{\left| V_{ry}(s) H_{yf}(s) f(s) \right|_2^2}{\left| V_{ry}(s) H_{yd}(s) d(s) \right|_2^2} \right)_{|d| \neq 0}
$$

The interpretation of this index is to maximise the signal over noise ratio in the residual, using the total power, i.e. over all frequencies.

This index cannot be easily optimised. If we, however, make a slight modification to the optimisation criterion, standard tools are available.
### 6.5 Optimization-Based Approach to Diagnosis

Application of the standard methods require a specific formulation of the problem, which is first illustrated using manipulation on the block diagram in Fig. 6.4 for the case, where the objective is to find a polynomial matrix $F(s)$ such that a signal $e(s)$ is insensitive to a disturbance $d(s)$.

\[
e(s) = (H_{zd}(s) - F(s)H_{yd}(s)) \cdot d(s)
\]

Fig. 6.4
6.5 Optimization-Based Approach to Diagnosis

6.5.2 Solution Using the Standard Setup Formulation

Definition 6.4 (Standard estimation setup) Let a system be given by input vector (known and unknown input) \( d \in \mathbb{R}^m \), state vector \( x \in \mathbb{R}^n \), an auxiliary output \( z \in \mathbb{R}^l \) and measured output vector \( y \in \mathbb{R}^p \) with state-space equation

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + E_x d(t) \\
z(t) &= C_z x(t) + E_z d(t) \\
y(t) &= C_y x(t) + E_y d(t)
\end{align*}
\]  
(6.78)

and, ignoring initial conditions, represented in the Laplace domain by

\[
\begin{align*}
z(s) &= H_{zd}(s) d(s) \\
y(s) &= H_{yd}(s) d(s),
\end{align*}
\]  
(6.79)

where

\[
H_{zd}(s) = \begin{pmatrix} A & E_x \\ C_z & E_z \end{pmatrix} = C_z(sI - A)^{-1}E_x + E_z
\]
(6.80)

\[
H_{yd}(s) = \begin{pmatrix} A & E_x \\ C_y & E_y \end{pmatrix} = C_y(sI - A)^{-1}E_x + E_y.
\]
(6.81)
6.5 Optimization-Based Approach to Diagnosis

6.5.2 Solution Using the Standard Setup Formulation

**Problem 6.4** (Standard estimation problem) Let \( \hat{z}(s) \) be an estimate of \( z(s) \). Denote the difference by \( e_z(s) = z(s) - \hat{z}(s) \). For the system defined by the standard problem setup (6.79), determine a stable transfer function matrix \( F(s) \) to provide an estimate of the auxiliary output given the measured output,

\[
\hat{z}(s) = F(s)y(s)
\]

subject to a suitable norm of the estimation error, \( |e_z(s)| \) being less than a chosen gain factor

\[
\sup_{F(s)} |e_z(s)| < \gamma \iff \sup_{F(s)} |H_{zd}(s) - F(s)H_{yd}(s)| < \gamma,
\]

where the norm can be of types \( \mathcal{H}_2 \) or \( \mathcal{H}_\infty \) for instance.

---

Remark 6.6 Different filtering and estimation problems can be easily formulated within this general estimation framework. The state can be estimated using \( C_z = I_{n,n} \) and \( E_z = O \). The input can be estimated using \( C_z = O \) and \( E_z = I_{1,1} \). The estimation setup will be used later for residual generation. \( \square \)
6.5 Optimization-Based Approach to Diagnosis

6.5.2 Solution Using the Standard Setup Formulation

**Definition 6.5** (Standard system setup) Let a system be described in the Laplace domain by the transfer function matrix \( P(s) \), and four vectors, input \( u(s) \in \mathbb{C}^m \), auxiliary input \( d(s) \in \mathbb{C}^{m_d} \), auxiliary output \( e(s) \in \mathbb{C}^{m_e} \) and measured output \( y(s) \in \mathbb{C}^p \). Input and output are related through \( P(s) \in \mathbb{C}^{(p+m_e) \times (m+m_d)} \) as

\[
\begin{pmatrix}
e(s) \\
y(s)
\end{pmatrix} = P(s) \begin{pmatrix}
d(s) \\
u(s)
\end{pmatrix} = \begin{pmatrix}
P_{ed}(s) & P_{eu}(s) \\
P_{yd}(s) & P_{yu}(s)
\end{pmatrix} \begin{pmatrix}
d(s) \\
u(s)
\end{pmatrix}
\]

Let the transfer function matrix \( F(s) \in \mathbb{C}^{m \times p} \) be a feedback controller for the system, between \( y \) and \( u \),

\[
u(s) = F(s)y(s)
\]

Using this setup and utilising solutions for two fundamental optimisation problems in the design of residual generators, the \( \mathcal{H}_\infty \) sub-optimal control problem and the \( \mathcal{H}_2 \) sub-optimal control problem.

