Élie Cartan

Exterior Differential Systems and its Applications

Translated form French by

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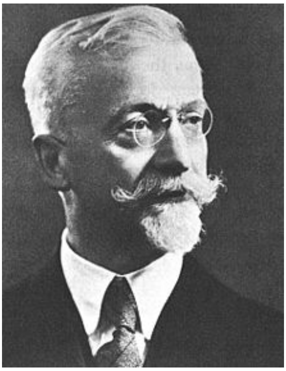

Les systémes différentiels extérieurs et leurs applications géométriques.
(Actualités Scientifiques et Industrielles, no. 994.) By Elie Cartan.
Paris, Hermann, 1945. Second ed, 1971.

The first part of this book contains the theory of integration of total differential equations connected with a general system of exterior differential forms (covariant alternating quantities). The symbolism used is the ω-method introduced in Cartans well-known publications [Ann. Sci. Ecole Norm. Sup. (3) 18, 24–311 (1901); 21, 153–206 (1904), in particular, chap. I] with some modifications due to E.Kahler [Einfuhrung in die Theorie der Systeme von Differential-gleichungen, Hamburger Math. Einzelschr., no. 16, Teubner, Leipzig-Berlin, 1934]. The first two chapters contain an exposition of the method. In chapter III, after introducing the important notions of closed systems and characteristic systems the theory of completely integrable systems is presented and applied to the ordinary problem of Pfaff. Chapter IV contains the definitions of the integral elements, the characters and the genus and two fundamental existence theorems. Systems in involution are defined in chapter V and this chapter contains several simple forms of the conditions for these systems. The theory of prolongation is dealt with in chapter VI. For the chief theorem of prolongation, proved by Cartan in 1904, another proof is given for the case $m = 2$. In No. 117 special attention is paid to the cases where the proof is not valid.

The second part of the book contains applications to several problems of differential geometry. Chapter VII deals with old and new problems of the classical theory of surfaces. In each case the degree of freedom of the solution is discussed. The last chapter contains problems with more than two independent variables. A new method is developed for orthogonal systems in n variables. The problem of the realization of a V^2 with a given ds^2 in an $\mathbb{R}^6$ is discussed elaborately and special attention is paid to the singular solutions of this problem.

Reviewed by J. A. Schouten

Contents

Preface

This book is the reproduction, quite radically changed, a course taught during the first semester 1936-1937 at the Faculty of Sciences of the University of Paris.

The first part of this work is devoted to discussion of the theory of systems of differential equations in total, which was the subject of several memoirs, already old, published mainly in the Annals of Ecole Normale Superieure[1] between years 1902 and 1908, this theory was the basis for my theory of the structure of infinite transformation groups, in the sense of S. Lie[2].

It has since been generalized by various authors, especially by E. Kähler[3], who has extended to any system of differential equations Exterior.

I adopt in this work the notation advocated by E. Kähler of designating by $d\omega$, and called exterior differential of an exterior differential form ω of any degree, what I called earlier by ω' and what I called the exterior derivative of the form ω.

After a first chapter, purely algebraic, the exterior forms and Exterior systems of equations, Chapter II is devoted to exterior differential forms (symbolic forms of E. Goursat[4]) and the operation of exterior differentiation.

Chapter III introduces the concept of closed exterior differential system and characteristic system, outlines the theory of completely integrable systems with applications to the classical problem of Pfaff[5].

Chapter IV introduces the concepts of integral element, character and gender, devoted also to two fundamental theorems of existence.

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[1] It was founded in 1864 by Louis Pasteur. For more informations, see: http://www.math.ens.fr/edition/annales/

[2] Marius Sophus Lie (1842–1899) was a Norwegian mathematician. For more informations, see: http://www-history.mcs.st-andrews.ac.uk/Biographies/Lie.html

[3] Erich Kähler (1906–2000) was a German mathematician. For more informations, see: http://www-history.mcs.st-andrews.ac.uk/Biographies/Kahler.html

[4] Édouard Jean-Baptiste Goursat (1858–1936) was a French mathematician. For more informations, see: http://www-history.mcs.st-andrews.ac.uk/Biographies/Goursat.html

[5] Johann Friedrich Pfaff (1765–1825) was a German mathematician. For more informations, see: http://www-history.mcs.st-andrews.ac.uk/Biographies/Pfaff.html

Chapter V is devoted to differential systems with independent variables imposed, especially for systems in involution with existence theorems relating to these systems and the latest indication of several simple criteria of involution, it is especially this Chapter which writing was the most altered.

Finally, Chapter VI introduces the notion of prolongation of a differential system, with applications that can make a search of soIutions a system that is not in involution.

Throughout this first part, from Chapter IV, equations of the systems considered involve only the analytic functions, the Cauchy-Kowalewski theorem, on that underlie the existence theorems that are proven, having no validity and even sene that if the data is analytic.

The second Part of the book is devoted to applications to problems in differential geometry. It consists of two chapters.

The problems addressed in Chapter VII are all related to the classical theory of surfaces. Many are old, quite a few others are new. In each case the degree of generality of the solution is given, and how we should pose the Cauchy problem, comprehending where the data are characteristic.

Chapter VIII refers to problems with more than two independent variables, containing the problem of orthogonal triple systems, the general problem of orthogonal systems p-tuples of p-dimensional Euclidean space: in the latter case, the problem of Cauchy is presented in a simple form which has not yet, I believe, been considered.

Finally the problem of realization of ds^2 has three variables in Euclidean space has six dimensions is discussed in detail, with indications of singular solutions of this problem.

Throughout this second Part, use is made almost exclusive method of moving frame, which is particularly suitable ii using the theory of differential systems in involution outside.

In writing this book, I used a first draft due to Luc Gauthier, responsible for research, after the notes taken during. This first draft has made me the greatest service and he is agrbable to express my gratitude Mr. Gauthier.

Élie Cartan

Part I
Theory of exterior differential systems

Chapter 1
Exterior Forms

1.1 Symmetric and alternating bilinear forms, Exterior algebraic quadratic forms

1. In classical algebra, a *bilinear form* of two sets of variables $u^1, u^2, \cdots, u^n$ and $v^1, v^2, \cdots, v^n$ of the same number n, is an expression[1]

$$F(u,v) = a_{ij} u^i v^j, \tag{1.1}$$

with coefficients a_{ij} belonging to a given field, we assume, for simplicity, be the field of real numbers $\mathbb{R} relationship$.

This form is called *symmetric* if it remains identical to itself when we replace variables u^i by variables v^i with the same index and vice versa, i.e. if one has

$$a_{ij} = a_{ji}, \qquad (i, j = 1, \cdots, n). \tag{1.2}$$

With any symmetric bilinear form we can associate a quadratic form

$$F(u) = a_{ij} u^i u^j. \tag{1.3}$$

has a single set of variables, the form (1.1) is the polar form of the quadratic form (1.2), that is

$$F(u,v) = v^i \frac{\partial F}{\partial u^i} = u^i \frac{\partial F}{\partial v^i}. \tag{1.4}$$

There is an intrinsic relation between the form (1.1) and the form (1.2), in the sense that if we perform on the variables u^i and the variables v^i a single linear transformation

[1] We adopt a convention once and for all, now classic, which is to remove the summation sign in front of an expression that contains the same index repeated twice, in the formula (1.1), it is two summation indices i and j, each repeated twice.

$$u^i = A^i_k U^k, \qquad v^i = A^i_k V^k, \tag{1.5}$$

the bilinear form (1.1) is transformed into bilinear form a $\Phi(U,V)$ still symmetric and the quadratic form (1.2) in the quadratic form $\Phi(U)$ associated, such that

$$\Phi(U,V) = a_{ij} A^i_k A^j_h U^k V^k, \qquad \Phi(U) = a_{ij} A^i_k A^j_h U^k U^k. \tag{1.6}$$

We check easily that the form $\Phi(U,V)$ is symmetric, since the coefficient of $U^k V^h$ which is the sum $a_{ij} A^i_k A^j_h$ where i and j are two indices of summation independent, can be written $a_{ji} A^j_k A^i_h = a_{ij} A^i_h A^j_k$, which is the coefficient, precisely. But this verification can be avoided if we note as exchanging the two sets of variables U^i and V^i exchange returns has two sets of variables u^i and v^i, which does not change the initial form $F(u,v)$.

2. The form (1.1) is called *alternating* if it changes sign with exchange two sets of variables u^i and v^i:

$$F(v,u) = -F(u,v); \tag{1.7}$$

this results in the anti-symmetry of the coefficients:

$$a_{ji} = -a_{ij}. \tag{1.8}$$

If we put $v^i = u^i$ we get this time a form identical zero. Yet we can associate in the alternating form $F(u,v)$ a quadratic form, but non-commutative multiplication.
Note why if we shall meet the two terms of $a_{12} u^1 v^2 - a_{21} u^2 v^1$, we obtain

$$a_{12}(u^1 v^2 - u^2 v^1) = a_{12} \begin{vmatrix} u^1 & u^2 \\ v^1 & v^2 \end{vmatrix}, \tag{1.9}$$

can we agree to write $a_{12}\, u^1 \wedge u^2$, the notation $u^1 \wedge u^2$ recalling the determinant whose first row is formed of two variables u^1, u^2, the second of two variables v^1, v^2. The expression $a_{12}\, u^1 \wedge u^2$ can be regarded as a monomial quadratic multiplication non-commutative Grassmann, the product of two variables u^1, u^2 changing sign with the order of factors. We can therefore, in this case has involved the alternating bilinear form

$$F(u,v) = a_{ij} u^i v^j, \qquad (a_{ij} = -a_{ji}), \tag{1.10}$$

the quadratic form has exterior multiplication or, more briefly, the *exterior quadratic form*

$$F(u,v) = \frac{1}{2} a_{ij}\, u^i \wedge u^j, \qquad (a_{ij} = -a_{ji}). \tag{1.11}$$

We put the numerical factor $\frac{1}{2}$ because the second order product $u^1 \wedge u^2$ appeared two times, in the form $u^1 \wedge u^2$ and as $u^2 \wedge u^1 = -u^1 \wedge u^2$, which gives the total coefficient $\frac{1}{2}a_{12} - \frac{1}{2}a_{21} = a_{12}$.

There is a correspondence between the intrinsic bilinear form alternating and the quadratic form associated exterior, this correspondence subsisting linear substitution effected simultaneously on two sets of variables u^i and v^i. If we set

$$u^i = A^i_k U^k, \qquad v^i = A^i_k V^k, \tag{1.12}$$

the form $F(u,v)$ becomes

$$\Phi(U,V) = a_{ij} A^i_k A^j_h U^k V^h, \tag{1.13}$$

with antisymmetric coefficients $U^i V^i$ products, and the form $F(u)$ becomes

$$\Phi(U) = \frac{1}{2} a_{ij} A^i_k A^j_h U^k \wedge U^h; \tag{1.14}$$

we see that we deduce the form of $F(u)$ by replacing everywhere u^i by $A^i_k U^k$ and performing the multiplication under the rules of algebraic multiplication, but taking care not to invert the order of the variables U^i in the partial products that arise.

3. There are certain similarities between the classical quadratic forms, which we call algebraic, and exterior quadratic forms. Define the partial derivative of the quadratic form exterior $F(u)$ by the relation

$$\frac{\partial F}{\partial u^i} = a_{ik} u^k. \tag{1.15}$$

The derivative of each monomial is zero if u^i is not among the factors of the monomial; u^i appears as if the first factor, as in the term $a_{ij}[u^i u^j]$, the derivative is $a_{ij} u^j$; appears as a second factor, we have the same rhgle, but taking care, first, to pass u^i in front, the derivative with respect to a u^i of $a_{ij}[u^i u^j]$ is that of - $a_{ji}[u^i u^j]$, that is to say - $a_{ji} u^j = a_{ij} u^j$. The derivative with respect to u^i of the derivative with respect to a_{ij} is after this and we write

$$\frac{\partial^2 F}{\partial u^j \partial u^i} = a_{ij} = -\frac{\partial^2 F}{\partial u^i \partial u^j}. \tag{1.16}$$

Note that the sum $u^i \frac{\partial F}{\partial u^i} = a_{ij} u^i u^k$ is zero, whereas if the quadratic form is algebraic sum $u^i \frac{\partial F}{\partial u^i}$, after of Euler's theorem, is equal to $2F$.

Now suppose that instead of multiplying the following traditional rules u^i and $\frac{\partial F}{\partial u^i}$, the exterior multiplication is performed, we will, as it is easy to see, following the form F is algebraic or exterior

$$u^i \wedge \frac{\partial F}{\partial u^i} = 0, \qquad u^i \wedge \frac{\partial F}{\partial u^i} = 2F. \tag{1.17}$$

Theorem. *Any algebraic quadratic form F satisfies the relations*

$$u^i \frac{\partial F}{\partial u^i} = 2F, \qquad u^i \wedge \frac{\partial F}{\partial u^i} = 0. \tag{1.18}$$

Instead any exterior quadratic form F satisfies relations

$$u^i \frac{\partial F}{\partial u^i} = 0, \qquad u^i \wedge \frac{\partial F}{\partial u^i} = 2F. \tag{1.19}$$

4. The exterior multiplication of the two forms indices $f(u)$, $\phi(u)$ same variables gives the quadratic form external associated to the alternating bilinear form

$$f(u)\phi(v) - \phi(u)f(v) = \begin{vmatrix} f(u) & \phi(u) \\ f(v) & \phi(v) \end{vmatrix}; \tag{1.20}$$

be denoted by the notation $f(u) \wedge \phi(u)$ the quadratic form F and the multiplication result of term end of the two forms $f(u)$ and $\phi(u)$, but taking care not to invert the order of factors:

$$F = a_i u^i \wedge b_j u^j = a_i b_j \, u^i \wedge u^j. \tag{1.21}$$

Note that in the second member $a_i b_j$ the coefficient is not antisymmetric, although the form of the second member is exterior, but we can write, by exchanging the two indices of summation,

$$F = a_i b_j \, u^i \wedge u^j = a_j b_i \, u^j \wedge u^i = -a_j b_i \, u^i \wedge u^j, \quad \text{then} \quad F = \frac{1}{2}(a_i b_j - a_j b_i) \, u^i \wedge u^j, \tag{1.22}$$

and now the coefficient of $u^i \wedge u^j$ is antisymmetric.

We still here two theorems that correspond.

Theorem. *If $f^1, f^2, \cdots, f^p$ are p independent linear forms with n variables $u^1, u^2, \cdots, u^n$, the relation*

$$f^1 \phi_1 + f^2 \phi_2 + \cdots + f^p \phi_p = 0, \tag{1.23}$$

where $\phi_1, \phi_2, \cdots, \phi_n$ are p forms are the same variables, implies that the ϕ_i are linear combinations of forms to f^i coefficients antisymmetric

$$\phi_i = \alpha_{ih} f^h, \qquad (\alpha_{ij} = -\alpha_{ji}); \tag{1.24}$$

on the contrary causes the relation

$$f^1 \wedge \phi_1 + f^2 \wedge \phi_2 + \cdots + f^p \wedge \phi_p = 0, \tag{1.25}$$

that are linear combinations ϕ_i forms f^i to have coefficients α_{ij} symmetric.

Proof. Suppose first $p = n$, in which case the forms are independent f^i any linear form can be expressed as a linear combination of f^i. By asking $\phi_i = \alpha_{ik} f^k$ was then $f^i \phi_i = \alpha_{ik} f^i f^k$, and $f^i \wedge \phi_i = \alpha_{ij} f^i \wedge f^k$. The sum $f^i \phi_i$ is zero if $\alpha_{ij} = -\alpha_{ji}$ and the sum $[f^i \phi_i]$ is zero if $\alpha_{ij} = \alpha_{ji}$, which proofs the theorem.

Now assume $p < n$, then, introduce new $n-p$ forms $f^{p+1}, \cdots, f^n$ mutually independent and independent of the first p given forms. We can then apply the theorem proved in the case $p = n$ by taking the functions $\phi_{p+1}, \cdots, \phi_n$ identically zero. We will then, since the form ϕ_{p+1} is zero, $\alpha_{p+1,k} = 0$ $(j = 1, 2, \cdots, n-p; k = 1, 2, \cdots, n)$ where, in both cases, $\alpha_{k,p+j} = 0$. The indices $p+1, p+2, \cdots, n$ are therefore not included in the coefficients α_{ij} which do fall under the terms of ϕ_i through f^i and the theorem is well proved in the general case. □

Note. The two equations $f^i \phi_i = 0$ and $f^i \wedge \phi_i = 0$ assumed verified if and only if all $\phi_i = 0$.

5. It is known that any algebraic quadratic form can be put in a canonical form, in which the all rectangular coefficients are zero. There are even a canonical form for exterior quadratic forms.

Theorem. *Any quadratic form can be reduced to exterior form*

$$F = U^1 \wedge U^2 + U^3 \wedge U^4 + \cdots + U^{2p-1} \wedge U^{2p}, \tag{1.26}$$

where U^i being the $2p$ independent linear forms.

Proof. The proof is very simple. Suppose the form $F(u)$ not identically zero and, for example $a_{12} \neq 0$. We can write

$$F = \left(u^1 + \frac{a_{23}}{a_{12}} u^3 + \cdots + \frac{a_{2n}}{a_{12}} u^n\right) \wedge (a_{12} u^2 + a_{13} u^3 \cdots + a_{1n} u^n) + \Phi, \tag{1.27}$$

the form Φ containing only the variables $u^3, u^4, \cdots, u^n$. Just then suppose

$$U^1 = u^1 + \frac{a_{23}}{a_{12}} u^3 + \cdots + \frac{a_{2n}}{a_{12}} u^n, \qquad U^2 = a_{12} u^2 + a_{13} u^3 \cdots + a_{1n} u^n, \tag{1.28}$$

for an exterior quadratic form $F - U^1 \wedge U^2$ which depends only on variables of $u^3, \cdots, u^n$, since n forms $U^1, U^2, u^3, \cdots, u^n$ being independent. If the form Φ is identically zero the theorem is proved, the integer p is equal to 1. Otherwise we will perform on Φ the same operation as that performed on F and so on. □

One can find the integer p without needing to actually the previous reduction. Indeed note that for any linear substitution of variables, the system of linear equations

$$\frac{\partial F}{\partial u^1} = 0,\ \frac{\partial F}{\partial u^2} = 0,\ \cdots,\ \frac{\partial F}{\partial u^n} = 0, \tag{1.29}$$

is retained. Because if we set $u^i = A^i_k U^k$, the determinant of the coefficients A^i_k being non-zero, we immediately see that we have

$$\frac{\partial F}{\partial U^i} = A^k_i \frac{\partial F}{\partial u^k}, \tag{1.30}$$

and as the determinant of A^k_i is non-zero, the vanishing of partial derivatives $\frac{\partial F}{\partial u^i}$ implies vanishing the $\frac{\partial F}{\partial U^i}$ and vice versa.

But if we put F in the canonical form

$$F = U^1 \wedge U^2 + U^3 \wedge U^4 + \cdots + U^{2p-1} \wedge U^{2p}, \tag{1.31}$$

the system (1.29) becomes, with the variables U^i,

$$U^1 = U^2 = \cdots = U^{2p-1} = U^{2p} = 0. \tag{1.32}$$

The integer $2p$ is the rank of the table of coefficients forms. This table is symmetrical left, and it is well known that the rank of such a table is always even.

The rank $2p$ states at the same time the minimum number of variables that can be included in the form F, if it is transformed by a suitable linear substitution of variables effected.

There are similar theorems for algebraic quadratic forms. The rank of the table of coefficients forms $\frac{\partial F}{\partial u^i}$ also indicates the minimum number of variables that can be included in the form F; the easiest way to recognize it is to use the decomposition of the form of a sum of squares.

1.2 Exterior forms of arbitrary degree

6. We will assume in general exterior forms of any degree. A cubic exterior form, for example, is written as

$$F = \frac{1}{6} a_{ijk} u^i \wedge u^j \wedge u^k, \tag{1.33}$$

where the coefficients a_{ijk} are anti-symmetric: this means that if one performs a permutation of three indices i, j, k, the coefficient is equal to itself, or it changes sign depending on whether the permutation is even or odd. The symbol $[u^i u^j u^k]$ can be regarded as a product but which changes sign if we interchange two factors, this product remains equal to itself by an even permutation of the three factors, becomes equal to its objects by an odd permutation of these factors.

