Acta Automatica Sinica
Vol. 34, No. 1 January, 2008
DOI: 10.3724/SP.J.1004.2008.00085

Existence Condition on Solutions to the Algebraic Riccati Equation

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Abstract First, the existence conditions on the solutions to the algebraic Riccati equation are reviewed. Then, a strict proof is presented for a necessary and sufficient condition on the existence of a unique optimal positive definite solution to this equation. By using this condition, some untrue results on the design of robust decentralized controllers are corrected.

Key words Riccati equation, optimal control, robust control, stability, decentralized control

1 Introduction

The algebraic Riccati equation (ARE) has been widely used in control system syntheses[1−2], especially in optimal control[3−5] , robust control[6−7] and the LMI-based design[7−8]. As the solution to this equation may not be unique[9], the existence conditions of solutions have been considerably investigated[10−15]. In [15], we proposed a necessary and sufficient condition on the existence of a unique optimal positive definite solution to this equation. But the proof given in [15] is not strict. Moreover, with using this equation, quite a few results published before are found to be incorrect (See [16], for example).

First, this note reviews the existence conditions on the solutions to the ARE. Then, we present a strict proof for a necessary and sufficient condition on the existence of a unique optimal positive definite solution. Some wrong results appearing in [16] are also corrected.

2 Existence condition on solutions to the ARE

Consider the algebraic Riccati equation

$$A^{\mathrm{T}}P + PA - PBB^{\mathrm{T}}P + Q = 0,\ Q \geq 0 \tag{1}$$

where A and B are $n \times n$ and $n \times m$ real matrices, respectively. $Q \geq 0$ means that Q is a positive semidefinite real symmetric matrix. Without loss of generality, let $Q = C^{\mathrm{T}}C$, where C is a $p \times n$ matrix. It is well known that (1) is associated with the following linear system:

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= A\boldsymbol{x}(t) + B\boldsymbol{u}(t) \\ \boldsymbol{x}(0) &= \boldsymbol{x}_0 \end{aligned} \tag{2}$$

with the state feedback control

$$\boldsymbol{u}(t) = -K\boldsymbol{x}(t),\ K = B^{\mathrm{T}}P \tag{3}$$

and the performance index

$$J = \int_0^\infty (\boldsymbol{x}^{\mathrm{T}}Q\boldsymbol{x} + \boldsymbol{u}^{\mathrm{T}}\boldsymbol{u})\mathrm{d}t \tag{4}$$

For convenience, the following terms[12] are employed in this note: 1) the stabilizing solution P_s is defined as the positive semidefinite solution $P \geq 0$ to (1) such that the closed-loop system matrix $A - BK$ is asymptotically stable; 2) the optimal solution P_0 is defined as the positive semidefinite solution $P \geq 0$ to (1) such that control (3) is optimal with respect to index (4). Further, we introduce the $2n \times 2n$ Hamiltonian matrix

$$M = \left[\begin{array}{cc} A & -BB^{\mathrm{T}} \\ -C^{\mathrm{T}}C & -A^{\mathrm{T}} \end{array}\right] \tag{5}$$

Firstly, we review some existence conditions on solutions to the ARE (1), which are useful and fundamental for establishing the necessary and sufficient condition on the existence of a unique optimal positive definite solution (Theorem 1).

Lemma 1[10−11]**.** If (A, B) is stabilizable, then there exists a positive semidefinite solution $P \geq 0$ to (1).

Lemma 2[11−12]**.** (1) has a positive semidefinite solution $P \geq 0$ which is the stabilizing one if and only if (A, B) is stabilizable and $\mathrm{Re}\lambda \neq 0$ for every eigenvalue λ of the Hamiltonian matrix M.

Lemma 3[13]**.** $\mathrm{Re}\lambda \neq 0$ for every eigenvalue λ of M if and only if every eigenvalue β of A satisfying $\mathrm{Re}\beta = 0$ is a controllable mode of (A, B) and an observable mode of (C, A) .

Lemma 4[12]**.** (1) has a unique positive semidefinite solution $P \geq 0$ and the solution is the optimal one as well as the stabilizing one if and only if (A, B) is stabilizable and (C, A) is detectable.