Fault D. and FTCS
by: Dr B. Moaveni

---

6.5 Optimization-Based Approach to Diagnosis

6.5.2 Solution Using the Standard Setup Formulation

**Problem 6.5** (\( \mathcal{H}_\infty \) sub-optimal control) Given a system in form of the standard system setup of Definition 6.5. Design a stabilising controller \( F(s) \) such that the norm of the closed-loop transfer function \( T_{ed}(s) \) from auxiliary input \( d(s) \) to auxiliary output \( e(s) \) is lower than a specified bound \( \gamma \):

\[
\sup_F |T_{ed}|_\infty < \gamma \Leftrightarrow \sup_{F(j\omega)} \tilde{\sigma}(T_{ed}(j\omega)) < \gamma,
\]

where \( \tilde{\sigma} \) denotes the largest singular value.

The \( \mathcal{H}_\infty \) norm gives the maximum sinusoidal gain of the system (energy gain or induced \( L_2 \) system gain).

Fault D. and FTCS
by: Dr B. Moaveni
6.5 Optimization-Based Approach to Diagnosis

6.5.2 Solution Using the Standard Setup Formulation

Problem 6.6 (H₂ sub-optimal control problem) Given a system in form of the standard system setup in Definition 6.5. Design a stabilising controller $F(s)$ such that the $H₂$ norm of the closed-loop transfer function $T_{ed}(s)$ from auxiliary input $d(s)$ to auxiliary output $e(s)$ is minimised.

The standard estimation problem of Fig. 6.4 can be formulated in the standard setup. The generalised system $P(s)$ then takes the form

$$P(s) = \begin{pmatrix} H_{zd}(s) - I \\ H_{yd}(s) \\ O \end{pmatrix}.$$  

Note that there is no feedback through the system since $P_{yu}(s) = O$.

6.5 Optimization-Based Approach to Diagnosis

6.5.3 Residual Generation

The above result can be applied in connection with detection, isolation and estimation of faults. We aim at making $\hat{z}(s)$ a residual signal. We investigate two problems.
- The first is to suppress disturbances as well as possible.
- The second is to make a balanced optimization, where the fault signature is preserved in the residual while disturbances are suppressed to the extent possible.

Both results follow from appropriate formulation of the standard problem. The strategy is to select an auxiliary output $z(s)$ and give it the properties that the residual should have. This means the formulation of $z(s)$ is directly a specification of the residual. In designing the estimate $\hat{z}(s)$ to track $z(s)$ as closely as possible, according to a given criterion, a sub-optimal estimator is obtained for the ideal residual. The accuracy with which the specification is met is seen in the choice of the optimization coefficient $\gamma$. 
6.5 Optimization-Based Approach to Diagnosis

6.5.3 Residual Generation

The basic residual generator will have the form

\[ r(s) = F(s)(y(s) - H_{yu}(s)u(s)). \]  \hspace{1cm} (6.85)

The design problem is to determine the operator \( F(s) \).

Remark 6.7 (Relation to parity space formulation) The residual generator Eq. (6.43) had as prerequisite, following from Problem 6.2, that

\[ V_{ru} + V_{ry}H_{yu} = O \]

Hence

\[ r(s) = V_{ry}(s)y(s) + V_{ru}u(s) \]
\[ = V_{ry}(s)y(s) - V_{ry}(s)H_{yu}u(s) \]
\[ = V_{ry}(s)(y(s) - H_{yu}(s)u(s)). \]

Comparison with Eq. (6.85) shows that finding the solution \( F(s) \) in the standard setup is equivalent to determining the operator \( V_{ry}(s) \).
### 6.5 Optimization-Based Approach to Diagnosis

#### 6.5.3 Residual Generation

In the design, two requirements have to be combined.