With to these conventions, we see that if one considers the different monomials which appear in, the order was near, the three factors u^1, u^2, u^3, we get six different terms, but whose sum is $a_{123}[u^1u^2u^3]$. It is hardly worth noting that the monomials that have two identical factors must be regarded as zero.

One can consider a 3−linear alternating form F, as a form with three sets of variables u^i, v^i, w^i

$$\frac{1}{6}a_{ijk}\begin{vmatrix} u^i & u^j & u^k \\ v^i & v^j & v^k \\ w^i & w^j & w^k \end{vmatrix}. \tag{1.34}$$

More generally we may consider exterior forms of degree 4, 5, etc., which could be combined alternating forms with 4, 5, etc., sets of variables.

7. *Addition and multiplication of exterior forms.* The sum of two exterior forms of the same degree is the form of same degree whose coefficients are the sums of the coefficients of the two given types. For example, the sum of the form (1.33) and of the form

$$\Phi = \frac{1}{6}a_{ijk}\,u^i \wedge u^j \wedge u^k \tag{1.35}$$

is the sum

$$F + \Phi = \frac{1}{6}(a_{ijk} + b_{ijk})\,u^i \wedge u^j \wedge u^k. \tag{1.36}$$

Called *exterior product* of two forms of exterior degrees equal or not, for example, two forms

$$F = \frac{1}{2}a_{ij}\,u^i \wedge u^j, \qquad \Phi = \frac{1}{6}b_{ijk}\,u^i \wedge u^j \wedge u^k. \tag{1.37}$$

the form obtained by multiplying exterior of all possible ways each monomial of the first form of each monomial of the second, taking care to respect the order of two monomials and adding the resulting monomials[2]

$$F \wedge \Phi = \frac{1}{12}a_{ij}b_{khl}\,u^i \wedge u^j \wedge u^k \wedge u^h \wedge u^l. \tag{1.38}$$

The second member of relation (1.38) does not have its coefficients antisymmetric with respect to five indices i, j, k, h, l. But we can arrange to bring up the antisymmetric coefficients by performing as we have done to the exterior product of two linear forms.

[2] The numerical coefficients of the forms can be placed at any position, being the law of commutative property holds for these coefficients.

The exterior product of forms F, Φ may depend on the order in which one arranges the factors F and Φ. Indeed change the order of these factors is to replace the monomial $u^i \wedge u^j \wedge u^k \wedge u^h \wedge u^l$ the monomial $u^k \wedge u^h \wedge u^l \wedge u^i \wedge u^j$ to make this change we can advance each of the three factors of two places to the left , resulting in 2×3 successive changes of sign.

Generally, if F and Φ are of respective degrees p and q, then the relation

$$F \wedge \Phi = (-1)^{pq}\, \Phi \wedge F, \tag{1.39}$$

holds, or

Theorem. *The exterior product of two forms of exterior degrees p and q does not change when one reverses the order of two factors, except if both factors are of odd degree, in which case the product changes sign.*

Another important theorem is on the distributivity of multiplication over addition. This theorem leads to the general equality

$$\big(F_1 + F_2 + \cdots + F_h\big) \wedge \big(\Phi_1 + \Phi_2 + \cdots + \Phi_k\big) = \sum_{i,j} F_i \wedge \Phi_j, \tag{1.40}$$

where we assume the forms $F_1, F_2, \cdots, F_h$ are of the same degree p and forms $\Phi_1, \Phi_2, \cdots, \Phi_k$ are of the same degree p.

8. *Exterior forms monomials.* An exterior form of degree p will be called *monomial* if it can be represented as the exterior product of p linear forms. We will look at what conditions must meet the coefficients of the form so that it is monomial.

Consider why the system of linear equations (associated system) obtained by vanishing all the partial derivatives of order $p-1$ form. These derivatives are defined in the case of a form of any degree as a quadratic form, which depend on the order in which the derivations are performed, but in fact that order does not matter here since the only change that can be subjected the derivative of the change of the order of derivations is optionally a change of sign.

The associated system we consider has an *intrinsic* meaning in the sense that if a substitution is made linear non-zero determinant variables on transforming the form $F(u^1, u^2, \cdots, u^p)$ in the form $\Phi(U^1, U^2, \cdots, U^p)$, the same linear substitution transforms the system associated to F in the associated system of Φ; this is that the linear substitution

$$u^i = A^i_k U^k \tag{1.41}$$

leads

$$\frac{\partial \Phi}{\partial U^i} = A^i_k \frac{\partial F}{\partial u^k}, \qquad \frac{\partial^2 \Phi}{\partial U^i \partial U^j} = A^i_k A^j_h \frac{\partial^2 F}{\partial u^k \partial u^h}, \cdots \tag{1.42}$$

This posed assume a monomial exterior form, we can make a linear substitution of variables such that this form contains only one term $a\,u^1 \wedge u^2 \wedge u^3 \wedge \cdots \wedge u^p$ (Unless the form is identically zero, if we exclude). The system combines this form is obviously

$$u^1 = 0, \quad u^2 = 0, \quad \cdots, \quad u^p = 0; \tag{1.43}$$

it therefore reduces to p independent equations.

Conversely if the system associated to a form F of degree p reduces to p independent equations, one can, for a linear substitution on variables, make the system is associated

$$u^1 = 0, \quad u^2 = 0, \quad \cdots, \quad u^p = 0; \tag{1.44}$$

but then no non-zero coefficient of the form may not contain an index other than $1, 2, \cdots, p$, as a non-zero coefficient $a_{i_1 i_2 \cdots i_{p-1}(p+1)}$ as for example, would involve the variable in the derivative $\partial^{p-1} F / \partial u^{i_1} \partial u^{i_2} \cdots \partial u^{i_{p-1}}$, in contrast to the hypothesis. The form is $u^1 \wedge u^2 \wedge \cdots \wedge u^p$, that is to say a monomial.[3][4]

Theorem. *For an exterior form of degree p is monomial if and only if its associated system is of rank p.*

9. Let us apply this criterion for necessary and sufficient conditions to be met by an exterior form factors for it to be monomial. We start with the simple case of quadratic forms. Is a quadratic form

$$F = \frac{1}{2} a_{ij}\, u^i \wedge u^j, \tag{1.45}$$

we do not assume identically zero. Suppose for definiteness $a_{12} \neq 0$. Among the equations of system associated the two equations are

$$\frac{\partial F}{\partial u^1} = a_{12}u^2 + a_{13}u^3 + \cdots + a_{1n}u^n = 0, \tag{1.46}$$

$$\frac{\partial F}{\partial u^2} = a_{22}u^2 + a_{23}u^3 + \cdots + a_{2n}u^n = 0. \tag{1.47}$$

These two equations are independent and gives

[3] The same argument shows that if the system is associated of rank r, it is possible, by a change of variables, find an expression of the form which is made only r variables and it is obviously impossible to involve less otherwise the number would be associated exterior system rank r.

[4] The theorem has already been practically proves for $p = 2$ thanks to the introduction of the canonical form.

$$u^1 = \frac{1}{a_{12}}\Big(a_{23}u^3 + \cdots + a_{2n}u^n\Big), \tag{1.48}$$

$$u^2 = \frac{-1}{a_{12}}\Big(a_{13}u^3 + \cdots + a_{1n}u^n\Big). \tag{1.49}$$

Looking coefficient u^k in $\partial F/\partial u^i$ taking into account the values of u^1, u^2 that we just wrote. We have

$$\frac{\partial F}{\partial u^i} = \Big(a_{ik} + \frac{a_{i1}}{a_{12}}a_{2k} - \frac{a_{i2}}{a_{12}}a_{1k}\Big)u^k = \frac{1}{a_{12}}\Big(a_{12}a_{ik} - a_{1i}a_{2k} + a_{1k}a_{2i}\Big)u^k. \tag{1.50}$$

The second member must be zero if one requires that the system associated is of rank 2, or the necessary and sufficient conditions, in case $a_{12} \neq 0$,

$$a_{12}a_{ik} - a_{1i}a_{2k} + a_{1k}a_{2i} = 0. \tag{1.51}$$

This relation was demonstrated assuming $a_{12} \neq 0$, but it is true even if $a_{12} = 0$; indeed the indices $1, 2, i, k$, all play the same role as any permutation of these four indices leaves unaltered the relation . It is still true if any of the coefficients $a_{12}, a_{2i}, a_{2k}, a_{1i}, a_{1k}, a_{ik}$ is nonzero, it is a fortiori if they are all zero. One can thus replace the indices 1, 2 by any other evidence without the relation (1.51) ceases to take place.

Theorem. *If a quadratic exterior form be monomial, there are relations between its coeficients*

$$a_{ij}a_{kh} - a_{ik}a_{jh} + a_{ih}a_{jk} = 0, \qquad (i, j, k, h = 1, 2, \cdots, n). \tag{1.52}$$

Conversely, if these relations are verified, the form is monomial, because if we assume $a_{12} \neq 0$, the relations (1.51), which are taken from the relations (1.52), are checked and we have seen that in these conditions form is monomial.

10. The same method applies a form of any degree. Suppose to fix our ideas $p = 5$, and the coefficient nonzero a_{12345}. The associated system of the form contains five independent equations

$$\begin{aligned}
\frac{\partial^4 F}{\partial u^2 \partial u^3 \partial u^4 \partial u^5} &= a_{12345}u^1 + a_{2345k}u^k = 0,\\
\frac{\partial^4 F}{\partial u^1 \partial u^3 \partial u^4 \partial u^5} &= -a_{12345}u^2 + a_{1345k}u^k = 0,\\
\frac{\partial^4 F}{\partial u^1 \partial u^2 \partial u^4 \partial u^5} &= a_{12345}u^3 + a_{1245k}u^k = 0,\\
\frac{\partial^4 F}{\partial u^1 \partial u^2 \partial u^3 \partial u^5} &= -a_{12345}u^4 + a_{1235k}u^k = 0,\\
\frac{\partial^4 F}{\partial u^1 \partial u^2 \partial u^3 \partial u^4} &= a_{12345}u^5 + a_{1234k}u^k = 0.
\end{aligned} \tag{1.53}$$

For the form be monomial, it is necessary and sufficient that all equations of the associated system are of the consequences of the five previous ones, that is to say, that there replacing u^1, u^2, u^3, u^4, u^5 by their values from of these five equations, other equations are identically verified. For example, consider the equation

$$\frac{\partial^4 F}{\partial u^{i_1} \partial u^{i_2} \partial u^{i_3} \partial u^{i_4}} = 0; \tag{1.54}$$

the coefficient of u^k in the first member, once replaced u^1, u^2, u^3, u^4, u^5 by their values, will, when multiplied by a_{12345}, equal to

$$\begin{aligned}
a_{12345}a_{i_1 i_2 i_3 i_4 k} - a_{i_1 i_2 i_3 i_4 1}a_{2345k} + a_{i_1 i_2 i_3 i_4 2}a_{1345k}\\
- a_{i_1 i_2 i_3 i_4 3}a_{1245k} + a_{i_1 i_2 i_3 i_4 4}a_{1235k} - a_{i_1 i_2 i_3 i_4 5}a_{1234k}.
\end{aligned} \tag{1.55}$$

We see that in this expression are 10 indexes, which are divided into two groups: the indices i_1, i_2, i_3, i_4 will form the first group and indices $1,2,3,4,5,k$ the second group within each group all the cues are the same role. The term is antisymmetric with respect to the indices i_1, i_2, i_3, i_4 the first group and antisymmetric with respect to the indices of the second group. The resulting expression is invalid if the form is given monomial, supposing $a_{12345}0$, but it is even though $a_{12345} = 0$, unless the coefficients $a_{1234k}, a_{1235k}, a_{1234k}, a_{1345k}, a_{2345k}$, are all zero at the same time, because if one of them is not zero, the first example, since the indices $1,2,3,4,5,k$ play the same will be found rble the same expression starting from the hypothhse $a_{1234k} \neq 0$, if we write the conditions for the form is monomial. It is obvious, moreover, that if the 5 coefficients $a_{1234k}, a_{1235k}, a_{1234k}, a_{1345k}, a_{2345k}$ are zero, the expression is zero itself. This brings us to the following theorem:

Theorem. *For a form of degree 5 is monomial, it is necessary and sufficient as the $\binom{n}{4}\binom{n}{6}$ equations*

$$H_{i_1i_2i_3i_4,j_1j_2j_3j_4j_5j_6} \equiv a_{i_1i_2i_3i_4j_1}a_{j_2j_3j_4j_5j_6} - a_{i_1i_2i_3i_4j_2}a_{j_1j_3j_4j_5j_6} + a_{i_1i_2i_3i_4j_3}a_{j_1j_2j_4j_5j_6} - a_{i_1i_2i_3i_4j_4}a_{j_1j_2j_3j_5j_6} + a_{i_1i_2i_3i_4j_5}a_{j_1j_2j_3j_4j_6} - a_{i_1i_2i_3i_4j_6}a_{j_1j_2j_3j_4j_5}, \tag{1.56}$$

are all zero.

The condition is necessary: show this fact. It is sufficient, because if it is done, and if the coefficient qp, is not no one among the equations (1.56) are the relations,

$$H_{i_1i_2i_3i_4,12345k} = 0 \tag{1.57}$$

which we have seen, are necessary and sufficient, in hypothesis $a_{12345} \neq 0$ to the form is monomial.

We see that these conditions all result in quadratic relations between coefficients of the form given.

If the shape was cubic, one would consider a $\frac{n(n-1)}{2}\frac{n(n-1)(n-2)(n-3)}{24}$ relations

$$H_{ij,khlm} := a_{ijk}a_{hlm} - a_{ijh}a_{klm} + a_{ijl}a_{khm} - a_{ijm}a_{khl} = 0 \tag{1.58}$$

but there are discounts for if $k = i,\ j = h$, the expression $H_{ij,khlm}$ itself is zero. If $k = i$, the expression has only three terms instead of four:

$$H_{ij,ihlm} = -a_{ijh}a_{ilm} + a_{ijl}a_{ihm} - a_{ijm}a_{ihl}. \tag{1.59}$$

1.3 Exterior differential systems

11. Consider a system of equations obtained by canceling one or more external forms built with n variables $u^1, u^2, \cdots, u^n$. Each of these equations is to some degree, but they are not all necessarily the same degree. Let us look $u^1, u^2, \cdots, u^n$ such as the cartesian coordinates of a point in n dimensional space. We say that a variety plane through the origin of coordinates (we do not consider other) satisfies the given system, or is a solution of this system, if the equations of the system are all verified taking into account the equations of the flat variety, equations that are linear and homogeneous with respect to variables. For example, if the variete is flat $p-$dimensions (we say it is a $p-$plane), it is defined by $n-p$ linear independent relations between the coordinates, if one takes $n-p$ of coordinates, such as $n-p$ last, according to other $u^1, u^2, \cdots, u^p$, will be replaced in the first members of the given system of equations, $u^{p+1}, \cdots, u^n$ by values based on $u^1, u^2, \cdots, u^p$, and forms will be obtained in exterior forms $u^1, u^2, \cdots, u^p$ which should be identically zero. More generally we can express $u^1, u^2, \cdots, u^p$, be a $p-$linear forms as variables $t^1, t^2, \cdots, t^p$ and the result will be in exterior forms $t^1, t^2, \cdots, t^p$ to be identically zero. A first remark is to be done, and leads to

Theorem. *Any exterior equation of degree p is automatically verified by any variety of flat less than p dimensions.*

This results from an exterior form of degree $q < p$ variables is automatically void.

To express a $p-$plane solution is a system of equations exterior, it is therefore unnecessary to consider the system of equations which are of degree greater than p.

12. To investigate whether a $p-$plane is a solution of given system of exterior equations, one can start from the consideration of what is called the *Plückerian*[5] coordinates or better *Grassmannian*[6] coordinates of a $p-$plane (through the origin). Such a $p-$plane is defined completely if one takes p independent vectors from the origin are respectively $\xi_1^i, \xi_2^i, \cdots, \xi_p^i$ components of these p vectors. Let us form table with p rows and n columns

$$\begin{pmatrix} \xi_1^1 & \xi_1^2 & \cdots & \xi_1^n \\ \xi_2^1 & \xi_2^2 & \cdots & \xi_2^n \\ \vdots & \vdots & & \vdots \\ \xi_p^1 & \xi_p^2 & \cdots & \xi_p^n \end{pmatrix}, \tag{1.60}$$

and denote by $u^{i_1 i_2 \cdots i_p}$ the determinant formed by the p rows of the table and the columns of order $i_1, i_2, \cdots, i_p$. These quantities $u^{i_1 i_2 \cdots i_p}$ are antisymmetric with respect to their p indices. If we replace these p given vectors by other p independent vectors taken in the same $p-$plane, with components $\bar{\xi}_1^i, \bar{\xi}_2^i, \cdots, \bar{\xi}_p^i$, the components will be deducted $\bar{\xi}_1^i, \bar{\xi}_2^i, \cdots, \bar{\xi}_p^i$ descomposantes $\xi_1^i, \xi_2^i, \cdots, \xi_p^i$ (given i) by a linear substitution, the same regardless of i. We then see that new determinants $\bar{u}^{i_1 i_2 \cdots i_p}$ by multiplying the former with the same indexes determinants $u^{i_1 i_2 \cdots i_p}$ by the same factor, namely the determinant of the substitution which increased from $\xi_1^i, \xi_2^i, \cdots, \xi_p^i$ to $\bar{\xi}_1^i, \bar{\xi}_2^i, \cdots, \bar{\xi}_p^i$. The coordinates of the $p-$plane are well defined within a factor near zero: it is the *Plückerian coordinates* of $p-$plan and are still overdetermined and homogeneous if $p > 1$.

Conversely coordinates Pluckerian knowledge - of a $p-$plane completely determines the p-plane, since its equations are obtained by expressing the vector from the origin, components u^i, is a linear combination of vectors $\xi_1^i, \xi_2^i, \cdots, \xi_p^i$, the determinants of degree $p+1$ of tableau

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[5] Julius Plücker (1801–1868) was a German mathematician and physicist.

[6] Hermann Günther Grassmann (1809–1877) was a German polymath, renowned in his day as a linguist and now also admired as a mathematician.

$$\begin{matrix} u^1 & u^2 & \cdots & u^n \\ \xi_1^1 & \xi_1^2 & \cdots & \xi_1^n \\ \xi_2^1 & \xi_2^2 & \cdots & \xi_2^n \\ \vdots & \vdots & & \vdots \\ \xi_p^1 & \xi_p^2 & \cdots & \xi_p^n \end{matrix} \tag{1.61}$$

are all nule, yet each of these determinants is a linear combination of $u^1, u^2, \cdots, u^n$ one whose coefficients are all determinants $u^{i_1 i_2 \cdots i_p}$.