Lemma 5[10−11]**.** If (A, B) is stabilizable and (C, A) is observable, then there is a unique positive definite solution $P > 0$ to (1) and it is the stabilizing one.

Lemma 6[14]**.** (1) has a positive definite solution $P > 0$ and the solution is the stabilizing one if and only if (A, B) is stabilizable and $(C, -A)$ is detectable.

Secondly, we present a strict proof for the result proposed in [15] in the following.

Theorem 1. (1) has a unique positive definite solution $P > 0$ and the solution is the optimal one as well as the stabilizing one if and only if (A, B) is stabilizable and (C, A) is observable.

Proof.

1) Sufficiency

Since (A, B) is stabilizable and (C, A) is observable, it follows from Lemma 5 that the ARE (1) has a unique positive definite solution $P_+ > 0$ and the solution is the stabilizing one. Meanwhile, one can see that the conditions of Lemma 4 are satisfied as well in this case. From Lemma 4, we know that the positive definite solution $P_+ > 0$ above is also the unique positive semidefinite solution $P \geq 0$ to the ARE (1) and it is the optimal one as well as the stabilizing one.

Received November 15, 2006; in revised form June 2, 2007
Supported by National Natural Science Foundation of China (60774002, 90405017) and National Basic Research Program of China (973 Program) (2002CB312205)
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DOI: 10.3724/SP.J.1004.2008.00085

2) Necessity

Since the ARE (1) has a positive definite solution $P > 0$ and the solution is the stabilizing one, we know from Lemma 6 that (A, B) is stabilizable and $(C, -A)$ is detectable.

On the other hand, it is shown in [12] that if more than one solution $P \geq 0$ to the ARE (1) exist, then we always have

$$P_s \geq P_0\ ,\ P_s \neq P_0$$

where $P_s \geq P_0$ means that $P_s - P_0 \geq 0$, or $P_s - P_0$ is a positive semidefinite real symmetric matrix. Now, since the positive definite solution P_+ is the optimal one P_0 as well as the stabilizing one P_s, i.e.,

$$P_0 = P_s = P_+$$

it is necessitated[12] that the ARE (1) has only one positive semidefinite solution P_1. Clearly,

$$P_1 = P_+$$

Thus, it is concluded that the ARE (1) has a unique positive semidefinite solution P_+ and it is the optimal one P_0 as well as the stabilizing one P_s. From Lemma 4, it is straightforward that (A, B) is stabilizable and (C, A) is detectable.

Now, if one can show that (C, A) is observable from the results above, i.e., both (C, A) and $(C, -A)$ are detectable, the proof will be completed immediately.

In fact, it is known[12] that (C, A) is detectable if and only if every unobservable mode of (C, A) is necessarily associated with an asymptotically stable eigenvalue of matrix A . Under the conditions that both $(C, -A)$ and (C, A) are detectable, we assume that (C, A) is not observable. Then there exists at least one eigenvalue λ of matrix A and the corresponding eigenvector $\boldsymbol{\xi} \neq 0$ satisfying

$$\begin{aligned} A\boldsymbol{\xi} &= \lambda\boldsymbol{\xi} \\ C\boldsymbol{\xi} &= \mathbf{0} \end{aligned}$$

Obviously, this will contradict the condition that (C, A) is detectable in the case of $\mathrm{Re}\lambda \geq 0$ or that $(C, -A)$ is detectable in the case of $\mathrm{Re}\lambda < 0$ or $\mathrm{Re}(-\lambda) > 0$. It is noted that $-\lambda$ is an eigenvalue of matrix $-A$, and $\boldsymbol{\xi}$ is the corresponding eigenvector. Therefore, under the conditions that both $(C, -A)$ and (C, A) are detectable, (C, A) is necessarily observable.

This completes the proof. □

3 Application to the decentralized control

To show the significance of the necessary and sufficient condition, we use Theorem 1 to correct some results appearing in a decentralized control scheme for interconnected systems presented by [16].