**Residual generation with specification on fault sensitivity and disturbance suppression.**

The goal is now to have the residual replicating a fault through a specified dynamical relation while the disturbance should be suppressed as far as possible. Therefore, we include the fault vector \( f(s) \) in the system description and define the auxiliary output \( z(s) \) to be dependent only of the fault vector:

\[
\begin{align*}
y(s) &= H_{yd}(s)d(s) + H_{yf}(s)f(s) \\
z(s) &= H_{zf}(s)f(s) \\
\hat{z}(s) &= V_{ry}(s)y(s) \\
ev_z(s) &= z(s) - \hat{z}(s)
\end{align*}
\]

The selection of the auxiliary output reflects directly the properties that the residual should have, \( H_{zd} = 0 \) is chosen because we wish to interpret \( d(s) \) as a disturbance and decouple it from the residual.

The specification of \( H_{zf}(s) \) is a design choice. There may not exist a solution \( V_{ry}(s) \) for all arbitrary specifications, so \( H_{zf}(s) \) is the key design parameter.

The performance that should be achieved is that the residual follows \( z(s) \) as close as possible, hence, the relation

\[
|H_{zf}(s) - V_{ry}(s)H_{yf}(s)|_\infty < \gamma_s
\]

should hold, where \( \gamma_s \) characterises the desired fault sensitivity (or tracking) performance.
6.5 Optimization-Based Approach to Diagnosis

6.5.3 Residual Generation

Simultaneously, the effect of the disturbance should be below a certain level, hence

\[ \left| V_{y}(s)H_{yd}(s) \right|_{\infty} < \gamma_r \]

where \( \gamma_r \) is a measure of robustness with respect to input effects.

Combining the two, the physically motivated optimisation problem yields:

\[ \left| (-V_{y}(s)H_{yd}(s)) \left( H_{zf}(s) - V_{y}(s)H_{yf}(s) \right) \right|_{\infty} < \gamma. \] (6.86)

Since

\[
\begin{align*}
    z(s) - V_{y}(s)y(s) &= \left( H_{zf}(s) - V_{y}(s)H_{yf}(s) \right) f(s) - V_{y}(s)H_{yd}(s)d(s) \\
    &= \left( (-V_{y}(s)H_{yd}(s)) \left( H_{zf}(s) - V_{y}(s)H_{yf}(s) \right) \right) \begin{pmatrix} d(s) \\ f(s) \end{pmatrix}
\end{align*}
\]

Equation (6.86) is equivalent to

\[
\sup_{d \neq 0} \frac{\left| z(j\omega) - V_{y}(j\omega)y(j\omega) \right|_2^2}{\left| d(j\omega) \right|_2^2} < \gamma \iff \sup_{d \neq 0} \frac{\left| e_{z}(j\omega) \right|_2^2}{\left| d(j\omega) \right|_2^2} < \gamma
\]

Fault D. and FTCS
by: Dr B. Moaveni
6.5 Optimization-Based Approach to Diagnosis

6.5.3 Residual Generation

\[ y(s) = H_yd(s)d(s) + H_yf(s)f(s) \]
\[ z(s) = H_zf(s)f(s) \]
\[ \hat{z}(s) = V_{ry}(s)y(s) \]
\[ e_x(s) = z(s) - \hat{z}(s) \]

Fig. 6.5 Residual generator depicted in a standard setup formulation with specifications \( H_{zf} \) and \( H_{yd} \) in the upper part of the figure. \( H_{zd} \) is specified as zero in the design problem shown in the lower part of the figure.
6.5 Optimization-Based Approach to Diagnosis

6.5.3 Residual Generation

**Problem 6.7** (Residual generation with specification on fault sensitivity and disturbance suppression)

Given, an LTI system with input $u(s)$, unknown input (disturbances) $d(s)$ and faults $f(s)$ and let input-output relations of the system be described by $(H_{yu}, H_{yd}, H_{yf})$. Introduce an auxiliary variable $z(s)$ and specify a transfer function matrix $H_{zf}$ and a real number $\gamma_s$.