13. That said we propose first to recognize if a Pluckerian coordinates $u^{i_1 i_2 \cdots i_p}$ of $p-$plane nulls a given exterior form of degree p

$$F = \frac{1}{p!} a_{i_1 i_2 \cdots i_p} u^{i_1} \wedge u^{i_2} \wedge \cdots \wedge u^{i_p}. \tag{1.62}$$

To introduce this in the p-plane coordinates of $v^1, v^2, \cdots, v^p$, obtained by putting

$$u^i = v^k \xi_k^i, \qquad (i = 1, 2, \cdots, n), \tag{1.63}$$

the summation index k taking the values $1, 2, \cdots, p$. We will

$$u^{i_1} \wedge u^{i_2} \wedge \cdots \wedge u^{i_p} = \xi_{k_1}^{i_1} \xi_{k_2}^{i_2} \cdots \xi_{k_p}^{i_p} v^{k_1} \wedge v^{k_2} \wedge \cdots \wedge v^{k_p}; \tag{1.64}$$

in the second part there is to consider only non-zero monomial terms, that is to say where the indices $k_1, k_2, \cdots, k_p$, are separate: the monomial $v^{k_1} \wedge v^{k_2} \wedge \cdots \wedge v^{i_p}$ is not other than $v^1 \wedge v^2 \wedge \cdots \wedge v^p$ precedes the $+$ or $-$ sign according as the permutation of p indices $k_1, k_2, \cdots, k_p$, is even or odd, the coefficient of $v^1 \wedge v^2 \wedge \cdots \wedge v^p$ will be, as is easily seen, the determinant

$$\begin{vmatrix} \xi_1^{i_1} & \xi_1^{i_2} & \cdots & \xi_1^{i_p} \\ \xi_2^{i_1} & \xi_2^{i_2} & \cdots & \xi_2^{i_p} \\ \vdots & \vdots & & \vdots \\ \xi_p^{i_1} & \xi_p^{i_2} & \cdots & \xi_p^{i_p} \end{vmatrix} = u^{i_1 i_2 \cdots i_p}. \tag{1.65}$$

The result is

$$F = \frac{1}{p!} a_{i_1 i_2 \cdots i_p} u^{i_1 i_2 \cdots i_p} v^1 \wedge v^2 \wedge \cdots \wedge v^p. \tag{1.66}$$

The required condition for the $p-$plane is given exterior solution of the equation $F = 0$ is that F is zero when we replace the exterior product $u^{k_1} \wedge u^{k_2} \wedge \cdots \wedge u^{i_p}$ by the coordinate Plückerian $u^{i_1 i_2 \cdots i_p}$ of $p-$plane.[7]

[7] We see that it suffices to replace the monomial $u^{i_1} \wedge u^{i_2} \wedge \cdots \wedge u^{i_p}$ the $p-$linear alternating form associated or p sets of variables ξ_1^i, ξ_2^i, ..., and ξ_p^i.

Theorem. *For a $p-$plane with Pluckerian coordinates $u^{i_1 i_2 \cdots i_p}$ be a solution of the equation*

$$F \equiv \frac{1}{p!} a_{i_1 i_2 \cdots i_p} u^{k_1} \wedge u^{k_2} \wedge \cdots \wedge u^{i_p} = 0. \quad (1.67)$$

it is necessary and sufficient that

$$a_{i_1 i_2 \cdots i_p} u^{i_1 i_2 \cdots i_p} = 0. \quad (1.68)$$

14. Now looking the conditions for a $p-$plane given by its Plückerian coordinates annuls an exterior form Φ of degree $q < p$.

Start with some geometric remarks. If a $p-$plane annuls the form Φ, all $q-$plans contained in the $p-$plane ($q < p$) the form has also cannulsed because the form Φ is zero when one takes into account the equations of $p-$plan will be no more so when you take into account also the additional equations, with the first, define the $q-$plane. Conversely, assume that the form Φ is zero for all $q-$plans contained in the $p-$given plan. Taking into account the equations of $p-$plane, for example resolved over a $u^{p+1}, \cdots, u^n$ the form Φ becomes a form Ψ with variables $u^1, u^2, \cdots, u^p$. Say that it vanishes for all $q-$plans contained in the $p-$plane given, that is to say that it vanishes when p binds variables $u^1, u^2, \cdots, u^p$ by any $p-q$ independent linear relations. But if we take the linear relations

$$u^{q+1} = u^{q+2} = \cdots = u^p = 0, \quad (1.69)$$

will remain a single term in Ψ, that in $u^1 \wedge u^2 \wedge \cdots \wedge u^q$; the coefficient of the monomial $u^1 \wedge u^2 \wedge \cdots \wedge u^q$ in Ψ is zero, it will be the same for all other coefficients.

Theorem. *The necessary and sufficient condition for a $p-$plan vanish an exterior form Φ of degree $q < p$ is any plan contained in the given $p-$plane vanish this form.*

Analytically we can proceed in the following manner. If the form Φ vanishes for $p-$plan gives all forms of degree p

$$\Phi \wedge u^{\alpha_1} \wedge u^{\alpha_2} \wedge \cdots \wedge u^{\alpha_{p-q}} \qquad (\alpha_1, \alpha_2, \cdots, \alpha_{p-q} = 1, 2, \cdots, n) \quad (1.70)$$

annuls a fortiori for $p-$plane. These conditions are nessesary and sufficient; Indeed, suppose that, taking into account, for example, the $p-$plane equations solved with respect to $u^{p+1}, \cdots, u^n$, the form Φ becomes to a form Ψ. If each form $\Psi \wedge u^{\alpha_1} \wedge u^{\alpha_2} \wedge \cdots \wedge u^{\alpha_{p-1}}$ of degree p obtained by taking for $u^{\alpha_1}, u^{\alpha_2}, \cdots, u^{\alpha_{p-q}}$ any $p-q$ of the variables $u^1, u^2, \cdots, u^p$ is zero, is that each of the coefficients of the form p is zero and hence the $p-$form Φ annuls plan.

Theorem. *For a given $p-$plane annuls an exterior form of degree $q < p$*

$$\Phi = \frac{1}{q!} b_{i_1 i_2 \cdots i_q} u^{i_1} \wedge u^{i_2} \wedge \cdots \wedge u^{i_q}, \tag{1.71}$$

it is necessary and sufficient that the Plückerian coordinates of $p-$plan satisfies the $\binom{n}{p-q}$ linear equations

$$b_{i_1 i_2 \cdots i_q} u^{i_1 i_2 \cdots i_q \alpha_1 \alpha_2 \cdots \alpha_{p-q}} = 0 \qquad (\alpha_1, \alpha_2, \cdots, \alpha_{p-q} = 1, 2, \cdots, n). \tag{1.72}$$

For example, if a tri-plane be a solution of equations

$$a_i u^i = 0, \quad a_{ij} u^i \wedge u^j = 0, \quad a_{ijk} u^i \wedge u^j \wedge u^k = 0, \tag{1.73}$$

then, its Plückerian coordinates satisfy the $\frac{n^2+n+2}{2}$ equations

$$a_i u^{i\alpha\beta} = 0, \qquad (\alpha, \beta = 1, 2, \cdots, n), \tag{1.74}$$

$$a_{ij} u^{ij\alpha} = 0, \qquad (\alpha = 1, 2, \cdots, n), \tag{1.75}$$

$$a_{ijk} u^{ijk} = 0. \tag{1.76}$$

15. Conditions for the antisymmetric quantity $u^{i_1 i_2 \cdots i_p}$ are the Plückerian coordinates of $p-$plan. - If we give a priori a system of numbers ?? antisymmetric with respect to the p iodises $i_1, i_2, \cdots, i_p$ taken from the integers $1, 2, \cdots, n$, these numbers will usually contact Plückerian coordinates of a $p-$plane.

Theorem. *For a system of numbers $u^{i_1 i_2 \cdots i_p}$ antisymmetric with respect to p indices are the Plückerian coordinates of a $p-$plane it is necessary and sufficient that the exterior form of degree p*

$$F = \frac{1}{p!} u^{i_1 i_2 \cdots i_p} z_{i_1} \wedge z_{i_2} \wedge \cdots \wedge z_{i_p} \tag{1.77}$$

with n variables $z_1, z_2, \cdots, z_n$ is a monomial form.

Proof. Indeed if $u^{i_1 i_2 \cdots i_p}$ are the homogeneous coordinates of a $p-$plane determined by p independent vectors $\xi_1^i, \xi_2^i, \cdots, \xi_p^i$ was

$$F = f_1 \wedge f_2 \wedge \cdots \wedge f_p, \tag{1.78}$$

by asking

$$f_1 = \xi_1^i z_i, \quad f_2 = \xi_2^i z_i, \quad \cdots, \quad f_p = \xi_p^i z_i. \tag{1.79}$$

Conversely if F can be written in the previous form, $f_1, f_2, \cdots, f_p$ being arbitrarily given p linear forms, the Plückerian coordinates $u^{i_1 i_2 \cdots i_p}$ of a $p-$plane determined by the vectors $\xi_1^i, \xi_2^i, \cdots, \xi_p^i$. [8] □

We can now apply the conditions found at Paragraph No. 10. Especially for a system of quantities u^{ijk} antisymmetric with respect to the three indices i, j, k, are the coordinates of a 3−plane, it is necessary and sufficient that we have relations

$$H^{ij,klm} \equiv u^{ijk}u^{hlm} - u^{ijh}u^{klm} + u^{ijl}u^{khm} - u^{ijm}u^{khl} = 0$$
$$(i, j, k, h, l, m = 1, 2, \cdots, n). \tag{1.80}$$

We obtained in previous issues the linear equations to be satisfied by Plückerian coordinates of a $p-$plane for this $p-$plane is a solution of a given exterior system of equations, must, to be complete, add to these linear relations quadratic relations which express that these coordinates are those of a $p-$plane.

1.4 Algebraically equivalence of exterior systems of equations

16. *Ring of exterior forms.* We call ring determined by a number of h homogeneous exterior forms in n variables $F_1, F_2, \cdots, F_n$ all the exterior forms

$$\Phi = F_1 \wedge \phi^1 + F_2 \wedge \phi^2 + \cdots + F_h \wedge \phi^h, \tag{1.81}$$

where ϕ^is are homogeneous exterior forms subject has the sole condition that the terms $[F_i \phi^i]$ of the second member the sum of the degrees, positive or zero, the two factors is the same for all terms.

It is obvious that any form Φ, the ring determined by given h forms $F_1, F_2, \cdots, F_h$ vanishes for any solution of system

$$F_1 = 0, \quad F_2 = 0, \quad \cdots, \quad F_h = 0. \tag{1.82}$$

But the converse is not always true. We will give a rather general case in which the converse is true, and then we will indicate an example that uses the default reciprocal.

Theorem. *For an external form Φ vanishes for any solution of system of linear equations*

[8] The linear relations $u^{i_1 i_2 \cdots i_{p-k}} z_k$ give the necessary and sufficient conditions to be satisfied by so that the quantities z_k hyperplane $z_1 u^1 + z_2 u^2 + \cdots + z_k u^k = 0$ contains considered $p-$plane.

$$F_1 = 0, \quad F_2 = 0, \quad \cdots, \quad F_h = 0, \tag{1.83}$$

must be independent and sufficient that Φ belongs to the ring determined by the F_i forms. In addition the necessary and sufficient condition for that Φ belong to this ring is that the form $F_1 \wedge F_2 \wedge \cdots \wedge F_h \wedge \Phi$ is zero.

Proof. We have already seen that if Φ belongs to the ring in question, the form Φ vanishes for any solution of linear equatione (1.83). The condition is necessary; indeed reduce by a change of variables forms F_i with variables $u^1, u^2, \cdots, u^h$; to say that the form Φ, vanishes when one annuls the variables $u^1, u^2, \cdots, u^h$ is to say that every monomial of Φ contains at least one factor variables $u^1, u^2, \cdots, u^h$. Denote by ϕ^1 then the coefficient of u^1 in the set of terms of the variable containing u^1, once we have passed u^1 first in each of these terms; denote by ϕ^2 the form obtained in a similar manner compared to the variable u^2 starting from the form $\Phi - u^1 \wedge \phi^1$ and so on, we will obviously

$$\Phi = u^1 \wedge \phi^1 + u^2 \wedge \phi^2 + \cdots + u^h \wedge \phi^h. \tag{1.84}$$

The first part of the theorem is thus proved.

Clearly then the form

$$F_1 \wedge F_2 \wedge \cdots \wedge F_h \wedge \Phi \tag{1.85}$$

is identically zero, since by substituting expression (1.85), each term of the product will contain two identical factors of the first degree and as a result will be zero.

Conversely if the form $F_1 \wedge F_2 \wedge \cdots F_h \wedge \Phi$ is zero and that we make it again, by a suitable change of variables in order to have $F_i = u^i$ $(i = 1, 2, \cdots, h)$, any monomial which enters into the expression can be all of different factors of $u^1, u^2, \cdots, u^h$, and consequently will vanish with $u^1, u^2, \cdots, u^h$. □

18. Now consider the following example. Consider non-linear forms

$$F_1 = u^1 \wedge u^3, \quad F_2 = u^1 \wedge u^4, \quad F_3 = u^1 \wedge u^2 - u^3 \wedge u^4. \tag{1.86}$$

All 2−plane coordinates u^{ij} annulling these three forms, taking into account the quadratic relations

$$u^{13}u^{24} - u^{14}u^{23} - u^{12}u^{34} = 0, \tag{1.87}$$

will have its coordinated u^{12} zero, meaning that it will satisfy the equation

$$\Phi \equiv u^1 \wedge u^2 = 0. \tag{1.88}$$

It will be the same for all $p-$plane which cancels the same forms F_1, F_2, F_3, because all biplanes contained in this p-plane to cancel the form Φ is the form Φ to annihilate the $p-$plane.

The form Φ, which vanishes for any solution of equations $F_1 = F_2 = F_3 = 0$, not, however, of the ring determined by the forms F_1, F_2, F_3.

Definition. *We say that exterior system of equations is complete if every form which vanishes for any solution of this system belongs to the ring determined by the first members of system of equations.*

19. Definition. *Two systems of equations are called exterior algebraically equivalent if the first members of equations of any of these systems belong to the ring determined by the first members of equations of the other system.*

It is clear that two algebraically equivalent systems admit the same solutions. If a system is complete, any other system admitting the same solutions will be algebraically equivalent, the converse may not be true.

It is easy to form the most general system algebraically equivalent to a given system. Consider for example the system of degree three

$$\begin{cases} F_1 \equiv A_{1i}u^i = 0, \\ F_2 \equiv A_{2i}u^i = 0, \\ \Phi \equiv \frac{1}{2}B_{ij}\,u^i \wedge u^j = 0, \\ \Psi \equiv \frac{1}{6}C_{ijk}\,u^i \wedge u^j \wedge u^k = 0. \end{cases} \tag{1.89}$$

Suffice it to form the equations

$$\begin{cases} \bar{F}_1 \equiv aF_1 + bF_2 = 0, \\ \bar{F}_2 \equiv a'F_1 + b'F_2 = 0, \\ \bar{\Phi} \equiv c\Phi + \omega^1 \wedge F_1 + \omega^2 \wedge F_2 = 0, \\ \bar{\Psi} \equiv h\Psi + \omega^3 \wedge \Phi + \psi^1 \wedge F_1 + \psi^2 \wedge F_2 = 0, \end{cases} \tag{1.90}$$

where a, b, a', b', c, h are constants ($ab' - ba' \neq 0$, $c \neq 0$, $h \neq 0$) and $\omega^1, \omega^2, \omega^3$ are of arbitrary linear forms, ψ^1, ψ^2, exterior arbitrary quadratic forms.

1.5 Associated system with an exterior system of equations

20. We have already considered (Paragraph No. 8) that have so called the associated system with an exterior form. We now define the associated system associated to an exterior system of equations.

Definition *We say that a straight line Δ from the origin is a* **characteristic** *for a exterior system of equations Σ if be a solution of this system and moreover, given a flat variety V, solution of the system Σ, the smallest flat variety containing V and Δ is also a solution of Σ.*

We will seek the necessary and sufficient conditions for a line of parameters is characteristic for a given exterior system of equations.

Given an system Σ, we assume, for simplicity, of degree three

$$\begin{cases} F_\alpha \equiv A_{\alpha i}u^i = 0, & (\alpha = 1,2,\cdots,r_1), \\ \Phi_\alpha \equiv \frac{1}{2}B_{\alpha ij}\,u^i \wedge u^j = 0, & (\alpha = 1,2,\cdots,r_2), \\ \Psi_\alpha \equiv \frac{1}{6}C_{\alpha ijk}\,u^i \wedge u^j \wedge u^k = 0. & (\alpha = 1,2,\cdots,r_3). \end{cases} \tag{1.91}$$

For a line (ξ^i) is characteristic, it must satisfy three types of conditions (Paragraphs No. 13 and 14):

1) It must be itself a solution of Σ, that is to say, it annuls the forms F_α:

$$A_{\alpha i}\xi^i = 0 \qquad (\alpha = 1,2,\cdots,r_1); \tag{1.92}$$

2) It should be that given any straight line (u^i) which is a solution of Σ, the 2−plane determined by this line (u^i) and the line (ξ^i) annuls forms Φ_α, that is to say that we have

$$A_{\alpha ij}u^i\xi^j = 0 \qquad (\alpha = 1,2,\cdots,r_2); \tag{1.93}$$

whenever components u^i satisfy in r_1 relations

$$A_{\beta i}u^i = 0 \qquad (\beta = 1,2,\cdots,r_1); \tag{1.94}$$

3) For any 2−plane with coordinates u^{ij} which is a solution of Σ, the 3−plane determined by this 2−plane and line (ξ^i) must annuls the forms Ψ_α; thus, coordinates of u^{ijk} of this 3−plane must be

$$u^{ijk} = u^{ij}\xi^k - u^{ik}\xi^j + u^{jk}\xi^i; \tag{1.95}$$

So we must have

$$A_{\alpha ijk}u^{ij}\xi^k = 0 \qquad (\alpha = 1,2,\cdots,r_3); \tag{1.96}$$

whenever the coordinates given u^{ij} 2−plane satisfy the $nr_1 + r_2$ relations

$$\begin{cases} A_{\beta i}u^{i1} = A_{\beta i}u^{i2} = \cdots = A_{\beta i}u^{in} = 0 & (\beta = 1,2,\cdots,r_1); \\ A_{\gamma ij}u^{ij} = 0 & (\gamma = 1,2,\cdots,r_2). \end{cases} \tag{1.97}$$

These necessary conditions are sufficient. V is indeed a $p-$plane solution of the system Σ and W be the $(p+1)-$plane determined by V and the line ξ^i. For this $(p+1)-$plane satisfies the system Σ if and only if every triplane contained in W is a solution of Σ. But it is indeed the case if it belongs to the triplane V, otherwise it is determine the line (ξ^i) and V contained in a biplane, biplane this is therefore a solution of Σ, and the conditions for the triplane is a solution of Σ are specifically provided by the equations (1.92), (1.93) and (1.96) from which we take into account (1.94) and (1.97).

Conclusion. *For a the line (ξ^i) is characteristic for system Σ, it is necessary and sufficient to satisfy the equations (1.92), (1.93) and (1.96), the coefficients u^i which are in equations (20) being subject to the sole condition to annul the forms F_α, and u^{ij} coefficients which are in equations (1.96) being subject to the condition to satisfy*

1) quadratic equations that express that these coefficients are the coordinates of a $2-$plane ,
2) to equations (1.97) that express this $2-$plane is a solution of Σ.

They would see how easy search of characteristic lines would if the system was to give any degree.

21. *Associated system to an exterior system of equations.* Suppose to the conditions (1.97) to be satisfied by quantities u^{ij} which are in equations (1.96) not adding the quadratic equations that express the u^{ii} are the coordinates of a biplane. We will ξ^i for a system of conditions (1.92), (1.93) and (1.96), which may be less restrictive than those in extracts, which give the lines characteristic. We can demonstrate that the conditions obtained new exterior forms that express $u^1, u^2, \cdots, u^n$,

$$\xi^i \frac{\partial F_\alpha}{\partial u^i}, \qquad \xi^i \frac{\partial \Phi_\alpha}{\partial u^i}, \qquad \xi^i \frac{\partial \Psi_\alpha}{\partial u^i}, \tag{1.98}$$

which belong to the ring of forms F_α, Φ_α, Ψ_α, that is to say, the ring system Σ.

Definition. *We call associated system to an exterior system of equations Σ, the set of all linear equations $\xi^1, \xi^2, \cdots, \xi^n$, which any exterior form expression $\xi^i \frac{\partial H}{\partial u^i}$, where H is an arbitrary first member of Σ, belongs to the ring of given system.*

The system associated to an exterior form F, introduced in No. 8, can also be defined as the set of linear equations in $\xi^1, \xi^2, \cdots, \xi^n$, that express the considered form $\xi^i \frac{\partial F}{\partial u^i}$ in exterior form $u^1, u^2, \cdots, u^n$, is identically zero, that is to say up to the ring F.[9]

22. We easily see that both systems have the same system algebraically equivalent associates and also associates the system is intrinsically linked to Σ. We will prove the following theorem:

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[9] Examples of cases where the system is less restrictive than associated system that provides the characteristic lines, let us consider the equations of system Σ (cf. No. 18) $u^1 \wedge u^3 = u^1 \wedge u^4 = u^1 \wedge u^2 - u^3 \wedge u^4 = u^1 \wedge u^2 \wedge u^3 - u^3 \wedge u^4 \wedge u^6 = 0$. Its associated system is $\xi^1 = \xi^2 = \xi^3 = \xi^4 = \xi^5 = \xi^6 = 0$, while the its characteristic lines are provided by the equations $\xi^1 = \xi^2 = \xi^3 = \xi^4 = 0$.