In Section 3 of [16], the following extended system is employed ((17) therein):

$$\dot{\bar{\boldsymbol{X}}}_i(t) = \bar{A}_i\bar{\boldsymbol{X}}_i(t) + \bar{B}_i\bar{\boldsymbol{U}}_i(t) \tag{6}$$

where

$$\begin{aligned} \bar{A}_i &= \begin{bmatrix} A_i & I_i & -\tilde{L}_i \\ 0 & A_{zi} & 0 \\ 0 & 0 & \tilde{A}_i \end{bmatrix} \\ \bar{B}_i &= \begin{bmatrix} B_i & 0 \\ 0 & I_i \\ 0 & 0 \end{bmatrix} \end{aligned} \tag{7}$$

with the pair $(\bar{A}_i,\ \bar{B}_i)$ being stabilizable.

The associated performance index is

$$J_i = \frac{1}{2}\int_0^{\infty}[\bar{\boldsymbol{X}}_i^{\mathrm{T}}(t)\bar{Q}_i\bar{\boldsymbol{X}}_i(t) + \bar{\boldsymbol{U}}_i^{\mathrm{T}}(t)\bar{R}_i\bar{\boldsymbol{U}}_i(t)]\mathrm{d}t \tag{8}$$

where

$$\begin{aligned} \bar{Q}_i &= \begin{bmatrix} Q_i & 0 & 0 \\ 0 & Q_{ui} & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \bar{R}_i &= \begin{bmatrix} R_i & 0 \\ 0 & R_{vi} \end{bmatrix} \end{aligned} \tag{9}$$

where $\bar{Q}_i = \bar{D}_i^{\mathrm{T}}\bar{D}_i$ is a positive semidefinite matrix with $(\bar{D}_i, \bar{A}_i)$ being detectable and $\bar{R}_i$ is a positive definite one.

The optimal control with respect to index (8) is given by

$$\bar{\boldsymbol{U}}_i(t) = -\bar{G}_i\bar{\boldsymbol{X}}_i(t) \tag{10}$$

where

$$\begin{aligned} \bar{G}_i &= \bar{R}_i^{-1}\bar{B}_i^{\mathrm{T}}\bar{P}_i = \\ &\begin{bmatrix} G_{i1} & G_{i2} & G_{i3} \\ H_{i1} & H_{i2} & H_{i3} \end{bmatrix} \end{aligned} \tag{11}$$

and $\bar{P}_i$ is a steady state solution of the following Riccati equation:

$$\bar{A}_i^{\mathrm{T}}\bar{P}_i + \bar{P}_i\bar{A}_i - \bar{P}_i\bar{B}_i\bar{R}_i^{-1}\bar{B}_i^{\mathrm{T}}\bar{P}_i + \bar{Q}_i = 0 \tag{12}$$

Based on (10)∼(12), a decentralized controller is developed in [16], as shown in Fig. 1. Then, a sufficient condition, Theorem 1, on the system stability is presented. However, in the proof for this theorem, the following is used as a candidate Lyapunov function

$$\begin{aligned} V[\boldsymbol{X}(t)] &= \sum_{i=1}^{s} V_i[\boldsymbol{X}_i(t)] = \\ &\sum_{i=1}^{s} \boldsymbol{X}_i^{\mathrm{T}}(t)\bar{P}_i\boldsymbol{X}_i(t) \end{aligned} \tag{13}$$

where the matrix $\bar{P}_i$ is the same as that in (11), namely the optimal solution of the ARE (12).

It is noticed that, in view of the structure of $\bar{Q}_i$ in (9), $(\bar{D}_i,\ \bar{A}_i)$ can be only detectable rather than observable. Thus, it is impossible for the optimal solution of the ARE (12) to be positive definite, according to our Theorem 1. In other words, $\bar{P}_i$ in (13) is not a positive definite matrix. $V[\boldsymbol{X}(t)]$ is not a Lyapunov function, either. Obviously, the proof presented in [16] for Theorem 1 therein is wrong.

To fix the problem above, according to our Theorem 1, one must manage to choose such a matrix $\bar{Q}_i$ in (9) that $(\bar{D}_i,\ \bar{A}_i)$ is observable. Yet, since there exist some more incorrect results in the proof of [16], we conclude that Theorem 1 of [16] is not true.

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