Let $z(s) = H_{zf}f(s)$. Determine $V_{ry}$ such that the maximal deviation between $\hat{z}(s) = V_{ry}y(s)$ and $z(s)$ is bounded by $\gamma_s$:

$$|z(s) - \hat{z}(s)| < \gamma_s \quad (6.87)$$

---

6.5 Optimization-Based Approach to Diagnosis

6.5.3 Residual Generation

The solution to this problem of residual generation design has the following form:

1. Define the standard problem setup:

   aux.input : $d(s) \leftarrow \begin{pmatrix} d(s) \\ f(s) \end{pmatrix}$

   input : $u(s) \leftarrow r(s)$

   aux.output : $e(s) \leftarrow e_z(s) = z(s) - r(s)$

   output : $y(s) \leftarrow y(s)$

   $P(s) \leftarrow \begin{pmatrix} O & H_{zf}(s) \\ H_{yd}(s) & H_{yf}(s) \end{pmatrix}^{-1} I$

   $F(s) \leftarrow V_{ry}(s)$
6.5 Optimization-Based Approach to Diagnosis

6.5.3 Residual Generation

2. Use software that solves the standard problem to determine a solution in form of a stable transfer function \( V_{ry}(s) \) that satisfies the inequality

\[
\sup_{\left(\begin{array}{c}
 d \\
 f
\end{array}\right) \neq 0} \left| \left| \varepsilon \right|_{2} \right| < \gamma
\]

which is equivalent to finding a solution to

\[
\left| \left| \left( -V_{ry}H_{yd}H_{zf} - V_{ry}H_{yf} \right) \right|_{\infty} \right| < \gamma.
\]

If a result exists, which is not guaranteed, the result is strong in the sense it provides the residual generator with optimal weighting between suppression of disturbances and specified sensitivity to faults.

In practice it is worthwhile to start a design with investigating the extent to which disturbances can be suppressed using the disturbance suppression problem. When insight in the problem has been gained, continue with supplying a specification to the problem and iterate until a suitable compromise has been found between disturbance suppression and fault tracking.
6.5 Optimization-Based Approach to Diagnosis

6.5.3 Residual Generation

Fault detection.
When the purpose is to design a pure fault detection filter, a sensible way to specify $H_{zf}(s)$ is to require that it is a row vector with non-zero causal and stable entries. When a residual vector is sought the specification becomes:

$$\forall \omega; j\omega \neq z_k : \left\{ \begin{array}{l} \text{rank} \,(H_{zf}(j\omega)) \geq 1 \\ \forall i : h_i(j\omega) \neq 0, \end{array} \right.$$

where $h_i(j\omega)$ stands for the $i$th column of $H_{zf}(j\omega)$ and $z_k$ are the zeros of $H_{zf}(s)$.

Fault isolation.
If the number of faults to be isolated is $n_f$ and simultaneous faults can occur, $H_{zf}(s)$ has to fulfil the requirement:

$$\text{rank } H_{zf}(s) = n_f$$

When simultaneous faults are not considered, a vector $z$ of size $l$ is sufficient to isolate $2^l - 1$ faults by considering suitable coding sets. This translates into the following specification for matrix $H_{zf}(s)$. To isolate $n_f$ faults, choose a matrix $H_{zf}(s)$ such that:

$$\text{rank } H_{zf}(s) \geq \log_2(n_f + 1)$$

$$\text{rank } (h_i(s) h_j(s)) = 2 \text{ with } i = 1, \ldots, n_f \quad i \neq j, \quad j = 1, \ldots, n_f.$$
6.5 Optimization-Based Approach to Diagnosis

6.5.3 Residual Generation

Fault estimation.
Fault estimation can be obtained by specifying $Hzf(s) = I$. In this case,

$$z(s) = If(s)$$

and

$$e(s) = z(s) - \hat{z}(s)$$

In the ideal situation, where no disturbance exists, this specification aims at assuring that $\hat{z}$ tracks the fault $f$ by guaranteeing that

$$\sup \frac{|\hat{z} - z|}{|f|} < \gamma.$$
Design considerations.
In connection with using H2 or H∞ optimisation to design the residual generator, a weight function can further be included in the setup to some advantage of the designer. The weight function W(s) can be applied to specify the frequency range(s), where detection, isolation or estimation should be obtained most effectively. The way to include a weight matrix in the design is to modify the P(s) system matrix to

\[
P(s) = \begin{pmatrix} O & WH_f \\ H_{yd} & H_{yf} \end{pmatrix} - W \begin{pmatrix} O \\ H_{yd} & H_{yf} \end{pmatrix}
\]