Theorem. *If the system associates has rank p and, by a suitable change of variables, reduces to the equations* $\xi^1 = \xi^2 = \cdots = \xi^p = 0$*, there is a system algebraically equivalent to* Σ *in the equations which only includes the variables* $u^1, u^2, \cdots, u^p$*.*

Proof. Let us consider the the line of which all parameters ξ^i are zero except $\xi^{p+1} = 1$. Forms $\frac{\partial F_\alpha}{\partial u^{p+1}}$ which are constants, can not belong to the ring of Σ, if they are null: the F_α does not contain the variable u^{p+1}. Forms

$$\Phi_\alpha^* = \Phi_\alpha - u^{p+1} \wedge \frac{\partial \Phi_\alpha}{\partial u^{p+1}}, \qquad \Psi_\alpha^* = \Psi_\alpha - u^{p+1} \wedge \frac{\partial \Psi_\alpha}{\partial u^{p+1}}, \tag{1.99}$$

obviously does not contain the variable u^{p+1}. Now the form $\frac{\partial \Phi_\alpha}{\partial u^{p+1}}$ belonging to the ring of the form F_α and the form $\frac{\partial \Psi_\alpha}{\partial u^{p+1}}$ to the ring of F_α and Φ_α, the system

$$F_\alpha, \qquad \Phi_\alpha, \qquad \Psi_\alpha, \tag{1.100}$$

is algebraically equivalent to Σ: or equations of this system does not contain u^{p+1}. One could also get step by step system algebraically equivalent to Σ and containing only the variables $u^1, u^2, \cdots, u^p$. □

Remark. The integer p is obviously the minimum number of variables with which the equations can be written as a system algebraically equivalent to Σ, because if such a system not involving that $q < p$ variables, its associated system that is the same as that of Σ is at most rank q. We see more tan the minimum number p variables involved in a system algebraically equivalent to Σ are determined as a whole.

Particular case. If the system Σ is only of order two, we easily see that the associated system is obtained by adding to first-degree equations of Σ, the associated system of equations r_2 forms

$$F_1 \wedge F_2 \wedge \cdots \wedge F_{r_1} \wedge \Phi_\alpha \qquad (\alpha = 1, 2, \cdots, r_2), \tag{1.101}$$

equations which are obtained by setting the derivatives of order $r_1 + 1$ of the first members.

Chapter 2
Exterior Differential Systems

2.1 Definition. Exterior Differential

23. Consider an $n-$dimensional space, or at least a certain area $\mathscr{D}$ of this space. We assume that this is the Euclidean space, although this is by no means necessary, but this hypothesis will at least simplify the language. We denote by $x^1, x^2, \cdots, x^n$ the current coordinates. We will call exterior differential form of degree p an exterior form in which the variables will be the differential of $dx^1, dx^2, \cdots, dx^n$, the coefficients being functions of the coordinates $x^1, x^2, \cdots, x^n$ defined in the domain $\mathscr{D}$. A differential form of the first degree, or linear, will thus be written as

$$a_i(x)\,dx^i, \tag{2.1}$$

a quadratic differential form is written as

$$\frac{1}{2}a_{ij}(x)\,dx^i \wedge dx^j, \qquad (a_{ij} = -a_{ji}), \tag{2.2}$$

and so on.

By abuse of language that will be convenient, we will consider a function of $x^1, x^2, \cdots, x^n$, as a differential form of degree zero.

By a change of coordinates, which translates aur variables dx^i by linear substitution of determinant not zero, any form of external differential degrh p changes into another form of differential same degree. It must be assumed that the new coordinates naturally admit partial derivatives with respect to the former.

24. *Exterior differential of a differential form.* If the form is of degree zero, that is to say a function $f(x)$, its exterior differential is by definition its ordinary differential df.

The exterior differential of linear differential form

$$\omega = a_i\,dx^i \tag{2.3}$$

is by convention, the second degree form

$$d\omega = da_i \wedge dx^i; \tag{2.4}$$

same for exterior differential form of degree two

$$\omega = a_{ij}\, dx^i \wedge dx^j \tag{2.5}$$

we have

$$d\omega = da_{ij} \wedge dx^i \wedge dx^j, \tag{2.6}$$

and so on.

This definition assumes that the coefficients of the forms considered admit partial derivatives of first order, but we shall see that there are cases where we can define the exterior differential of a form, even if the coefficients are not derivable.

General Definition. *If ω is an exterior differential form of order p*

$$\omega = \frac{1}{p!} a_{i_1 i_2 \cdots i_p}\, dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_p}, \tag{2.7}$$

its exterior differential $d\omega$ is

$$\omega = \frac{1}{p!} da_{i_1 i_2 \cdots i_p} \wedge dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_p}. \tag{2.8}$$

25. The previous definition needs to be legitimized and we will prove the following theorem.

Theorem. *If a differential form $\omega(x, dx)$ is transformed by a change of coordinates to the form $\omega(y, dy)$, the exterior differential $d\omega$ of the form ω is transformed by the same change of coordinates, the exterior differential dm of the form ω.*

Prior to the demonstration, we will demonstrate some lemmas, however important by themselves.

Lemma I. *The exterior differential of the differential of a function $f(x)$ is zero.*

Proof. Indeed, let the exterior differential form $\omega = \frac{\partial f}{\partial x^i} dx^i$; Its exterior differential is by definition has the form

$$d\omega = d\left(\frac{\partial f}{\partial x^i}\right) \wedge dx^i = \frac{\partial^2 f}{\partial x^i \partial x^j} dx^j \wedge dx^i; \tag{2.9}$$

the second member is zero because of the symmetry of coefficients of $dx^i \wedge dx^j$ relative to their indices. □

Lemma II. *The exterior differential of the exterior product of two forms ω, ϖ respectively of degrees p and q is*

$$d(\omega \wedge \varpi) = d\omega \wedge \varpi + (-1)^p \omega \wedge d\varpi. \tag{2.10}$$

Proof. [1] Indeed, let

$$\omega = \frac{1}{p!} a_{i_1 i_2 \cdots i_p} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_p}, \qquad \varpi = \frac{1}{q!} b_{j_1 j_2 \cdots j_q} dx^{j_1} \wedge dx^{j_2} \wedge \cdots \wedge dx^{j_q}. \tag{2.11}$$

then, we have

$$\omega \wedge \varpi = \frac{1}{p!q!} a_{i_1 i_2 \cdots i_p} b_{j_1 j_2 \cdots j_q} dx^{i_1} \wedge \cdots \wedge dx^{i_p} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_q}, \tag{2.12}$$

and, then

$$d(\omega \wedge \varpi) = \frac{1}{p!q!} d\left(a_{i_1 i_2 \cdots i_p} b_{j_1 j_2 \cdots j_q}\right) \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_q}, \tag{2.13}$$

Now we have $d(ab) = b\,da + a\,db$, from which

$$\begin{aligned} d(\omega \wedge \varpi) =& \frac{1}{p!q!} b_{j_1 j_2 \cdots j_q} da_{i_1 i_2 \cdots i_p} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_q} \\ &+ \frac{1}{p!q!} a_{i_1 i_2 \cdots i_p} db_{j_1 j_2 \cdots j_q} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_q} \end{aligned} \tag{2.14}$$

The first sum of the second member is none other than the exterior product $d\omega \wedge \varpi$; as to the second sum, it is multiplied by $(-1)^p$ if one puts the factor $db_{j_1 j_2 \cdots j_q}$ after the pth factor dx^{i_p}, and it becomes equal to $\omega \wedge \varpi$; and, we deduce the formula (2.10). □

This lemma generalizes to the product of any number of factors. It has for example

$$d(\omega \wedge \varpi \wedge \chi) = d\omega \wedge \varpi \wedge \chi + (-1)^p \omega \wedge d\varpi \wedge \chi + (-1)^{p+q} \omega \wedge \varpi \wedge d\chi, \tag{2.15}$$

[1] In the particular case where one of the forms is of degree 0, that is to say a function a, we have $d(a\omega) = da \wedge \omega + a\,d\omega$, and $d(a\omega) = d\omega \wedge a + (-1)^p \omega \wedge da$.

assuming ω of degree p and ϖ of degree q.

26. Come now to the proof of the theorem. Consider a differential form of the second degree, and

$$\omega = a_{ij}\, dx^i \wedge dx^j \tag{2.16}$$

be one of the terms of this form. Express the variables x^i by new variables y^i. The term in question is an exterior product of three factors, the first a_{ij}, is a form of degree zero, the second and third are linear forms. We will therefore, passing to new variables y^i and calling the resulting form as ϖ, we have

$$d\varpi \wedge = da_{ij} \wedge dx^i \wedge dx^j + a_{ij} \wedge (ddx^i) \wedge dx^j - a_{ij} \wedge dx^i \wedge (ddx^j); \tag{2.17}$$

but by Lemma I, the exterior differential of dx^i and dx^j are zero and consequently the form $d\varpi$ is obtained by substituting in the expression $da_{ij} \wedge dx^i \wedge dx^j$, whose coordinates x^i are functions of y^i. □

27. One could, in regard to linear differential forms $\omega = a_i\, dx^i$ attach the exterior differentiation to the notion of covariant bilinear. Introduce a second symbol of differentiation δ, we consider the symbols $d\sigma x^i$, as preveusly we can look, and consider the symbols δdx^i, to constitute two new sets of variables; but we can agree that the variables δdx^i identical to variables $d\delta x^i$: this convention is legitimate in that it is respected by a change in any variable, as expressing the new variables y^i based on the old ones x^i, was

$$dy^i = \frac{\partial y^i}{\partial x^k}\, dx^i, \qquad \delta dy^i = \frac{\partial^2 y^i}{\partial x^h \partial x^k}\, \delta x^h\, dx^k + \frac{\partial y^i}{\partial x^k}\, \delta dx^k, \tag{2.18}$$

and

$$\delta y^i = \frac{\partial y^i}{\partial x^k}\, \delta x^i, \qquad d\delta y^i = \frac{\partial^2 y^i}{\partial x^h \partial x^k}\, dx^h\, \delta dx^k + \frac{\partial y^i}{\partial x^k}\, d\delta x^k; \tag{2.19}$$

comparison of δdy^i and $d\delta y^i$ shows immediately their equality, taking into account the properties of symmetry of second derivatives with respect to $\frac{\partial^2 y^i}{\partial x^k \partial x^h}$ two indices derivation.

Given this, denote by $\omega(d)$ and $\omega(\delta)$ the differential form given, depending on whether one uses the symbol of differentiation d or δ, and take the expression

$$\begin{aligned} d(\omega(\delta)) - \omega(d(\delta)) &= d(a_i \delta x^i) - \delta(a_i dx^i) \\ &= da_i\, \delta x^i - \delta a_i\, dx^i + a_i(d\delta x^i - \delta dx^i). \end{aligned} \tag{2.20}$$

We obtain an alternating bilinear form with two sets of variables dx^i and δx^i. In this alternating bilinear form, which can be written

$$\begin{vmatrix} da_i & dx^i \\ \delta a_i & \delta x^i \end{vmatrix}, \tag{2.21}$$

is associated with the exterior form $da_i \wedge dx^i$, which is nothing other than what we called the exterior differential of the form $a_i dx^i$. The intrinsic character (covariant) clear expression $d\omega(\delta) - \delta\omega(d)$ implies immediately the intrinsic character of the form $da_i \wedge dx^i$.

One could also associate with the quadratic exterior differential form ω, exterior alternating 3−linear form

$$d_1\omega(d_2,d_3) - d_2\omega(d_1,d_3) + d_3\omega(d_1,d_2), \tag{2.22}$$

where three symbols of differentiation changed between them.

28, Theorem of Poincare. *The exterior differential of the exterior differential of a differential form is zero.*

Proof. The proof is simple. Just check the theorem on monomial form $\omega = a\,dx^1 \wedge dx^2 \wedge \cdots \wedge dx^p$; In this case, we have $d\omega = da \wedge dx^1 \wedge dx^2 \wedge \cdots \wedge dx^p$ since each factor as the second member is an exact differential, the formula for the exterior differential of a product makes the theorem obvious. □

Take, for example, as verification, the form

$$\omega = P\,dx + Q\,dy + R\,dz; \tag{2.23}$$

then, we have

$$\begin{aligned} d\omega &= dP \wedge dx + dQ \wedge dy + dR \wedge dz \\ &= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy \wedge dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dz \wedge dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy. \end{aligned} \tag{2.24}$$

With exterior differentiating second times, we obtain the cubic form $d(d\omega)$

$$\begin{aligned} & d\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) \wedge dy \wedge dz + d\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) \wedge dz \wedge dx + d\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \wedge dx \wedge dy \\ &= \left\{ \frac{\partial}{\partial x}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \right\} dx \wedge dy \wedge dz \\ &= 0. \end{aligned} \tag{2.25}$$

The Theorem of Poincare admits an inverse which we do not have the rest to serve us, and reads the statement.

Theorem. *Given a differential form of degree p whose exterior differential is zero, defined within a domain $\mathscr{D}$ homeomorphic to the interior of a hypersphere, there is a differential form of degree $p-1$ defined in this field, and is given the shape of which the exterior differential.*

Suffice it to verify the theorem for $p = 1$. The form (2.23) for example has its exterior differential zero if we have, from (2.24),

$$\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} = 0, \qquad \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} = 0, \qquad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0; \tag{2.26}$$

or are the necessary and sufficient conditions for the form $P\,dx + Q\,dy + R\,dz$ is an exact differential, that is to say the exterior differential of a form of degree zero or a function $f(x,y,z)$.

2.2 The exterior differentiation and generalized Stokes formula

29. The classical formulas of Cauchy-Green, Stokes and Ostrogradsky show a remarkable link between the operation of exterior differentiation and integral calculus operation which consists in passing an integral calculated on boundary of one domain of $(p+1)$-dimensional space equal to an integral calculated on this domain. For example, consider the formula Cauchy-Green

$$\oint_C P\,dx + Q\,dy = \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy, \tag{2.27}$$

in which the first member is a line integral calculated on a closed contour C of the plan and the second member is an integral double-calculated on the area A bounded by the contour. As the differential of exterior form $\omega = P\,dx + Q\,dy$ is seen that the Cauchy formula can be written as

$$d\omega = dP \wedge dx + dQ \wedge dy = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy, \tag{2.28}$$

we see then, that the Cauchy formula can be written as

$$\oint_C P\,\omega = \iint_A d\omega. \tag{2.29}$$

Note however that for this formula has a sense, we must orient line C and the area A in a consistent manner, the sign of each the integrals being defined only by specifying the direct on line and on the area which are applied. In fact here the rule is as follows: It first directs the area A willingness to agreeing that the area of the parallelogram constructed on two vectors $\mathbf{e}_1, \mathbf{e}_2$, arranged in the order $\mathbf{e}_1, \mathbf{e}_2$ is positive; the direction for the line C is as follows: there is a vector $\mathbf{e}_1$ at each point of C leads outside the area and it then leads the vector $\mathbf{e}_2$, tangential to the contour C so that the $(\mathbf{e}_1, \mathbf{e}_2)$ is positively oriented system.

The Stokes formula starts also in the form (2.29) for taking the form ω (2.23) and $d\omega$ for the form (2.24), the first integral is extended to a contour C bounding a surface portion A, the second integral is extended to the surface portion; there also to the choice of consistent guidelines for the contour and the area, which is, by changing those things which need to be changed, one that was suitable for the Cauchy formula.

Finally Ostrogradsky formula has the same form (2.29) by setting

$$\begin{aligned} \omega &= P\,dy\wedge dz + Q\,dz\wedge dx + R\,dx\wedge dy, \\ d\omega &= dP\wedge dy\wedge dz + dQ\wedge dz\wedge dx + dR\wedge dx\wedge dy, \\ &= \Big(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\Big) dx\wedge dy\wedge dz; \end{aligned} \tag{2.30}$$

the integral $\int\omega$ is extended to a closed surface S limiting volume V and the integral $\int d\omega$ is calculated on this volume. The guiding question is resolved as follows: It directs the volume by building a trihedron $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ we should call *the right handed trihedral*, and the volume of the parallelepiped oriented built on the three vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ will be measured positively. To orient the surface S, will be conducted by a point M on this surface $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ have three vectors form a right handed trihedral, the first being outside the volume V, the two other tangent to the surface S, then it must be admitted that the area of a parallelogram oriented constructed on $\mathbf{e}_2$ and $\mathbf{e}_3$, is as positive, which orients the surface.

It shows that, with similar orientation conventions, we have a most general form of Stokes formula

$$\int\omega = \int d\omega; \tag{2.31}$$

the first integral is calculated on the boundary $\partial\Omega$ of a $(p+1)-$dimensional domain Ω, and second integral calculated on that domain Ω, in other words, this domain covers the domain of the second integral.

30. If the exterior of a differential form ω of degree p is zero (we will say with de Rham that this form is closed), the integral $\int\omega$ calculated on the boundary of any $(p+1)-$dimensional domain is zero. When the conditions that we have implicitly allowed to form the exterior differential are not realized (differentiable coefficients), it may happen as the integral $\int\omega$ calculated on the boundary of any $(p+1)-$dimensional domain is zero. More generally, given a form ω of degree p, if there is a form $d\omega$ of degree $p+1$, such that the generalized Stokes formula (2.29) is valid for any $(p+1)-$dimensional domain A, we can say that $d\omega$ is the exterior differential of the given form ω, even if the coefficients of ω are simply continuous functions.

31. One can easily demonstrate the Poincare theorem in a case where the coefficients of the exterior differential $d\omega$ of a form are not differentiable: it suffices to assume the existence of this differential. Indeed suppose a form ω of degree p; the generalized Stokes formula tells us that the integral $\int\omega$ calculated on any $(p+1)-$dimensional domain is equal to the integral $\int d\omega$ calculated on the boundary of this area. Thus, let Σ be a closed $(p+1)-$dimensions variety; divide it in two parts Σ_1 and Σ_2 by a closed $p-$dimensional variety boundary C; the integral $\int d\omega$ calculated on the Σ_1

and the integral $\int d\omega$ calculated on the Σ_2 are equals, and equal to the integral $\int d\omega$ calculated on C, but the first one with a certain orientation of C, and the second one with the opposite orientation. The total integral $\int \omega$ calculated on Σ is zero: the form $d\omega$ is closed, which expresses the same theorem of Poincare.

32. It can be shown as the converse of the Theorem of Poincare stated in No. 22. We will limit ourselves to the case $p = 2$, which will be enough to understand how the demonstration can be done in the general case.[2]

Chapter 3
Exterior differential systems, Characteristic system

3.1 General facts. Completely integrable systems

35. Differential systems we propose to study are obtained by annulling a number of functions of n variables $x^1, x^2, \cdots, x^n$, which we always consider the coordinates of a point in $n-$dimensional space or an domain $\mathscr{D}$ of this space, and a number of exterior differential forms defined in this area and may have any degrees. We will be forced, in the general case, to assume that essentially functions that are introduced are analytic, and we also assume in the real domain (See for details in relation the analytical functions of real variables, G. Valiron [26]). The problems which we will ask knots and theorems which we will always have to demonstrate to local character. Clearly tied will reserve the right to make changes of coordinates, but in problems where analytical data is assumed, the new variables will necessarily be analytic functions of the former.

36. The differential systems of linear equations have been studied extensively. Let

$$\theta_\alpha \equiv A_{\alpha i}\, dx^i = 0, \qquad (\alpha = 1, 2, \cdots, r) \tag{3.1}$$

are the equations of such a system, we assume that the linear forms θ_α are linearly independent, whenever the variables x^i in the coefficients $A_{\alpha i}$ has generic values. We say in general that the point (x^i) of space is *generic* if the rank of the tableau of coefficients $A_{\alpha i}$ at this point is equal to r.

We call *integral manifold* of system (3.1) a variety defined by a number of relationships between variables and such that the relations between the variables and relations that deduced by differentiation forms θ_α vanish identically. This naturally assumes that the first members of equations of the manifold are differentiable.

In particular, consider the integral manifolds with dimension $n-r$. If there seek an integral manifold of this species passing through a point M_0 given generic coordinate (x^i). Suppose that at this point the determinant of degree r built with r rows and r last columns of tableau $A_{\alpha i}$ is different from zero. Near this point the equations (3.1) can be solved with respect to differential coordinates r, which we call, for convenience, $z^1, z^2, \cdots, z^r$. We see that the integral manifold, if it exists, may be defined by giving the functions $z^1, z^2, \cdots, z^r$ of $x^1, x^2, \cdots, x^{n-r}$, in a suitable manner.

37. Theorem I. *If there is an integral manifold passing through a given generic point , it can be obtained by integrating a system of ordinary differential equations and integral this variety is unique.*

Proof. Let $x^1, \cdots, x^h$ $(h = n-r)$ be the coordinates of a point of a $h-$dimensional Euclidean space and denote by O, a point of this space with coordinate $x_0^1, x_0^2, \cdots, x_0^h$.

We put inside of the hypersphere Σ with center O and radius R of this space and construct some different rays of the hypersphere from point O, each being determined by the parameters $a^1, a^2, \cdots, a^h$ the unit vector carries with him. Integral required for any variety, the z^α $(\alpha = 1, 2, \cdots, r)$ are functions of the coordinates of the stream interior of Σ, these coordinates can be written as

$$x^1 = a^1 t, \quad x^2 = a^2 t, \quad \cdots, x^h = a^h t, \quad (0 \leq t \leq R). \tag{3.2}$$

By moving along a radius, the unknown functions satisfy the equations $z^a lpha$ obtained by replacing, in the forms θ_α, x^i by $a^i t$ and dx^i by $a_i dt$. This will provide a system of ordinary differential equations

$$\frac{dz^\alpha}{dt} = \phi^\alpha(a^1, a^2, \cdots, a^h, t), \qquad (\alpha = 1, 2, \cdots, r), \tag{3.3}$$

that will integrate with the initial conditions

$$z^\alpha = (x^{n-r+\alpha})_0 \quad \text{for} \quad x^i = (x^i)_0. \tag{3.4}$$

For each ray we are sure that the integration can be done for a certain interval $(0, t_0)$, t_0 is a continuous function of $a^1, a^2, \cdots, a^h$; t_0 admit therefore a lower bound will be reached: this is the lower bound that we take the value of R.

Thus we see that if there is an integral manifold for $(n-r)-$dimensional passing through the M_0, it is given, within the hypersphere Σ, by integrating a system of differential equations ordinary and it is unique. □

36. Definition. *Completely integrable differential systems. - The system (3.1) is called* completely integrable *if it goes an $(n-r)$dimensional integral manifold, through each generic point of space and a sufficiently small neighbourhood of this point.*

We, in the preceding number, saw how one could find this integral manifold in case it exists. To find the conditions for complete integrability of system (3.1) make the following remark, which will play a fundamental role in the general theory of differential systems, which could not be easily shown that *any variety that annuals* *a differential form exterior, at the same time annuls the resulting form by exterior differentiation.* Any integral manifold of system (3.1) must annihilate the r forms θ_α. But if we calculate these forms we can, instead of expressing them as quadratic forms of $dx^1, dx^2, \cdots, dx^h, dz^1, dz^2, \cdots, dz^r$, be expressed as quadratic forms of the following r independent forms on a neighbourhood of M_0,

$$dx^1, dx^2, \cdots, dx^h, \theta_1, \theta_2, \cdots, \theta_r. \tag{3.5}$$

Therefore, we have formula

$$d\theta_\alpha = \frac{1}{2} C_{\alpha ij} dx^i \wedge dx^j + D^\lambda_{\alpha i} dx^i \wedge \theta_\lambda + \frac{1}{2} E^{\lambda\mu}_\alpha \theta_\lambda \wedge \theta_\mu, \tag{3.6}$$

where in summation indices i and j vary from 1 to h and the indices of summation λ and μ of 1 to r. Any integral manifold through M_0, which annihilate the forms θ_α, must annihilate the forms $\frac{1}{2} C_{\alpha ij} dx^i \wedge dx^j$ and hence, for any point of the manifold in the neighbourhood of M_0, the coefficients $C_{\alpha ij}$ will be zero. As will be void as whatever the initial values of the functions z^α of $x^i = x^i_0$, we can conclude the functions $C_{\alpha ij}$ must be zero in a sufficiently small neighbourhood of M_0; hence, we have

Theorem. *For the system (3.1) is completely integrable, it is necessary that near any generic point of space, forms $d\theta_\alpha$, exterior differential forms θ_α, belong to the ideal forms of θ_α. We can express this condition by writing the congruency*

$$d\theta_\alpha \equiv 0 \mod (\theta_1, \theta_2, \cdots, \theta_r). \tag{3.7}$$

This can also be expressed in a more accurate, as we have already seen, by the existence of linear forms ϖ^i_α regular near the point considered generic space and such that we have

$$d\theta_\alpha = \theta_1 \wedge \varpi^1_\alpha + \theta_2 \wedge \varpi^2_\alpha + \cdots + \theta_r \wedge \varpi^r_\alpha. \tag{3.8}$$

39. Now prove the converse. Returning to the $(n-r)-$dimensional manifold defined by the differential system (3.3) and passing through the point M_0, of coordinates $x^1 - 0, z^\alpha_0$. If we replace the functions $z^a lpha$ by their values based on the arguments a^i, t, we shall have, after the way in which it was obtained

$$\theta_\alpha = P_{\alpha k}(a,t)\, da^k \qquad (\alpha = 1, 2, \cdots, r); \tag{3.9}$$

and also

$$\varpi^i_\alpha = Q^i_\alpha(a,t)\, dt + Q^i_{\alpha k}(a,t)\, da^k \qquad (\alpha = 1, 2, \cdots, r). \tag{3.10}$$

Relations (4) give us, in leaving to the two members that the terms containing dt,

$$\frac{\partial P_{\alpha k}}{\partial t} dt \wedge da^k = P_{\lambda k} Q_\alpha^k da^k \wedge dt, \tag{3.11}$$

or

$$\frac{\partial P_{\alpha k}}{\partial t} + Q_\alpha^k P_{\lambda k} = 0. \tag{3.12}$$

For each value of index i, the r functions $P_{\alpha i}(a,t)$, considered as functions of t, satisfy a system of linear and homogeneous equations (which is the system remains the same for all values of i); for $t = 0$ functions $P_{\alpha i}$ are all zero since, for $t = 0$, the functions x^i and z^α are fixed and independent of a^k, their differentials do not contain, when we make $t = 0$ in the coefficients, the term in $da^1, da^2, \cdots, da^h$.

The initial values of unknown functions $P_{\alpha i}$ system (3.6) being zero, these functions are identically zero and therefore the variety determined by integrating the equations (3.3) vanishes identically forms θ_α, so it is an integral manifold.

We will express this result as follows:

Theorem. *For the system (3.1) is completely integrable, it is necessary and sufficient that near any generic point of space, forms $d\theta_\alpha$, exterior differential forms θ_α, belong to the ideal generated by these forms θ_α.*

41 **Note I.** The condition of complete integrability only requires the existence of an integral manifold for $(n-r)-$dimensions through a generic point of any space, and it was only in the neighbourhood of a generic point that the forms $d\theta_\alpha$, must belong to the ring of forms θ_α. It may be not the same in the neighbourhood of a point non-generic. Thus equation $\theta \equiv x dy - y dx = 0$ is completely integrable, as any ordinary differential equation. But we can not confirm the existence of a form linear $\varpi = A dx + B dy$, regular and continuous in the vicinity of any point gives the plan and such that $d\theta = \theta \wedge \varpi$, or

$$2 dx \wedge dy = -(Ax + By) dx \wedge dy; \tag{3.13}$$

in effect for $x = y = 0$, we can not have $2 = -(Ax + By)$.

Note II. You can put the completely integrability condition in a general form which does not require specially consider the case of each generic point. Indeed equations (3.8) result relations

$$\theta_1 \wedge \theta_2 \wedge \cdots \wedge \theta_r \wedge d\theta_\alpha = 0 \qquad (\alpha = 1, 2, \cdots, r); \tag{3.14}$$

vice versa, from these relations it follows that, on a generic neighbourhood of each point, for which the linear forms $\theta_1, \theta_2, \cdots, \theta_\alpha$ are independent, the exist forms $\varpi_b^\alpha eta$ satisfying the relationships (3.8). As any point. non-generic can be regarded as the limit of a sequence of generic points, the relations (3.14), being true for any generic point, by continuity will be true also for any point other than generic.

Relations (3.14) therefore give the necessary and sufficient conditions of complete integrability of system (3.1), and in a form generally more convenient than those initially indicated.

Note III. If the system is completely integrable, there are r independent functions $\phi_i(x^1, x^2, \cdots, x^n)$ defined on a neighbourhood of a generic point of space and which remain constant on any integral manifold (*first integrals* of the system), so that the given system is equivalent to the system $d\phi_1 == d\phi_2 = \cdots = d\phi_r = 0$. The converse is obvious.

Note IV. Everything that has been said does not imply the analyticity of the coefficients $A_{\alpha i}$ given equations, but only the existence of these coefficients for continuous partial derivatives of first order;[1] the reason is that the search of integral manifolds is reduced to the integration of ordinary differential equations.

3.2 Closed differential systems. Characteristic system

42. Let us take a differential system obtained by annulling a number of exterior differential forms, some of which may be of degree zero, i.e. some functions of the variables. A solution of such a system may be regarded as representing analytically a variety of $n-$dimensional space, which we call *integral manifold*; it will be defined by a number of relationships between variables so that these relations, together with the linear relationships between the differential $dx^1, dx^2, \cdots, dx^r$ we deduce by differentiation, vanish identically differential forms which are the first members of the given system of equations. It is clear that any solution of the integral manifold is given system, which would be added all of the equations obtained by differentiating of the equations exterior system, because if a differential form is zero, its exterior differential is zero by itself. The new differential system obtained can obviously be extended by the same process, after the theorem of Poincaré.

42. Definition. *A differential system is said to be* closed with respect to the operation of the exterior differentiation*, or, more simply,* closed*, if the exterior differential equations of the first members of system belong to the ideal determined by these first members.*

Clearly, if we extend a system by adding the equations obtained by exterior differentiation, we obtain a closed system, since the exterior differential of the first member of an equation of the new system is zero or is a first member of the system. This derived system from the given system is called *closure* of that system.

[1] The existence of these partial derivatives is necessary to ensure the existence of forms θ_α.

We see easily that if two differential systems are algebraically equivalent systems that come from their closure are algebraically equivalent. Based on considerations of No. 41, we see that the system derived from the closure of a given system admits the same solutions that this system, whatever the size of the integral manifolds considered . As a result:

Principle. *The search for solutions of a differential system can always be reduced to finding solutions of a closed differential system.*

43. *Characteristic system of a differential system.* We saw in Chapter 1 that a system of equations can always be expressed outside, making necessary a change of variables and replacing as necessary the system given by an algebraically equivalent system, using a minimum number of variables; this number is well defined and is given by the rank of the associated system; on the variables, are linear combinations of original variables, which equalled to zero, provided the associated system.

When there is an exterior system of differential equations, one can equally wonder if we can not make a change of variables and replace the system by an algebraically equivalent system so that the new system does involved in both its coefficients in the differential therein, a minimum number of variables. We will see that this is possible and the solution of the problem is provided by the consideration of the system characteristic.

42. Definition. Characteristic system *of a given differential system is the differential system associated with the closure of system.*

We will demonstrate the following theorem:

44. Theorem. *The characteristic system of a system Σ is completely integrable differential. If in addition $y^1, y^2, \cdots, y^p$ be a system of independent first integrals, then there is a system algebraically equivalent to Σ, constructed by the differentials $dy^1, dy^2, \cdots, dy^p$, where the coefficients being functions $y^1, y^2, \cdots, y^p$.*

Proof. We assume for simplicity, which essentially does not restrict the generality, that the system Σ contains no ordinary function of $x^1, x^2, \cdots, x^n$. It is enough to demonstrate the theorem in case the system Σ is closed. System is therefore defined by the following equations, we assume the degree ≤ 3, such as

$$\begin{cases} \theta_\alpha \equiv A_{\alpha i}\, dx^i = 0, & (\alpha = 1,2,\cdots,r_1), \\ \phi_\alpha \equiv \frac{1}{2} A_{\alpha ij}\, dx^i \wedge dx^j = 0, & (\alpha = 1,2,\cdots,r_2), \\ \psi_\alpha \equiv \frac{1}{6} A_{\alpha ijk}\, dx^i \wedge dx^j \wedge dx^k = 0, & (\alpha = 1,2,\cdots,r_3). \end{cases} \tag{3.15}$$

If the rank of the system characteristic is equal to n, the theorem becomes trivial. If this rank is an integer $p < n$, add to the system characteristic equation, if $p < n-1$, other linear equations number $n-1-p$ are mutually independent and independent of the first. We obtain a system of ordinary differential equations which we can assume, with a change of coordinates, if necessary, the variables $dx^1, dx^2, \cdots, dx^{n-1}$ can be integrated first.

First we know that we can find a algebraically equivalent system to Σ and do not include the differential dx^n (No. 22).

Suppose this result already reached, the first members of equations (3.15) not involving dx^n. We notice then that the derivative with respect to x^n any of these early members, ϕ_α for example, is simply the derivative of the form $d\phi^n$, compared to dx^n, and hence belongs to the ring system. We thus have the congruencies

$$\begin{cases} \frac{\partial}{\partial x^n}\theta_\alpha \equiv 0, & \mod (\theta_1, \theta_2, \cdots, \theta_{r_1}), \\ \frac{\partial}{\partial x^n}\phi_\alpha \equiv 0, & \mod (\phi_1, \cdots, \phi_{r_2}, \theta_1, \cdots, \theta_{r_1}), \\ \frac{\partial}{\partial x^n}\psi_\alpha \equiv 0, & \mod (\psi_1, \cdots, \psi_{r_3}, \phi_1, \cdots, \phi_{r_2}, \theta_1, \cdots, \theta_{r_1}). \end{cases} \tag{3.16}$$

The first congruency (3.16) can be written as

$$\frac{\partial}{\partial x^n}\theta_\alpha = H_\alpha^\beta \theta_\beta. \tag{3.17}$$

Consider then the system of ordinary differential equations

$$\frac{\partial z^\alpha}{\partial x^n} = H_\alpha^\beta z_\beta, \tag{3.18}$$

the coefficients H_α^β are functions $dx^1, dx^2, \cdots, dx^n$. Let $\bar{z}_\alpha^{(1)}, \bar{z}_\alpha^{(2)}, \cdots, \bar{z}_\alpha^{(r_1)}$ be r_1 independent solutions of this system. There are independent linear forms of x^n, for example $\bar{\theta}_1, \bar{\theta}_2, \cdots, \bar{\theta}_{r_1}$, such that we have

$$\theta_\alpha = \bar{\theta}_1 \bar{z}_\alpha^{(1)} + \bar{\theta}_2 \bar{z}_\alpha^{(2)} + \cdots, \bar{\theta}_{r_1} \bar{z}_\alpha^{(r_1)}; \tag{3.19}$$

but the system of equations $\theta = 0$ is equivalent to the system of equations $\bar{\theta} = 0$ whose first members do not involved x^n or dx^n. As a result, we can assume, replacing a system by Σ algebraically equivalent, that forms θ_α. do not contain x^n nor dx^n.

Pass to the form ϕ_α. We have

$$\frac{\partial \phi_\alpha}{\partial x^n} = K_\alpha^\beta \phi_\beta + \varpi_\alpha^\gamma \wedge \theta_\gamma, \tag{3.20}$$

where ϖ_α^γ is the linear forms not containing dx^n and θ_γ, not dependent to x^n or dx^n.

Consider this time the system of differential equations

$$\frac{\partial u_\alpha}{\partial x^n} = K_\alpha^\beta u_\beta, \qquad (\alpha = 1, 2, \cdots, r_2) \tag{3.21}$$

and a system of r_2 independent solutions $\tilde{u}_\alpha^{(i)}$ $(i = 1, 2, \cdots, r_2)$. Let

$$\phi_\alpha = \phi_\beta^* \bar{u}_\alpha^{(\beta)}, \tag{3.22}$$

where ϕ_β^* is new exterior quadratic forms. The system (3.20) take the form

$$\frac{\partial}{\partial x^n} \phi_\alpha = (\varpi_\beta)^* \wedge \theta_\beta. \tag{3.23}$$

If we denote by χ_α^β a primitive function of $(\varpi_\beta)^*$, considered as a function of x^n, see that the form knotted

$$\phi_\alpha^* - \chi_\alpha^\beta \wedge \theta_\alpha \tag{3.24}$$

not depend on x^n or dx^n. But the system $\theta_\alpha = \phi_a lpha^*$ is algebraically equivalent to the system $\theta_\alpha = \phi_a lpha = 0$. We can therefore suppose, substituting Σ with an algebraically equivalent system, that the first members of equations of the first and second degree system (3.15) does not depend on x^n or dx^n.

We will continue in the same manner for the first members of equations of the third degree and step by step we will thus show the existence of a system algebraically equivalent to the given system and whose equations do not involve x^n or dx^n.

If $p < n-1$, should be repeated on this system on the same reasoning as Σ has wide algebraically equivalent system in which only appear as $n-2$ variables and so on, until we come to a system in which p variables and their differentials does not appear.

The theorem is then shown. For if the p variables are $y^1, y^2, \cdots, y^p$, then the characteristic system is

$$y^1 = 0,\ y^2 = 0\ , \cdots,\ y^p = 0; \tag{3.25}$$

it is completely integrable and its most general integral manifold is obtained by matching $y^1, y^2, \cdots, y^p$ to arbitrary constants.

It is also obvious that you can not find system algebraically equivalent to the given system and involving less than p variables and their differentials.[2] □

We will call *class* of a given exterior system of differential, the rank of the characteristic system.

45. Definition. Characteristic variety *is called any* $(n-p)-$*dimensional solution manifold of the characteristic system.*

The following property is obvious.

Theorem. *Given an integral manifold* V *of any* Σ *system, the manifold obtained by drawing through each point of the characteristic variety* V *which passes through this point is also an integral manifold.*

It follows in particular[3]

Theorem. *If the integral manifold* V *system* Σ *is contained in any integral manifold to a larger number of dimensions, it is generated by the characteristic varieties.*

Indeed if it were not so, the characteristic manifolds conducted by the different points of an integral manifold V would generate a larger number of dimensions as V.

46. If system Σ is not complete (No. 18) and we can complete it, it will feature the system to a new characteristic system of Σ whose rank can only be reduced; if the rank remains the same, the system characteristic is not changed.

Take as an example the system

$$dx^1 \wedge dx^3 = dx^1 \wedge dx^4 = dx^3 \wedge dx^4 - x^5\, dx^1 \wedge dx^2 = 0, \tag{3.26}$$

which is closed by the new equation

$$dx^1 \wedge dx^2 \wedge dx^5 = 0; \tag{3.27}$$

Its characteristic system is

[2] Applying the theorem of No. 38 to show that the system is completely integrable characteristic is an exercise in calculation is quite simple if the given system is linear. We leave this calculation aside.

[3] There would be exceptions to this theorem if, at all points of the integral manifold V, the rank was characteristic of the system decreases. We would be dealing with a singular integral manifold. An example is the singular solutions of partial differential equation of first order.

$$dx^1 = dx^2 = dx^3 = dx^4 = dx^5 = 0. \tag{3.28}$$

The system (3.26) is not complete, a complete system assuming the same solutions as the system (3.26) was provided by the equations

$$dx^1 \wedge dx^3 = dx^1 \wedge dx^4 = dx^3 \wedge dx^4 = dx^1 \wedge dx^2 = 0, \tag{3.29}$$

Its characteristic system is

$$dx^1 = dx^2 = dx^3 = dx^4 = 0. \tag{3.30}$$

3.3 Applications of Pfaff's problem

47. Let us apply the above considerations to the case of a linear differential equation (a Pfaff's equation)

$$\theta \equiv A_i dx^i. \tag{3.31}$$

The system consists of the characteristic equation (3.31) can be reached at which the associated system of exterior differential dx^i, where we have replaced one of its differentials have value of (3.31). This system is associated with even rank, it follows the (3.31).