The weighting matrix specifies which frequency ranges a designer emphasizes to meet the bound γ and where it can be relaxed.
6.5 Optimization-Based Approach to Diagnosis

Algorithm 6.4 Residual generator design

1. **Formulate problem:** Formulate the relevant version of the standard setup for the problem.
2. **Design specification:** Specify an initial qualified guess on the specification \( H_{xf}(s) \). The specification needs to be bounded from below, otherwise, the optimal solution will be \( F = O \) and \( H_{xf} = O \).
3. **Solve problem:** Find the function \( F(s) \) in the residual generator
   \[
   r(s) = F(s)(y(s) - H_{yu}(s)u(s)),
   \]
   where \( F(s) \) is the best obtainable solution to the problem given the specification \( H_{xf}(s) \).
4. **Iterate until converged:** Continue until the value of \( \gamma \) obtained has converged.
5. **Iterate in specification:** Based on this residual generator, specify a new \( H_{xf}(s) \) and repeat the design.

6.6 Residual Evaluation

Given, a residual generator for the deterministic case, i.e. there are only insignificant random disturbances or measurement noise, the purpose of this section is to find a method for residual evaluation that will determine whether a fault is present.

6.6.1 Residual - General Case

Let a set of residuals obtained from structural analysis have the form
\[
\mathbf{r} = (r_1, r_2, \ldots, r_n)^T
\]

Consider one of these residuals
\[
r_j(t) = p_j(k_i, c_i, t) \quad k_i \in K^{(j)}, c_i \in C^{(j)}, \quad j = 1, \ldots, n,
\]
where \( p_j \) is of the form in which the constraints in \( C^{(j)} \) were formulated: linear, nonlinear, tabular, quantised, logical or hybrid.
6.6 Residual Evaluation

As it is useful to categorise known variables into the natural categories input $u$, measured $y$, and parameters $\theta$, the parity relation is written as

$$r_j(t) = p_j(u_i, y_i, \theta_i, c_i, t) \quad u_i, y_i, \theta_i \in K^{(j)}, c_i \in C^{(j)}, \quad j = 1, \ldots, n. \quad (6.89)$$

The parity relations implemented for residual generation would not be the true system constraints $c_i$ nor the true parameters $\theta_i$ but would be estimates of those, $\hat{c}_i$, $\hat{\theta}_i$, respectively.

In order to shape the signatures of faults in the residuals or suppress noise, filtering of the raw parity relation Eq. (6.89) will usually take place, and also the filtered version is a residual,

$$r_j(t) = \int_{0}^{t} w_j(t - \tau) p_j(u_i, y_i, \hat{\theta}_i, \hat{c}_i, \tau) d\tau, \quad j = 1, \ldots, n, \quad (6.90)$$

where $w_j(t - \tau)$ is the impulse response of the filter applied to parity relation $j$.

Further, the vector of residuals could be constructed as a linear combination of the elements from the above residuals, Eq. (6.90),

$$r(t) = W \begin{pmatrix} r_1(t) \\ \vdots \\ r_n(t) \end{pmatrix}, \quad (6.91)$$

where $W \in \mathbb{R}^{n \times n}$, $\det(W) \neq 0$. 
6.6 Residual Evaluation

Uncertainty

In real life, \( \hat{c}_i \neq c_i \) , and \( \hat{\theta}_i \neq \theta_i \) , hence \( r_j(t) \) could be non-zero even though there was no violation of a constraint in relation \( j \), \( \forall c_i \in C^j : c_i = 0 \). In particular, actuator demand and disturbances could drive the residual away from zero when parameters and constraints are not exactly equal to those of the real object. In order to make residual evaluation under such uncertainty, it is necessary to accept that a residual can have some deviation from zero even in the no-fault case. However, the effect on \( r_j(t) \) has to be bounded, hence \( p_j(u_i, y_i, \hat{\theta}_i, \hat{c}_i, t) \) is bounded,

\[
\| p_j(u_i, y_i, \hat{\theta}_i, \hat{c}_i, t) \| \leq \alpha_j(u_i, y_i, t) \land 0 < \alpha(u_i, y_i, t) < \infty. \tag{6.92}
\]

LTI case.