Theorem. *The class of any linear differential equation is an odd number.*

If this class is equal to 1, that is characteristic that the system reduces to equation (3.31), which is completely integrable. If Z is a first integral, the equation is equivalent to $dZ = 0$.

Assume the general case, the class being $2p+1$. Let X, a special first integral feature of the system. If we binding the variable n by the relation $X^1 = C^1$, which C^1 is an arbitrary constant: it reduces at least one unit of the characteristic equations of the system and as the class is always odd is that this class is reduced to less two units. Next, let X^2 be a first integral of the new characteristic system, X^2 is a function of x^i and C^1 is a function of x^i also (if we replace C^1 by X^1).

By linking the two variables by the relations

$$X^1 = C^1, \qquad X^2 = C^2, \tag{3.32}$$

that is to say, the differential dx^i by both the class of relations

$$dX^1 = 0, \qquad dX^2 = 0, \tag{3.33}$$

system will be further reduced by at least two new units. We will continue step by step, there will exist p independent first integrals $X^1, X^2, \cdots, X^p$ as if it binding variables by the relations

$$X^1 = C^1, \quad X^2 = C^2, \quad \cdots, \quad X^p = C^p, \tag{3.34}$$

the equation $\theta = 0$ becomes completely integrable reducible due to the form $dZ = 0$.

This reduction is valid only because we assumed constant functions X^i. If we do suppose most consistent is that the equation $\theta = 0$ is reducible to form

$$dZ - Y_1\, dX^1 - Y_2\, dX^2 - \cdots - Y_p\, dX^p = 0, \tag{3.35}$$

the coefficients Y_i being conveniently chosen functions of p given variables.

Chapter 4
Integral elements, Character, Type, and Existence theorems

4.1 Integral elements of a differential system

51. We propose in this chapter to indicate some existence theorems for integral manifolds of a closed exterior differential system (we base that could always be reduced to this case). These existence theorems solve in some cases what is called the *Cauchy problems*, which we will clarify the statement. From this chapter we will be forced to assume, as we have already noted, that the functions entering the equations of the given system are analytic, while in the preceding chapter, it was enough to admit the existence continuous partial derivatives up to a low order (1 or 2).

The theory we will develop was first created by E. Cartan for linear differential equations, corresponding to closed systems containing differential equations of the at most second degree. It has been extended, especially by E. Kaehler, to systems of any degree.

52. The exterior differential systems that we have to consider are of the form[1]

$$\begin{cases} f_\alpha(x^1,x^2,\cdots,x^n)=0 & (\alpha=1,2,\cdots,r_0),\\ \theta_\alpha \equiv A_{\alpha i}dx^i=0 & (\alpha=1,2,\cdots,r_1),\\ \phi_\alpha \equiv \dfrac{1}{2}A_{\alpha ij}\,dx^i\wedge dx^j=0 & (\alpha=1,2,\cdots,r_2),\\ \psi_\alpha \equiv \dfrac{1}{6}A_{\alpha ijk}\,dx^i\wedge dx^j\wedge dx^k=0 & (\alpha=1,2,\cdots,r_3),\\ \cdots\cdots\cdots\cdots & \end{cases} \tag{4.1}$$

The system being closed, linear equations $df_\alpha=0$ must be among the equations (4.1), or better the form df_α must belong to the ring of forms θ_α; as the same, the exterior differentials $d\theta_\alpha$ must belong to the ring of forms θ_α and ϕ_α, and so on.

[1] As in the preceding chapters, the coefficients $A_{\alpha ij}, A_{\alpha ijk},\cdots$, are assumed antisymmetric with respect to their Latin indices $i,j,k,\cdots$

53. A key observation to make is about the equations $f_\alpha = 0$ contained in the system (4.1). They define an analytical manifold $\mathscr{V}$ whit a certain dimension ρ of the $n-$dimensional space . We suppose, in a simple point of $\mathscr{V}$, the rank of the matrix of partial derivatives f_α is equal to ρ (notice in passing that ρ can be less than the number r_0 of equations $f_\alpha = 0$, certain algebraic manifolds with $n-\rho$ dimensions front, for example, be defined by algebraic equations over ρ entries, if we do not want to miss any of their points). The condition we have to impose the equations $f_\alpha = 0$ would not be achieved if for example we had the single equation

$$f \equiv \left((x^1)^2 + (x^2)^2 + \cdots + (x^n)^2 - 1\right)^2 = 0. \tag{4.2}$$

In this case the reasoning we have to fall into default later.

If the above condition is realized and if we place ourselves in a simple point x_0^i of manifold $\mathscr{V}$, each equation $df_\alpha = 0$ must appears among the equations $\theta_\alpha = 0$, which are ρ linearly independent equations, for example those corresponding to rows of the matrix $\partial f_\alpha / \partial x^i = 0$ of that enter the main determinant for $x^i = x_0^i$. Any manifold satisfying these ρ equations $df_\alpha = 0$ and containing the point x_0^i of $\mathscr{V}$, is completely contained in $\mathscr{V}$, at least in the neighbourhood of this point.

54. *Integral plane elements.*
We will call $p-$dimensional plane element the set of a point $(x^1, x^2, \cdots, x^n)$ and a $p-$plane passing through that point. This point is called the *origin point of the element*. A $p-$dimensional plane element with a given origin, can be set through a system of $n-p$ independent linear relations between $x^1, x^2, \cdots, x^n$, regarded as coordinates in a common Cartesian frame of reference that would have the point (x^i) as origin can also define by p linearly independent vectors from the point (x^i) or by its Plückerian coordinates $u^{i_1 i_2 \cdots i_p}$ (No. 12).

A $p-$dimensional plane element is said *integral* if it satisfies the following two conditions:

1) Its origin point belongs to the manifold $\mathscr{V}$ (we can say that this is an *integral point*);
2) The exterior forms which are in the first members of equations of the system are annihilated by the considered plane element.

It is clear that, if a manifold is integral, each points of that is also integral and, furthermore all its tangent planes elements are integrals. The converse is obvious.

It is natural to say that:

instead of study integrals manifolds, we can study integral plane elements.

55. *When a $p-$dimensional plane element is integral?*
This is a problem that we solved in Chapter 1 (No. 12-14). Let us briefly recall the results therein.

An linear element with origin (x^i) and parameters u^i is integral if the origin point is integral and if the linear element (u^i) annihilates forms θ_α, the first degree of system (4.1):

$$A_{\alpha i}u^i = 0. \tag{4.3}$$

A two-dimensional plane element is integral if its origin is an integral if its Plückerian coordinates u^{ij} annihilate quadratic exterior forms $\theta_\alpha \wedge dx^1, \cdots, \theta_\alpha \wedge dx^n$ and that is if the Plückerian coordinates u^{ij} annihilate all exterior quadratic forms that belong the ring of the system (4.1).[2] More generally:

A p−dimensional plane element is integral if its origin is an integral point, and its Plückerian coordinates $u^{i_1 i_2 \cdots i_p}$ annihilate all forms of degree p belonging to the ring of the given system.

56. *Regular integral point, ordinary linear integral element.*
Let (x^i) be a simple generic point of $\mathscr{V}$. The linear integral elements having this point as origin are defined by the condition that their parameters u^i satisfy the equations

$$A_{\alpha i}u^i = 0, \qquad (\alpha = 1, 2, \cdots, r_1). \tag{4.4}$$

Let s_0 be the number of those of these equations are linearly independent or the rank of the matrix $A_{\alpha i}$, when simple point (x^i) is generic. *Point (x^i) will be called regular if for this point the number of equations (4.4) independent does not less than of s_0. An integral linear element called ordinary, if its point of origin is a regular point of $\mathscr{V}$.*

The integer s_0 is called the zero-order *character* of the system.[3]

Any regular point is necessarily a simple point of the variety $\mathscr{V}$, but the converse may not be true. As the condition for a point of not being regular results in additional equations for the coordinates of this point, any sufficiently small neighbourhood of a regular point within the manifold $\mathscr{V}$ contains only regular points. One can also say that if a point of $\mathscr{V}$ is not regular, every neighbourhood of this point inside of $\mathscr{V}$ contains an infinite number of points regular, since every point non-regular is limit of an infinite sequence of regular points.

Note again that the number of equations that define a linear integral with a given origin can never exceed s_0. Finally, for any point in space, integral or not sufficiently close to a regular point, the rank of equations (4.4) is at least equal to s_0 and may be higher.

---

[2] Recall that the coordinates u^{ij} annihilate a quadratic exterior form $H_{ij}\,dx^i \wedge dx^j$ if one has $H_{ij}u^{ij} = 0$. The ring of the system (4.1) is determined by the ring forms $\theta_\alpha, \phi_\alpha, \psi_\alpha, \cdots$ which appear as the first members of equations (4.1).

[3] If we solve the equations $f_\alpha = 0$ over a number of variables x^i manner as to leave in equations (4.1) as $n - \rho$ variables and their differentials, s_0 character would naturally be decreased accordingly.

57 *Regular integral linear element, two-dimensional ordinary integral element.*
Let (E_1) be an ordinary integral part with parameters u^i. To get some information on two-dimensional integral elements containing (E_1), we will form what is called the *polar element* of (E_1): this is the place of linear elements (dx^i) such that the element plane determined by (u^i) and (dx^i) is integral. The conditions with which the parameters dx^i are

$$A_{\alpha i} dx^i = 0 \qquad (\alpha = 1, 2, \cdots, r_1), \tag{4.5}$$

$$A_{\alpha ij} dx^j = 0 \qquad (\alpha = 1, 2, \cdots, r_2); \tag{4.6}$$

these equations are what we call the *polar system* of (E_1).

Let $s_0 + s_1$ the rank of the polar system of a *generic* ordinary integral element (E_1). This means that r_1 and r_2 equations (4.5) and (4.6) reduce to $s_0 + s_1$ independent if one takes into account that the u^i satisfy the equations

$$A_{\alpha i} u^i = 0. \tag{4.7}$$

The ordinary integral element (E_1) will be called regular if the rank of the polar system does not drop below its normal value $s_0 + s_1$. A two-dimensional integral element will be called ordinary *if it contains at least one regular linear integral.*

The integer s_1 is said *first order character* of a the given differential system.

Note that if $s_0 + s_1$ is greater than or equal to $n - 1$, the pole element of the regular linear integral element (E_1) reduces to that element itself. There is therefore no evidence in this case two-dimensional ordinary integral.

It can be observed as in the preceding number that any sufficiently small neighbourhood of a regular linear integral element within the range of linear integral contains only regular elements.

58 *Generalization.*
Suppose $s_0 + s_1 < n - 1$. Let (E_2) be an ordinary two-dimensional integral element, defined for example by two linear integral elements (u^i) and (v^i). The polar system of (E_2) is formed of equations that express that the linear integral element (dx^i) determines (E_2) a three-dimensional integral element. The polar equations of the system are

$$\begin{cases} A_{\alpha i} dx^i = 0 & (\alpha = 1, 2, \cdots, r_1), \\ A_{\alpha ij} u^i dx^j = 0, \quad A_{\alpha ij} v^i dx^j = 0 & (\alpha = 1, 2, \cdots, r_2), \\ A_{\alpha ijk} u^i v^j dx^k = 0 & (\alpha = 1, 2, \cdots, r_3). \end{cases} \tag{4.8}$$

Let $s_0 + s_1 + s_2$ be the rank of this system for an *generic* ordinary integral element (E_2). The ordinary integral element (E_2) will be called regular if the rank of the polar system does not drop dessourr of $s_0 + s_1 + s_2$, and all three dimensiona integral element will be called *ordinary* if it contains at least one element integral regular two-dimensional.

The integer s_2 is the character of order 2 for the given differential system.

If $s_0+s_1+s_2$ is superior or equal to $n-2$, the polar element (E_2) is two dimensional and there is no integral three-dimensional element which is ordinary.

Final element in a integral $p-$dimensional (E_p) is ordinary if it contains at least one regular integral element (E_{p-1}), it at least one regular integral element (E_{p-2}) and so on, until to a linear regular (E_1) to be regular for a point origin.

There is element of ordinary p-dimensional integral if one has

$$s_0+s_1+\cdots+s_{p-1}<n-p+1. \tag{4.9}$$

59 *Genus of a differential system closed.*
There will come a when will exist no ordinary integral element of a certain dimension, $h+1$. The integer h is called the genus of differential system and is the first integer for which

$$s_0+s_1+\cdots+s_h=n-h. \tag{4.10}$$

There are regular integral elements in h dimensions, but there is no element at integral ordinary $h+1$ dimensions.

Integers $s_0,s_1,s_2,\cdots,s_h$ are the characters of the differential system.

As mentioned, liver than in the manifold of integral elements $p<h$ dimensions, any sufficiently small neighborhood of a regular integral element contains only regular integral elements. An integral element (E_p) can be analytically defined by the coordinates of its point of origin $x^1,x^2,\cdots,x^n$ and coordinates Plückerian $u^{i_1 i_1\cdots i_p}$ subject the rest to satisfy a system of quadratic relations we have formed in Chapter 1; every neighborhood of an element (E_p) can be defined by the condition that the coordinates $x^i,u^{i_1 i_1\cdots i_p}$ of an element (E_p) in the neighborhood do not deviate a certain value of the coordinates of the same name of the element (E_p) gives.

4.2 Two existence theorems

60. We intend to demonstrate, given a system of differential Σ genus h, the existence of integral manifolds for any number $p\leq h$ dimensions. It does not mean that there is no integral manifold more than h dimensions; it does not mean that the $p-$dimensional integral manifolds which we will demonstrate use of all available varieties $p-$dimensional integrals. This is an application of *Cauchy-Kowalmski theorem*, which we state earlier, the existence theorems in question will be demonstrated.

Consider then a closed differential system Σ with genus h, and let $p\leq h$, we have

$$s_0+s_1+\cdots+s_{p-1}\leq n-p. \tag{4.11}$$

Consider a $p-$dimensional integral element $(E_p)_0$ which is *ordinary*. We can assume that his equations implies no linear relationship $dx^1, dx^2, \cdots, dx^p$. Then change the notations and denote by z^λ $(\lambda = 1,2,3,\cdots,n-p=\nu)$ the variables $x^{p+1},\cdots,x^n$; the element $(E_p)_0$ is defined by equations of the form

$$dz^\lambda = a_i^\lambda, \qquad (\lambda = 1,2,3,\cdots,\nu), \tag{4.12}$$

the summation index i varying from 1 to p.

There is a chain of elements regular integrals $(E_{p-1})_0, (E_{p-2})_0, \cdots, (E_1)_0$ each of which is contained in the prhcdent and $(E_p)_0$ and whose origin is a regular point of $\mathscr{V}$, and we can assume, for simplicity of exposition, that the first p coordinates x^i of this point are zero, the others being $z^\lambda = a^\lambda$. Finally we can assume, if necessary by a linear transformation with constant coefficients effected on p coordinates x^i, the equations of linear $(E_{p-1})_0, (E_{p-2})_0, \cdots$, are obtained by adding the equations (4.12) successively the equations

$$dx^p = 0, \quad dx^{p-1} = 0, \quad \cdots, \quad dx^2 = 0; \tag{4.13}$$

paramhtres of the $(E_1)_0$ are then

$$1,\ 0,\ \cdots,\ 0,\ a_1^1,\ a_1^2,\ \cdots,\ a_1^\nu. \tag{4.14}$$

These conventions are made, any $p-$dimensional integral manifold $\mathscr{V}$ tangent to the element $(E_p)_0$ will elements tangent planes near the origin point ($x^i = 0$, $a^\lambda = z^\lambda$) of $(E_p)_0$, ordinary integral elements. The integral manifold can be defined by ν equations

$$z^\lambda = \phi^\lambda(x^1, x^2, \cdots, x^p), \qquad (\lambda = 1,2,3,\cdots,\nu), \tag{4.15}$$

where ϕ^λ are holomorphic fonotions in the neiboorhood of the x^i values $x^i = 0$, and taking the values $a^\lambda = z^\lambda$.

Theorem. *There are at least one analytic integral manifold tangent to any given $p-$dimensional ordinary integral element $(E_p)_0$ and containing an $(p-1)-$ dimensionl integral manifold $\mathscr{V}_{p-1}$ tangent to any given regular $(p-1)-$dimensional integral element $(E_{p-1})_0$.*

62. *The Cauchy-Kowalewski theorem.*

We will build, to demonstrate, on a classical theorem, which we state as follows, which will suffice (See, for the general statement, E. Goursat [15]).

Theorem. *Given a system of q partial differential equations of first order between q unknown functions z^λ of p independent variables x^i, solved with re-*

spect to derivatives $\partial z^\lambda/\partial x^p$, the second members are functions of the arguments $x^i, z^\lambda, \partial z^\lambda/\partial x^1, \cdots, \partial z^\lambda/\partial x^{p-1}$ holomorphic near values $x^i = 0, z^\lambda = a^\lambda, \partial z^\lambda/\partial x^i = a_i^\lambda$, this system admits an analytical solution and one for which the unknown functions are holomorphic fontions $x^1, x^2, \cdots, x^p$ near $x^i = 0$, reducing themselves to $z^\lambda = \chi^\lambda(x^1, x^2, \cdots, x^{p-1})$ to holomorphic functions $x^1 = x^2 = \cdots = x^{p-1} = 0$ data, taking the values for a^λ, drivdcs $\partial z^\lambda/\partial x^i$ taking their values a_i^λ.

63. We start with donations prove the following theorem, which is a special case of the theorem stated in No. 61.

First Existance Theorem. *Let gives a closed differential system Σ for which*

$$s_0 + s_1 + \cdots + s_{p-1} = n - p. \tag{4.16}$$

Let $(E_p)_0$ an element intdgral ordinary $p-1$ dimensin V_p and an integral manifold for a $p-$dimensional plane tangent to an integral regular $(E_{p-1})_0$ contained in $(E_p)_0$. There are a variety $p-$dimensional integral and one containing V_{p-1} and this variety is tangent to the element $(E_p)_0$.

Let us point out immediately that the latter part of the statement is obvious, since the regular integral element $(E_p)_0$ it does not pass, because the value of $n-p$ of sum $s_0 + s_1 + \cdots + s_{p-1}$, one $p-$dimensional integral element, which is $(E_p)_0$.

We will demonstrate for $p = 3$, enough to give an idea of the proof in the general case.

64. *Opening remarks at the demonstration.* We suppose, as has been said in No. 61, that the element $(E_3)_0$ has its origin in the coordinates $x^i = 0$, $z^\lambda = a^\lambda$ $(\lambda = 1, 2, \cdots, n-3)$ and defined by the equations

$$dz^\lambda = a_1^\lambda\, dx^1 + a_2^\lambda\, dx^2 + a_3^\lambda\, dx^3, \qquad (\lambda = 1, 2, \cdots, \nu), \tag{4.17}$$

The regular integral element $(E_2)_0$ is obtained by adding the equations (4.17) the equation $dx^3 = 0$, and regular element $(E_1)_0$ by adding the equation $dx^2 = 0$ to it. Let

$$x^3 = 0, \quad z^\lambda = \Phi^\lambda(x^1, x^2), \qquad (\lambda = 1, 2, \cdots, \nu), \tag{4.18}$$

are equations of the two-dimensional integral manifold $\mathscr{V}_2$; for $x^1 = x^2 = 0$ functions Φ^λ and their partial derivatives $\partial\Phi^\lambda/\partial x^1, \partial\Phi^\lambda/\partial x^2$ take the values $a^\lambda, a_1^\lambda, a_2^\lambda$, respectively.

Let now $z^\lambda = F^\lambda(x^1, x^2, x^3)$ the equations of the unknown three-dimensional manifold $\mathscr{V}_3$, for $x^3 = 0$, the functions F^λ are reduced to functions Φ^λ data; more for $x^1 = x^2 = x^3 = 0$ we must have value $\partial F^\lambda/\partial x^3 = a_3^\lambda$.

The equations to be fulfilled are the functions F^λ, from equations (4.1) of system Σ,

$$
\begin{cases}
f_\alpha(x,z)=0 & (\alpha=1,2,\cdots,r_0),\\
H_{\alpha i}\equiv A_{\alpha i}+A_{\alpha\lambda}\dfrac{\partial z^\lambda}{\partial x^i}=0 & (i=1,2,3,\ \alpha=1,2,\cdots,r_1),\\
H_{\alpha ij}\equiv A_{\alpha ij}+A_{\alpha i\lambda}\dfrac{\partial z^\lambda}{\partial x^j}+A_{\alpha\lambda\mu}\dfrac{\partial z^\lambda}{\partial x^i}\dfrac{\partial z^\mu}{\partial x^j}=0 & (i=1,2,3,\ \alpha=1,2,\cdots,r_2),\\
H_{\alpha 123}\equiv A_{\alpha 123}+A_{\alpha ij\lambda}\dfrac{\partial z^\lambda}{\partial x^k}+A_{\alpha i\lambda\mu}\dfrac{\partial z^\lambda}{\partial x^j}\dfrac{\partial z^\mu}{\partial x^k} & \\
\qquad\qquad +A_{\alpha\lambda\mu\nu}\dfrac{\partial z^\lambda}{\partial x^1}\dfrac{\partial z^\mu}{\partial x^2}\dfrac{\partial z^\nu}{\partial x^3}=0. &
\end{cases}
\tag{4.19}
$$

In the expression of $H_{\alpha 123}$, the indices i,j,k appearing in the second and third terms are in turn the three permutations of indices $1,2,3$, that is 123, 231, and 312.

In the neighbourhood of of point M_0 origin of $(E_3)_0$ we can keep only ρ equations $F_\alpha(x,z)=0$, subject to the sole condition that the coefficient matrix of dz^λ in ρ differential df_α, has rank ρ; we will assume that it is the first ρ.

Equations (4.19) can be split into three groups

$$
\begin{array}{llllr}
f_\alpha(x,z)=0 & & & (\alpha=1,2,\cdots,\rho); & \text{(A)}\\
H_{\alpha 1}=0, & H_{\alpha 2}=0, & H_{\alpha 12}=0; & & \text{(B)}\\
H_{\alpha 3}=0, & H_{\alpha 13}=0, & H_{\alpha 23}=0, & H_{\alpha 123}=0. & \text{(C)}
\end{array}
$$

The variety $\mathscr{V}_2$ satisfies the equations (A) and (B). The variety $\mathscr{V}_3$ must also satisfy the equations (C).

65. If the coefficients in equations (C) are given respectively to the arguments $x^1, z^\lambda, \partial z^\lambda/\partial x^1, \partial z^\lambda/\partial x^2$ values $0, a^\lambda, a_1^\lambda, a_2^\lambda$ one obtains a system of linear equations with respect to $\partial z^\lambda/\partial x^3$, and is well easy to see that this syetem (C) is deduced from the polar system of $(E_2)_0$ by replacing dx^1 and dx^2 by 0 and dx^3 by 1 and dz^λ by $\partial z^\lambda/\partial x^3$.

The polar equations of this system are by hypothesis the number of $s_0+s_1+s_2$ independent with respect to dx^i and dz^λ, but these equations do not involve any linear relation between dx^1, dx^2, dx^3, otherwise this relation would be one of those that define the element $(E_3)_0$ which is not. The equations (C), considered as linear equations in $\partial z^\lambda/\partial x^3$, are the number of $s_0+s_1+s_2$ independent when, in the coefficients of these equations is replaced respectively $x^1, z^\lambda, \partial z^\lambda/\partial x^1, \partial z^\lambda/\partial x^2$ by $0, a^\lambda, a_1^\lambda, a_2^\lambda$. We will make a choice among these equations, $s_0+s_1+s_2$ independent, which we call *principal*.

If these in principal equations, arguments $x^1, z^\lambda, \partial z^\lambda/\partial x^1, \partial z^\lambda/\partial x^2$ are given values sufficiently close to $0, a^\lambda, a_1^\lambda, a_2^\lambda$, they will not cease to be linearly independent. Three cases are possibles:

1) *The given values of arguments define a two-dimensional integral element*; this element is necessarily regular if the argument values of arguments differ sufficiently near the values of $0, a^\lambda, a_1^\lambda, a_2^\lambda$. In this case the non-principal equations of (C) are consequences of principal equations of (C).
2) The values given only to the arguments x^i, z^λ, define an integral point, without the values given to the other arguments define a 2−dimensinal integral element. The non-principal equations of (C) are not consequences of principal equations of (C) and equations (B). In a more accurate non-principal equations of $H_{\alpha 3} = 0$, attached not to $\partial z^\lambda / \partial x^1$ or $\partial z^\lambda / \partial x^2$ are consequences of principal equations of $H_{\alpha 3} = 0$. The non-principal equations of equations $H_{\alpha 13} = 0$, not depending on the z^λ / x^1 nor z^λ / x^2 are consequences of the principal equations $H_{\alpha 13} = 0$ and $H_{\alpha 3} = 0$, and also equations $H_{\alpha 1} = 0$, expressing that the z^λ / x^1 define an linear integral element. Finally the non-principal equations of equations $H_{\alpha 23} = 0$, $H_{\alpha 123} = 0$ are consequences of non-principal of equations (C) and the set of equations (B). These results can be expressed in the following formulas, where α' is the index of non-principal equations of the equations (C), and α is the index of principal equation of equation (C) or one of the equations (B):

$$\begin{cases} H_{\alpha' 3} & = \{H_{\alpha 3}\} \\ H_{\alpha' 13} & = \{H_{\alpha 3}, H_{\alpha 13}, H_{\alpha 1}\} \\ H_{\alpha' 23} & = \{H_{\alpha 3}, H_{\alpha 13}, H_{\alpha 23}, H_{\alpha 123}, H_{\alpha 1}, H_{\alpha 2}, H_{\alpha 12}\} \\ H_{\alpha' 123} & = \{H_{\alpha 3}, H_{\alpha 13}, H_{\alpha 23}, H_{\alpha 123}, H_{\alpha 1}, H_{\alpha 2}, H_{\alpha 12}\} \end{cases} \tag{4.20}$$

the braces are linear combinations of expressions therein, the coefficients are homomorphic functions $x^1, z^\lambda, \partial z^\lambda / \partial x^1, \partial z^\lambda / \partial x^2$ in the neighbourhood of values $0, a^\lambda, a_1^\lambda, a_2^\lambda$ of these arguments.
3) *The given values of arguments x^i, z^λ do not define a integral point.* The non-principal equations of the equations (C) are consequences of principal equations of the equations (C), equations (B) and equations (A).

We should add two remarks. The first one is that we can assume that the equations $df_\alpha = 0$ are some of the equations $\theta_\alpha = 0$, that equations $d\theta_\alpha = 0$ are some of the $\phi_\alpha = 0$ and $d\phi_\alpha = 0$ equations are some of the equations $\psi_\alpha = 0$. The second is that we can assume that among the equations (C) of the principal form $H_{\alpha 3} = 0$ are the first ρ equations from $df_\alpha = 0$, that is to say

$$\frac{\partial f_\alpha}{\partial x^3} + \frac{\partial f_\alpha}{\partial z^\lambda} \frac{\partial z^\lambda}{\partial x^3} = 0, \qquad (\alpha = 1, 2, \cdots, \rho). \tag{4.21}$$

66. *Demonstration of the first existence theorem.* Consider the $s_0 + s_1 + s_2$ principal equations (C) . As $s_0 + s_1 + s_2 = n - 3$, number of unknown functions z^λ, they give them two other derivatives $\partial z^\lambda / \partial x^3$ based on $\partial z^\lambda / \partial x^1$ and $\partial z^\lambda / \partial x^2$. They form a system of Cauchy-Kowalewski. So they admit a homomorphic solution

$$z^\lambda = F^\lambda(x^1, x^2, x^3), \tag{4.22}$$

and one for which F^λ is reduced when there is $x^3=0$, for given functions $\Phi^\lambda(x^1,x^2)$. WE will show that the manifold $\mathscr{V}_3$ defined by equations (it) is integral.

Firstly, among the equations of the system of Cauchy-Kowalewski include ρ equations (4.21), which express that the manifold $\mathscr{V}_2$ functions $f_\alpha(x,z)$ are independent of x^3 or, for $x^3=0$, they are null because the manifold $\mathscr{V}_2$ is integral, so they are identically zero. The manifold $\mathscr{V}_3$ therefore satisfies the equations (A).

Secondly, the manifold $\mathscr{V}_3$ satisfying equations (A), all its points are integral points and consequently, after the part 2 of No. 65, all expressions $H_{\alpha 3}$, even non-principal ones, are identically zero on the manifold $\mathscr{V}_3$; respect to expressions $H_{\alpha 13}$, $H_{\alpha 23}$, $H_{\alpha 123}$, those are principal void in the same conditions, those that do non-principal satisfy according to (4.23) to equations of the form

$$\begin{cases} H_{\alpha' 3} &= \{H_{\alpha 1}\} \\ H_{\alpha' 23} &= \{H_{\alpha 1}, H_{\alpha 2}, H_{\alpha 12}\} \\ H_{\alpha' 123} &= \{H_{\alpha 1}, H_{\alpha 2}, H_{\alpha 12}\}. \end{cases} \tag{4.23}$$

A sufficient condition for the manifold $\mathscr{V}_3$ be integral, is that it satisfies the equations (B).

Third, the quantities $H_{\alpha 1}$, $H_{\alpha 2}$, $H_{\alpha 12}$ are zero for $x^3=0$. Consider a form θ_α; on the manifold $\sqsubseteq_3$ were

$$\theta_\alpha = H_{\alpha 1}\, dx^1 + H_{\alpha 2}\, dx^2, \tag{4.24}$$

from which

$$d\theta_\alpha = -\frac{\partial H_{\alpha 1}}{\partial x^3}\, dx^1 \wedge dx^3 - \frac{\partial H_{\alpha 2}}{\partial x^3}\, dx^2 \wedge dx^3 + \Big(\frac{\partial H_{\alpha 2}}{\partial x^1} - \frac{\partial H_{\alpha 1}}{\partial x^2}\Big)\, dx^1 \wedge dx^2. \tag{4.25}$$

As the equation $d\theta_\alpha = 0$ is one of the equations $\phi_\alpha = 0$, the coefficient is a $\partial H_{\alpha 1}/\partial x^3$ combination linear with coefficients homomorphic expressions $H_{\alpha 13}$ and result of after (4.22) expressions $H_{\alpha 1}$. The r_1 quantities $H_{\alpha 1}$ thus satisfy a system of linear differential equations with homomorphic coefficients, variable being independent of x^3; like functions $H_{\alpha 1}$ are zero for $x^3=0$, they are identically zero. The variety $\mathscr{V}_3$ therefore satisfies Hal equations $H_{\alpha 1}=0$ and hence the equations $H_{\alpha 13}=0$.

Fourthly the expression $\partial H_{\alpha 2}/\partial x^3$ is from (4.25), a linear combination of coefficients homomorphic $H_{\alpha 23}$, that is to say, according to (4.22) of $H_{\alpha 2}$ and $H_{\alpha 12}$.

On the other hand as we have $d\phi_\alpha = H_{\alpha 12}\, dx^1 \wedge dx^2 + H_{\alpha 23}\, dx^2 \wedge dx^3$, we will

$$d\phi_\alpha = \Big(\frac{\partial H_{\alpha 12}}{\partial x^3} + \frac{\partial H_{\alpha 23}}{\partial x^1}\Big)\, dx^1 \wedge dx^2 \wedge dx^3, \tag{4.26}$$

and as $d\phi_\alpha$ is a linear combination of ψ_α, the expression $\frac{\partial H_{\alpha 12}}{\partial x^3} + \frac{\partial H_{\alpha 23}}{\partial x^1}$ will be a linear combination of $H_{\alpha 123}$, that is to say, according to (4.22), of $H_{\alpha 2}$,and $H_{\alpha 12}$. As finally $\partial H_{\alpha 23}/\partial x^1$, according to (4.22) is a linear combination of $H_{\alpha 2}$, $H_{\alpha 12}$,

$\partial H_{\alpha 2}/\partial x^1$, $\partial H_{\alpha 12}/\partial x^1$, we see that $\partial H_{\alpha 12}/\partial x^3$ are linear combinations of coefficients homomorphic function $H_{\alpha 2}$ and $H_{\alpha 12}$ and their partial derivatives with respect to x^1. They therefore satisfy a system of Cauchy-Kowalewski and, as had vanish for $x^3 = 0$, they are identically zero, and, from (4.22) that the $H_{\alpha 23}$ and $H_{\alpha 123}$.

The variety $\mathscr{V}_3$ therefore satisfies all the equations (A), (B) and (C). □

Second Existance Theorem. *Let given a closed differential system Σ for which*

$$s_0 + s_1 + \cdots + s_{p-1} < n - p. \tag{4.27}$$

Let $(E_p)_0$ be an $n-$dimensional linear integral element and $\mathscr{V}_{p-1}$ be an $(p-1)-$dimensions integral manifold tangent to an element (E_{p-1}) contained in integral regular $(E_p)_0$. There are an infinite number of $p-$dimensional integral manifolds containing $\mathscr{V}_{p-1}$, and tangent to the element $(E_p)_0$. Each is uniquely determined if we choose an arbitrary $s_0 + s_1 + \cdots + s_{p-1}$ unknown functions on the sole condition of being reduced corresponding functions $\Phi^\lambda(x^1, x^2)$, for $x^3 = 0$.

Proof. The proof is easy and reduces to that of the first theorem. Indeed resume the hypothesis $p = 3$ and consider the system of equations (C) principal, and they are solvable with respect to $s_0 + s_1 + s_2$ derivatives $\partial z^\lambda/\partial x^3$. It remains in the second members $n - 3 - (s_0 + s_1 + s_2)$ derived $\partial z^\lambda/\partial x^3$; impute to $n - 3 - (s_0 + s_1 + s_2)$ functions z^λ values corresponding functions x^1, x^2, x^3 homomorphic in a neighbourhood of $x^i = 0$ and subject to the sole condition that they reduce to $x^3 = 0$ functions $\Phi^\lambda(x^1, x^2)$ of the same index. We obtain a Cauchy-Kowalewski system with a unique solution corresponding to given initial conditions $z^\lambda = \Phi^\lambda(x^1, x^2)$. □

68. We say that the integral manifolds whose existence is demonstrated by the two theorems of existence are *ordinary* integral manifolds. All the ordinary integral manifolds is what we call the *general solution* of the given differential system. The integral manifolds which are not normal at any point have a common tangent integral element. *regular integral manifold* are those that admit of elements integral regular tangent.

We can evaluate the degree of generality of ordinary integral manifold $\mathscr{V}_p$ admitting an given $p-$dimensional ordinary tangent integral element.

Indeed, maintaining the previous notations, section $\mathscr{V}_1$ of $\mathscr{V}_p$ by the planar manifold $x^p = x^{p-1} = \cdots = x^2 = 0$ depends on $n - p - s_0$ arbitrary functions of x^1 subject to the sole condition that for $x^1 = 0$ values of their derivatives are given a_1^λ. The integral manifold being chosen $\mathscr{V}_1$, section $\mathscr{V}_p$ by the planar manifold $x^p = x^{p-1} = \cdots = x^2 = 0$ depends on $n - p - s_0 - s_1$ arbitrary functions of x^1, x^2, subject to the unique condition of being reduced to $x^2 = 0$ to known functions of x^1. And so on. The variety V itself depends on $n - p - s_0 - s_1 - \cdots - s_p$, arbitrary functions of $x^1, x^2, \cdots, x^p$ the single subject requirement be reduced to $x^p = 0$ functions to data of $x^1, x^2, \cdots, x^{p-1}$.

Introduce, for reasons of symmetry, has an integer, (which is not a character) by the relation

$$s_0 + s_1 + \cdots + s_{p-1} + s_p + \sigma_p = n - p. \tag{4.28}$$

We can then say, roughly, that the $p-$dimensional ordinary integral manifold tangent to $(E_p)_0$ depends

$$\begin{aligned} s_1 + s_2 + \cdots + s_{p-1} + \sigma_p &\ \text{ arbitrary functions of } x^1, \\ s_2 + \cdots + s_{p-1} + \sigma_p &\ \text{ arbitrary functions of } x^1, x^2, \\ &\ \vdots \\ s_{p-1} + \sigma_p &\ \text{ arbitrary functions of } x^1, x^2, \cdots, x^{p-1}, \\ \sigma_p &\ \text{ arbitrary functions of } x^1, x^2, \cdots, x^{p-1}, x^p. \end{aligned}$$

69. *Remark.*
It is inappropriate to make sense of the preceding statement too absolute, which simply restates numerically all arbitrary functions that can be given to get the most $p-$dimensional general integral manifold of by successive applications of Cauchy-Kowalewski Theorem.

In reality, the only one of these, whole that makes sense is the absolute number of arbitrary functions at maximum number of variables (σ_p if $\sigma_p \neq 0$, $s_{p-1} = 0$, $\sigma_p = 0$, $s_{p-1} \neq 0$ etc.). Without wanting to justify this assertion, which makes sense for the rest for the differential system for analytical integrals analytic manifolds be content-tied systems to report on a simple example, how cautiously it should come forward in matters of this kind.

Let the equation

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial z}{\partial x}; \tag{4.29}$$

from our point of view, the general solution of this equation depends on two arbitrary functions of one variable, for example, homomorphic functions of x, such that z and $\partial z/\partial y$ are reduced for $x = 0$. But it could also be tempted to say that this general solution depends on a sense arbitrary function, namely the function $\Phi(y)$ which reduces to z for $x = 0$. Unfortunately if we are given for $\Phi(y)$ a homomorphic function of y in the vicinity of $y = 0$ for example, the equation can not admit any homomorphic solution around the point $x = 0$, $y = 0$: it suffices to take that $\Phi(y) = 1/(1-y)$, which leads for z to the series

$$z = \frac{1}{1-y} + \frac{2}{1}\frac{x}{(1-y)^2} + \frac{4!}{2!}\frac{x^2}{(1-y)^5} + \cdots + + \frac{(2n)!}{n!}\frac{x^n}{(1-y)^{2n+1}} + \cdots. \tag{4.30}$$

But this series in x is convergent for $x = 0$. We show that this negative result is found as the function $\Phi(y)$ is not entire, and the same for some entire functions $\Phi(y)$.

70. *Degree of generality of E_p ordinary integral elements have origin in a given regular integral point.*

If we make the same assumption as above with respect to integral elements $(E_{p-1})_0$, $\cdots,(E_1)_0$ contained in the ordinary regular element $(E_p)_0$, ordinary element (E_p) near the $(E_p)_0$ and with the same origin as $(E_p)_0$ will be obtained in a way and using a single p linear integral elements whose i^{th} its p components dx^k will all zero except for $dx^i = 1$, the other ν components being $dz^\lambda = t_1^\lambda$. But the components t_1^λ of the first linear element (E_1) are subject to satisfy equations s_0, components t_2^λ of the second, forming with the first integral one element (E_2) are subject to satisfy the s_0+s_1 equations of polar system (E_1); components t_3^λ forming with the third (E_2) an linear integral element (E_3) are subject to satisfy the $s_0+s_1+s_2$ equations of polar system (E_2) and so on. All these equations are independent and are among

$$\begin{aligned} s_0+(s_0+s_1)+(s_0+s_1+s_2)+\cdots+(s_0+s_1+\cdots+s_{p-1}) &= \\ = ps_0+(p-1)s_1+\cdots+s_{p-1}. \end{aligned} \tag{4.31}$$

By introducing the number σ_p, we see that the number of arbitrary parameters sought is equal to

$$\begin{aligned} p(n-p)-\left[ps_0+(p-1)s_1+\cdots+2s_{p-2}+s_{p-1}\right] &= \\ = p\left(s_0+s_1+\cdots+s_{p-1}+\sigma_p\right)-\left[ps_0+(p-1)s_1+\cdots+s_{p-1}\right] \\ = s_1+2s_2+3s_3+\cdots+(p-1)s_{p-1}+p\sigma_p. \end{aligned} \tag{4.32}$$

Theorem. *In a closed system of genus greater than or equal to p, any the p−dimensional ordinary integral element originating in a given regular integral point depends to $s_1+2s_2+\cdots+(p-1)s_{p-1}+p\sigma_p$ arbitrary parameters.*

71. *Particular case.*

If all integers s based on a certain rank $q<p$ are zero, we have the following important theorem:

Theorem. *If $s_q = s_{q+1} = \cdots = s_{p-1} = \sigma_p = 0$, it passes an integral manifold $\mathscr{V}_p$ ordinary only one by any regular integral manifold with p - 1 dimensions.*

This theorem is applicable for example to a completely integrable system of $n-p$ total differential equations in n variables. For then the closed system includes only equations $\theta_\alpha = 0$ and we have

$$s_0 = n-p, \quad s_1 = s_2 = \cdots = s_p = 0; \tag{4.33}$$

it effectively passes an integral manifold $\mathscr{V}_p$ and only one by any regular point of space.