If the object for diagnosis is linear and time-invariant (LTI), the residual generator could be LTI with a frequency representation

\[
r(s) = H_{ru}(s)u(s) + H_{rd}(s)d(s) + H_{rf}(s)f(s) \tag{6.93}
\]

being an explicit function of input, disturbances and faults. In an ideal case, residual generation is perfect and we have \( H_{ru}(s)=0 \) and \( H_{rd}(s)=0 \). Residual evaluation then reduces to investigating the properties of

\[
r(s) = H_{rf}(s)f(s). \tag{6.94}
\]

In the general case, still with an LTI system, model uncertainty and unmodelled dynamics will give rise to \( H_{ru} \neq 0 \) and \( H_{rd} \neq 0 \). Residual evaluation then be made such that false alarms are avoided from control input \( u(t) \) and disturbances \( d(t) \) within the normal range.
6.6 Residual Evaluation

6.6.2 Evaluation Against a Threshold

Validating that no fault is present is equivalent with checking that the residual vector is zero. Validating the presence of a fault means checking whether the residual is or has been different from zero. The two hypotheses and the associated condition on the residual vector are

\[ \mathcal{H}_0(0, t) : \text{no fault is present} \quad \| r \| = 0 \]
\[ \mathcal{H}_1(f_j, t_j) : \text{fault } f_j \text{ was present since time } t_j \quad \| r(t) \| \neq 0, \ t \geq t_j, \] \hspace{1cm} (6.95)

where \( \| r \| \) is an appropriate norm of the residual.

---

6.6 Residual Evaluation

Test function.

For generality, introduce a test function \( \varphi(r(t)) \), which provides a measure (norm) of the residual’s deviation from zero. Some common test functions are the following

- Absolute value
  \( \varphi(r_j(t)) = |r_j(t)| \). \hspace{1cm} (6.96)

- An approximation to the two-norm of the residual vector
  \( \varphi(r_j(t)) = \left( \frac{1}{T} \int_{t-T}^{t} |r_j(\tau)|^2 d\tau \right)^{\frac{1}{2}} \). \hspace{1cm} (6.97)

- Square root of filtered absolute value, squared,
  \( \varphi(r_j(t)) = \left( \int_{0}^{T} w^{(f)}(t-\tau) |r_j(\tau)| d\tau \right)^{\frac{1}{2}} \). \hspace{1cm} (6.98)
6.6 Residual Evaluation

Test function.

- Filtered mean square value of signal

\[
\varphi(r_j(t)) = \int_0^T w^j_\varphi(t - \tau) \left( r_j(\tau) - \frac{1}{T} \int_{\tau-T}^{\tau} r_j(\tau_2) d\tau_2 \right)^2 d\tau, \quad (6.99)
\]

where \( w^j_\varphi(t) \) is the impulse response of a filter used particularly for evaluation of residual \( j \). In this context, the test function given in Eq. (6.96) is considered further.

---

6.6 Residual Evaluation

Threshold function.

The next step in residual evaluation is to determine a threshold function \( \Phi(t) \) for evaluation of the test function \( \varphi(t) \). \( \Phi(t) \) should have the properties

- no fault: \( \forall t \geq 0 : f(t) = 0 \) : \( \varphi(r(t)) \leq \Phi(t) \)
- weakly detectable fault: \( \exists t \geq t_0 : f(t) \neq 0 \) : \( \varphi(r(t)) > \Phi(t) \)
- strongly detectable fault: \( \forall t \geq t_1 \geq t_0 : f(t) \neq 0 \) : \( t \geq t_0 : \varphi(r(t)) > \Phi(t) \)

LTI case. In the ideal LTI case, Eq. (6.94), \( \Phi(t) \) could be chosen constant and as close to zero as allowed by practical values of bias and noise in the residual.

In the non-ideal case, Eq. (6.93) applies and input and disturbances have some feed-through to the residual. With the test function \( \varphi(t) = \| r_j(t) \|_2 \), the threshold need be determined such that

\[
\Phi_j(t) \geq \sup_{f=0,\|u,d\|<\varepsilon} (\varphi(r_j(t)))
\]
6.6 Residual Evaluation

is achieved in the time domain. The fact that total power calculated in the time domain and in the frequency domain are equal is used to determine the threshold function,

\[ ||r(j\omega)||^2 = \frac{1}{2\pi} \int_0^\infty r(j\omega) r(-j\omega) \, d\omega = \lim_{T \to \infty} \int_0^T |r(t)|^2 \, dt = ||r(t)||^2 \]

From Eq. (6.93), the residual is given in the frequency domain. Component j of the residual is

\[ r_j(s) = (H_{ru}(s) u(s))^{(j)} + (H_{rd}(s) d(s))^{(j)} + (H_{rf}(s) f(s))^{(j)}. \]

With k control inputs and for all admissible u and d

\[
\|r_j(j\omega)\|_2 \leq \|H_{ru}(j\omega) u(j\omega)\|_2^{(j)} + \|H_{rd}(j\omega) d(j\omega)\|_2^{(j)} \\
\leq \sum_{i=1}^k \|H_{ru}(j\omega)\|_\infty^{(ji)} \|u_i(j\omega)\|_2 + \|H_{rd}(j\omega) d(j\omega)\|_\infty^{(j)} \tag{6.100}
\]

- The first term in the right hand side is a gain times input power.
- The second is the maximal contribution to the residual from disturbances.

Let the effect of disturbances on the residual be bounded by

\[ \|H_{rd}(j\omega) d(j\omega)\|_\infty^{(j)} < \beta_d^{(j)}, \tag{6.101} \]

then \( \Phi(t) \) should be chosen as the time-varying function

\[ \Phi_j(t) = \sum_{i=1}^k \beta_i \|u_i(t)\|_2 + \beta_d^{(j)} \tag{6.102} \]

\[ \beta_i = \|H_{ru}(j\omega)\|_\infty^{(ji)}. \]
6.6 Residual Evaluation

This threshold is a function of maximal gains from control inputs to residual and of the maximum gain from disturbances to residual. It is often referred to as a time-varying threshold in the literature. The term adaptive threshold has also been used.

If the time-varying threshold Eq. (6.102) is too conservative, a dynamical bound could be specified as

$$\Phi_j(t) = \sum_{i=1}^{k} \left( \int_0^t \hat{h}_{ri}^{(ji)}(t - \tau)u_i(\tau)d\tau \right) + \beta_d^{(j)}, \quad (6.103)$$

where $\hat{h}_{ri}^{(ji)}$ s an estimate of the maximum (envelope) of impulse response functions from input i to residual j for a given model uncertainty.

Example 6.10 Ship example (LTI case)

Assume the ship was LTI,

$$y_1(s) = \omega_3(s) + \omega_w(s) + f_\omega(s) = \frac{b}{s - b\eta_1}\delta(s) + \omega_w(s) + f_\omega(s) \quad (6.104)$$

$$y_2(s) = \psi(s) + f_\psi(s) = \frac{1}{s}(\omega_3(s) + \omega_w(s)) + f_\psi(s)$$

the design model was

$$\hat{\omega}_3(s) = \frac{\hat{b}}{s - \hat{b}\eta_1} \delta$$

$$\hat{\psi}(s) = \frac{1}{s} \hat{\omega}_3(s)$$
6.6 Residual Evaluation

Example 6.10

and a residual generator is chosen as

\[ r_1(s) = y_1(s) - \hat{\omega}_3(s) = \left( \frac{b}{s - b\eta_1} - \frac{\hat{b}}{s - \hat{b}\eta_1} \right) \delta(s) + \omega_w(s) + f_\omega(s) \]

\[ r_2(s) = \frac{\tau}{1 + s\tau} (sy_2(s) - y_1(s)) = \frac{s\tau}{1 + s\tau} f_\psi(s) - \frac{\tau}{1 + s\tau} f_\omega(s). \]

With no faults

\[ \| r_1(j\omega) \|_2 \leq \frac{b}{j\omega - b\eta_1} - \frac{\hat{b}}{j\omega - \hat{b}\eta_1} \| \delta(j\omega) \|_2 + \| \omega_w(j\omega) \|_\infty \]  

\[ \Phi_1(t) = \frac{b}{j\omega - b\eta_1} - \frac{j\omega}{j\omega - \hat{b}\eta_1} \| \delta(t) \|_2 + \| \omega_w(j\omega) \|_\infty \leq \beta_d. \]

\[ \Phi_1(t) = \left| \frac{\hat{\eta}_1 - \eta_1}{\eta_1 \hat{\eta}_1} \right| \| \delta(t) \|_2 + \beta_d = \beta_u \| \delta(t) \|_2 + \beta_d. \]

In real time, we evaluate

\[ \| r(t) \|_2 \leq \beta_u \| \delta(t) \|_2 + \beta_d \]  

using Eq. (6.97) as an approximation to the two norm. □
6.6 Residual Evaluation

**General case.** In the general case, if the parity relation is bounded by

\[ \varphi(p_j(u_i, y_i, \hat{\theta}_i, \hat{c}_i, t)) \leq \alpha_j(u_i, y_i, t) \wedge 0 < \alpha(u_i, y_i, t) < \infty. \]  

(6.109)

The threshold function can obviously be chosen as

\[ \Phi_j(t) \geq \alpha_j(u_i, y_i, t) \]  

(6.110)

If more detailed information is available, e.g.

\[ \alpha_j(u_i, y_i, t) \leq \beta_0 + \sum_{i=1}^{k} \beta_{ji} |u_i| \]  

(6.111)

such information should be utilised when specifying the threshold, in this case as

\[ \Phi_j(t) = \beta_0 + \sum_{i=1}^{k} \beta_{ji} |u_i|. \]  

(6.112)

---

**Algorithm 6.5 Test against time-varying threshold**

**Given:** A residual \( r_j = p_j(u_i, y_i, \hat{\theta}_i, \hat{c}_i, t) \) and the object for diagnosis assumed in the no-fault condition.

1. Determine a test function \( \varphi(r(t)) \) according to Eqs. (6.96) to (6.99).

2. Determine a threshold function \( \Phi_j(t) \) for the LTI case according to Eq. (6.102), for the general case according to Eq. (6.109) or Eq. (6.102) when specific information is available.

**Initialise:** \( H^{(j)} = H_0. \)

**Do:**

1. Calculate \( \varphi(r_j(t)) \) and \( \Phi_j(t) \).
2. If \( H^{(j)} = H_0, \forall j \):
   - If \( \varphi(r_j(t)) \geq \Phi_j(t) \) set hypothesis to \( H^{(j)} = H_1 \).
   - Else:
     - If \( \varphi(r_j(t)) < \gamma \Phi_j(t) \) for \( \forall j \) set hypothesis to \( H^{(j)} = H_0 \).
6.6 Residual Evaluation

Example 6.11 Time-varying threshold for ship

Let the ship’s true constraints be:

\[ c_1 : \dot{\omega}_3 = b\eta_1 \omega_3 + b\eta_3 \dot{\omega}_3 + b\delta \]
\[ c_2 : \ddot{\psi} = \omega_3 + \omega_w \]
\[ m_1 : y_1 = \dot{\psi} \]
\[ m_2 : y_2 = \psi \]  \hspace{1cm} (6.113)

And let a model used for design be

\[ \dot{c}_1 : \dot{\omega}_3 = \dot{b}\eta_1 \omega_3 + \dot{b}\delta \]
\[ \dot{c}_2 : \ddot{\psi} = \omega_3 \]
\[ m_1 : y_1 = \dot{\psi} \]
\[ m_2 : y_2 = \psi \]  \hspace{1cm} (6.114)

Using the model for design, a residual generator is suggested as

\[ r_1 = \frac{d}{dt} y_1 - \frac{d}{dt} \dot{y}_1 \]
\[ r_2 = \frac{d}{dt} y_2 - y_1 \]  \hspace{1cm} (6.115)

then, the real residual will vary with input and

\[ r_1(t) = (b\eta_1 - \dot{\delta})\omega_3 + b\eta_3 \dot{\omega}_3 + (b - \dot{\delta})\delta + \frac{d}{dt} \omega_w(t) \]
\[ r_2(t) = 0 \]

\[ |r_1(t)| \leq |b\eta_1 - \dot{\delta}| |\gamma_1| + |b\eta_3| |\gamma_3| + |b - \dot{\delta}| |\delta| + \left| \frac{d}{dt} \omega_w(t) \right|_{\sup} \]
\[ \leq \beta_1 |\gamma_1| + \beta_3 |\gamma_3| + \alpha_1 |\delta| + \beta_4 \leq \alpha_2 |\delta| + \beta_4. \]

Fault D. and FTCS
by: Dr. B. Moaveni