In the general case, the method indicated above would result, the integral manifold V_{q-1} being given, to include successive Cauchy-Kowalewski system with $p-q$. But we can limit ourselves with only one integer: simply assuming that V_{q-1} is located in the manifold

$$x^q = x^{q+1} = \cdots = x^p = 0, \tag{4.34}$$

to ask

$$x^q = a^q t, \quad x^{q+1} = a^{q+1} t, \quad \cdots, \quad x^p = a^p t, \tag{4.35}$$

looking $a^q, a^{q+1}, \cdots, a^p$ as arbitrary parameters and then to integrate the system, where z^p are regarded as unknown functions of q independent variables $x^1, x^2, \cdots, x^{q-1}, t$. It shall be reduced to a single system of Cauchy-Kowalewski with $n-p$ unknown functions of q independent variables. This integrated system will replace in the resulting expression of unknown function, t by 1, and $a^q, a^{q+1}, \cdots, a^p$, by $x^q, x^{q+1}, \cdots, x^p$.

The preceding process is basically the same as that indicated for the integration of a completely integrable system.

*Remark.*If the system (4.1) contains no exterior differential equation of degree greater than 2, the number series $s_0, s_1, s_2, \cdots$ is increasing, and if a s_q be zero, then all next s_rs are zero. About the whole σ_p, which is not a character, it must assume that we have

$$s_0 + s_1 + \cdots + s_{q-1} = n - p. \tag{4.36}$$

4.3 General solution and singular solutions, Characteristic

72. We say that a $p-$dimensional integral manifold is part of the *general solution* of a differential system, considered as p independent variables, if its generic $p-$dimensional tangent element is an ordinary integral element; it is an integral manifold which the fundamental theorem of existence (No. 67) proves its existence, at least locally, as a consequence of Cauchy-Kowalewski theorem.

An integral manifold which is not part of the general solution is called *singular*. A solution may be singular or general, because none of the points of the integral manifold is regular or integral elements because no one-dimensional or two-dimensional, or ets, or $(p-1)-$dimensional, is a regular integral element. So there can be different classes of singular integral varieties, decreasing the degree of singularity in some way when moving from one class to the next.

73. We have already introduced the concept of characteristic of a differential system , these characteristics entering into the generating of integral manifolds which are not contained in any integral manifold to a larger number of dimensions. These features do exist, moreover, that for some differential systems. We will refer to as the *Cauchy characteristic*.

There are other types of characteristics, usually in any given $p-$dimensional integral manifold there are some $p-$dimensional characteristics which are part of the general solution of the system. These are the manifolds with $q < p$ dimensions contained in the integral manifold $\mathscr{V}_p$ considered, and possessing the property that their elements tangent to q dimensions are not regular. *Their importance stems from the remark that the Cauchy-Kowalewski theorem falls into default if one seeks to determine the integral manifolds of $q+1$ dimensions that contain them.* Their research is linked to the problem of preliminary research integral elements $\mathscr{V}_p$ of dimensions q, which are not regular. The existence of such elements does not *ipso facto* (by the fact itself) remains of the existence of characteristic manifolds with dimensions q, except if $q = 1$, and this because of compatibility conditions which necessarily introduce for $q > 1$.

We will clarify these concepts by some examples from classical problems.

74. Example I. *Partial differential equations of first order.*
Let with the classical notations, the equation first order partial derivative

$$F(x,y,p,q) = 0 \tag{4.37}$$

for an unknown function two independent variables. Following the design of S. Lie, extend this problem to find solutions for two-dimensional closed differential system

$$\begin{cases} F(x,y,p,q) = 0, \\ F_x\,dx + F_y\,dy + F_z\,dz + F_p\,dp + F_q\,dq = 0, \\ dz - p\,dx - q\,dy = 0, \\ dx \wedge dp + dy \wedge dq = 0. \end{cases} \tag{4.38}$$

The character s0 is equal to the rank of the system

$$\begin{cases} F_x\,dx + F_y\,dy + F_z\,dz + F_p\,dp + F_q\,dq = 0, \\ dz - p\,dx - q\,dy = 0. \end{cases} \tag{4.39}$$

This rank is equal to 2; a point of an integral manifold is singular (that is to say non-regular) if the rank of the system (4.39) is less than 2, this is the case if one has

$$F_x + pF_z = 0, \quad F_y + qF_z = 0, \quad F_p = 0, \quad F_q = 0. \tag{4.40}$$

The singular solutions are those that satisfy the equations (4.40).

There is none else. Indeed suppose the regular generic point of the integral manifold. The pole piece of an integral linear components $\delta x, \delta y, \delta z, \delta p, \delta q$ is given by equations (4.39) which must be added the equation

$$\delta p\,dx + \delta q\,dy - \delta x\,dp - \delta y\,dq = 0. \tag{4.41}$$

Thus, the character s_1 is 1. A singular integral manifold in principle could arise if all elements linear tangent, the rank of systems (4.39) and (4.41) was reduces by one; it happens if one has

$$\frac{\delta x}{F_p} = \frac{\delta y}{F_q} = \frac{-\delta p}{F_x + pF_z} = \frac{-\delta q}{F_y + qF_z} = \frac{\delta z}{pF_p + qF_q}; \tag{4.42}$$

but the relations (4.42) show that a given point on the integral manifold there is only one singular linear element tangent. There is therefore singular solutions other than those, if any, that satisfy the equations (4.40).

The characteristics of an integral manifold $\mathscr{V}_2$ are, according to the definition given in No. 73, lines whose elements satisfy equations (4.42): what are the characteristics already encountered (No. 59). They depend in general arbitrary constants.

75. Example II. *Partial differential equations of second order.*
Any partial differential equation of second order $F(x,y,p,q,r,s,t) = 0$ for an unknown function z of two independent variables x, y, can be represented by the closed differential system

$$\begin{cases} F(x,y,p,q,r,s,t) = 0, \\ (F_x + pF_z + rF_p + sF_q)\,dx + (F_y + qF_z + sF_p + tF_q)\,dy \\ \qquad\qquad\qquad\qquad + F_r\,dr + F_s\,ds + F_t\,dt = 0, \\ dz - p\,dx - q\,dy = 0, \\ dp - r\,dx - s\,dy = 0, \\ dq - s\,dx - t\,dy = 0, \\ dx \wedge dr + dy \wedge ds = 0, \\ dx \wedge ds + dy \wedge dt = 0. \end{cases} \tag{4.43}$$

The character s_0 is 4, in place of the linear system $dx, dy, , \cdots, dt$ shaped by the four equations (4.43) after the first. This rank is reduced by one at the points where one has

$$F_x + pF_z + rF_p + sF_q = 0, \quad F_y + qF_z + sF_p + tF_q = 0, \quad F_r = F_s = F_t = 0. \tag{4.44}$$

Were a first class of singular solutions, those that satisfy equations (4.44).

Let now characteristic of a regular point. The polar system of an integral linear ordinary components $(\delta x, \delta y, \delta z, \delta p, \delta q, \delta r, \delta s, \delta t)$ is given by the four equations (4.43) that follow the first one, which we add the two equations

$$\begin{cases} \delta x\,dr + \delta y\,ds - \delta r\,dx - \delta s\,dy = 0, \\ \delta x\,ds + \delta y\,dt - \delta s\,dx - \delta t\,dy = 0. \end{cases} \tag{4.45}$$

We have $s_0 + s_1 = 6$, hence $s_1 = 2$, and rank 6 of polar system reduces by one if the rank of the matrix

$$\begin{pmatrix} F_x + pF_z + rF_p + sF_q & F_y + qF_z + sF_p + tF_q & F_r & F_s & F_t \\ -\delta r & -\delta s & \delta x & \delta y & 0 \\ -\delta s & -\delta t & 0 & \delta x & \delta y \end{pmatrix} \tag{4.46}$$

is equal to 2. This can not occur for all two-dimensional tangent linear integral manifold, at least if there is such manifold no relation between x and y.[4] Indeed from the hypothesis result in particular the relation

$$F_t\,\delta x^2 - F_s\,\delta x\delta y + F_r\,\delta y^2 = 0, \tag{4.47}$$

where the consequence of

$$F_t = F_s = F_r = 0 \tag{4.48}$$

and, according to the first equation (4.43), the relation

$$F_x + pF_z + rF_p + sF_q = 0, \quad F_y + qF_z + sF_p + tF_q = 0; \tag{4.49}$$

no point of the integral manifold would be regular. It is therefore no other singular integral manifold as those that satisfy the equations (4.44).

On an integral manifold part of the solution, the characteristics are the one-dimensional manifolds for which the rank of the matrix (4.46) is equal to 2; it was particularly while moving along such a characteristic, the relation

$$F_t\,\delta x^2 - F_s\,\delta x\delta y + F_r\,\delta y^2 = 0. \tag{4.50}$$

For every point of the variety he spends two, if $F_r F_t > 0$ satisfied. The rest is verified that if one moves over the integral manifold in order to satisfy the relation (**??**), the rank of the matrix (4.46) automatically becomes equal to 2.

Unlike what happens for partial differential equations of first order, the characteristics of a partial differential equation of second order in general dependent on an infinite number of arbitrary parameters (in fact, arbitrary function), it characteristics are not Cauchy.

75. Example III. *System of two partial differential equations of first order with two unknown functions z^1, z^2, of two independent variables x, y.*
This system can be represented by the closed differential system

[4] Cases where such a relation would exist are without interest: either there have a relation between x and y, for example y be a function of x, so, according to the three equations (4.43) that follow the second, z, p, q are also functions of x and we have the equations the variety in the form

$$r + 2sy' + ty'^2 = z'' - qy'', \qquad F(x,y,z,p,q,r,s,t) = 0;$$

either x and y are constants and consequently also z, p, q; the equation the variety is $F(x,y,z,p,q,r,s,t) = 0$.

$$\begin{cases} F(x,y,z^1,z^2,p_1,q_1,p_2,q_2)=0, \\ \Phi(x,y,z^1,z^2,p_1,q_1,p_2,q_2)=0, \\ (F_x+p_1F_{z^1}+p_2F_{z^2})\,dx+(F_y+q_1F_{z^1}+q_2F_{z^2})\,dy \\ \qquad\qquad +F_{p_1}\,dp_1+F_{q_1}\,dq_1+F_{p_2}\,dp_2+F_{q_2}\,dq_2=0, \\ (\Phi_x+p_1\Phi_{z^1}+p_2\Phi_{z^2})\,dx+(\Phi_y+q_1\Phi_{z^1}+q_2\Phi_{z^2})\,dy \\ \qquad\qquad +\Phi_{p_1}\,dp_1+\Phi_{q_1}\,dq_1+\Phi_{p_2}\,dp_2+\Phi_{q_2}\,dq_2=0, \\ dz^1-p_1\,dx-q_1\,dy=0, \\ dz^2-p_2\,dx-q_2\,dy=0, \\ dx\wedge dp_1+dy\wedge dq_1=0, \\ dx\wedge dp_2+dy\wedge dq_2=0. \end{cases} \tag{4.51}$$

The character s_0 is equal to 4; is the rank of the linear system of four equations formed (4.51) following the second. This rank can not be reduced if whether one has

$$\begin{aligned} F_x+p_1F_{z^1}+p_2F_{z^2}&=0, & \Phi_x+p_1\Phi_{z^1}+p_2\Phi_{z^2}&=0, \\ F_x+q_1F_{z^1}+q_2F_{z^2}&=0, & \Phi_x+q_1\Phi_{z^1}+q_2\Phi_{z^2}&=0, \\ F_{p_1}=F_{q_1}=F_{p_2}=F_{q_2}&=0, & \Phi_{p_1}=\Phi_{q_1}=\Phi_{p_2}=\Phi_{q_2}&=0, \end{aligned} \tag{4.52}$$

or whether

$$\begin{aligned} \frac{F_x+p_1F_{z^1}+p_2F_{z^2}}{\Phi_x+p_1\Phi_{z^1}+p_2\Phi_{z^2}} &= \frac{F_x+q_1F_{z^1}+q_2F_{z^2}}{\Phi_x+q_1\Phi_{z^1}+q_2\Phi_{z^2}} \\ &= \frac{F_{p_1}}{\Phi_{p_1}}=\frac{F_{q_1}}{\Phi_{q_1}}=\frac{F_{p_2}}{\Phi_{p_2}}=\frac{F_{q_2}}{\Phi_{q_2}} \end{aligned} \tag{4.53}$$

So there is possibility of two kinds of singular integral varieties, the according to equations (4.52) or equations (4.53) are verified.

Now calculate the character s_1. For the equations of the polar component of the full linear element components $\delta x,\delta y,\delta z^1,\delta z^2,\delta p_1,\delta q_1,\delta p_2,\delta q_2$, must be added to the four linear equations that appear in the system (4.51), equations

$$\begin{aligned} \delta x\,dp_1+\delta y\,dq_1-\delta p_1\,dx-\delta q_1\,dy&=0, \\ \delta x\,dp_2+\delta y\,dq_2-\delta p_2\,dx-\delta q_2\,dy&=0, \end{aligned} \tag{4.54}$$

We deduce $s_1=2$. The integral linear element is considered singular if the six equations that define the polar element of this linear integral element are reduced to five. But on a two-dimensional integral manifold not involving any relations between x and y, these equations not involve any linear relation between dx and dy as the polar element contains all the elements considered linear tangent to the manifold. It is therefore necessary and sufficient for the linear element is considered singular, the equations

$$\begin{cases} F_{p_1}\,dp_1 + F_{q_1}\,dq_1 + F_{p_2})\,dp_2 + F_{q_2}\,dq_2 = 0, \\ \Phi_{p_1}\,dp_1 + \Phi_{q_1}\,dq_1 + \Phi_{p_2})\,dp_2 + \Phi_{q_2}\,dq_2 = 0, \\ \delta x dp_1 + \delta y dq_1 = 0, \\ \delta x dp_2 + \delta y dq_2 = 0, \end{cases} \tag{4.55}$$

are reduced to three, which gives immediately

$$\frac{D(F,\Phi)}{D(q_1,q_2)}\,\delta x^2 - \left(\frac{D(F,\Phi)}{D(p_1,q_2)} - \frac{D(F,\Phi)}{D(p_2,q_1)}\right)\delta x\delta y + \frac{D(F,\Phi)}{D(p_1,p_2)}\,\delta y^2 = 0. \tag{4.56}$$

We deduce from this result two conclusions:

1) One can have a second class of singular integral varieties, those that satisfy the three equations

$$\frac{D(F,\Phi)}{D(q_1,q_2)} = \frac{D(F,\Phi)}{D(p_1,q_2)} - \frac{D(F,\Phi)}{D(p_2,q_1)} = \frac{D(F,\Phi)}{D(p_1,p_2)} = 0. \tag{4.57}$$

2) On a regular integral manifold, that is to say part of the general solution, there are generally two families of characteristic lines defined by the differential equation (4.56).

Theorem. *Given a system of two partial differential equations of first order with two unknown functions z^1, z^2 of two independent variables x, y, there can be three classes of singular solutions, they satisfy the following equations (4.52), with equations (4.53) or to equations (4.57). In addition, each integral manifold admits two general families of characteristic lines of the finite differential equation (4.56).*

77. *Partial differential equations of second order in an unknown function z of three independent variables x^1, x^2, x^3.*

We denote by pi and $pi_{ij} = P_{ji}$ the partial derivatives of first and second order of z, indicating the indices by reference to which the variables take place derivations. If $F(x^i, z, p_i, p_{ij}) = 0$ is the given equation, we will put

$$\begin{cases} A_i = \dfrac{\partial F}{\partial x^i} + p_i\dfrac{\partial F}{\partial z} + p_{ik}\dfrac{\partial F}{\partial p_k}, \\ A^{ij} = m\dfrac{\partial F}{\partial p_{ij}}, \qquad (m = 1 \text{ if } i = j,\ \ m = \dfrac{1}{2} \text{ if } i \neq j). \end{cases} \tag{4.58}$$

The given equation is represented by the closed differential system of 9 equations

$$\begin{cases} F = 0, \\ A_i\,dx^i + A^{ij}\,dp_{ij} = 0, \\ dz - p_i\,dx^i = 0, \\ dp_i - p_{ij}\,dx^k = 0, & (i = 1,2,3), \\ dx^k \wedge dp_{ik} = 0, & (i = 1,2,3). \end{cases} \tag{4.59}$$

The character s_0 is equal to 5, rank of the system formed by the linear equations (4.59) which follow the equation $F = 0$. Non-regular integral point is characterized by the relations

$$A_i = 0, \qquad A^{ij} = 0, \tag{4.60}$$

It was a first class possible to singular integrals, i.e. integrals which satisfy equations (4.61).

The polar system of a regular linear integral element contains $s_0 + s_1 = 8$ equations, namely the five equations that define the linear integral and in addition the three equations

$$\delta x^k\,dp_{ik} - \delta p_{ik}\,dx^k = 0, \qquad (i = 1,2,3). \tag{4.61}$$

The integral element $(\delta x^k, \delta z, \cdots)$ will be singular if the four equations

$$A_i\,dx^i + A^{ij}\,dp_{ij} = 0, \qquad \delta x^k\,dp_{ik} - \delta p_{ik}\,dx^k = 0, \tag{4.62}$$

are reduced to three. As a three-dimensional integral manifold for which the independent variables are x^1, x^2, x^3, equations (4.62) not involve any linear relation between dx^i, a linear singular integral element will be characterized by the property that the four equations

$$A^{ij}\,dp_{ij} = 0, \qquad \delta x^k\,dp_{ik} = 0, \tag{4.63}$$

reduced to three. If $\delta x^i = a^i$ be such a singular integral element, assuming it exists. If, in equations (4.63), dp_{ij} is replaced by the product $\xi_i\xi_j$ two new variables ξ we see that the equation $a^i\xi_i = 0$ will result in $A^{ij}\xi_i\xi_j = 0$. As a result the quadratic form $A^{ij}\xi_i\xi_j$ must be decomposed into a product of two linear factors: $A^{ij}\xi_i\xi_j = a^k b^h \xi_k \xi_h$, hence

$$A^{ij} = \frac{1}{2}(a^i b^j - b^i a^j). \tag{4.64}$$

Conversely if the quantities A^{ij} are of the form (4.63), we easily verify that there are two elements at each point linear singular integral, of respective components and have $\delta x^i = a^i$ and $\delta x^i = b^i$.

We see that if an integral manifold does not annihilate all A^{ij}, it is impossible that all its linear tangent elements are singular.

We see further that if the discriminant of the quadratic form $A^{ij}\xi_i\xi_j$ is zero, any non-singular integral manifold admits two families of characteristic lines separate or combined, formed one of the trajectories of the vector field a^i have the other involving trajectories of the vector field b^i. If instead the discriminant of the quadratic form $A^{ij}\xi_i\xi_j$ is not zero, there is no characteristic lines.

Turning finally to the character s_2. The polar element of a two-dimensional integral element is given by the regular five equations that define the linear integral, which must be added six more equations that are written, omitting the terms in dx^1, dx^2, dx^3,

$$\delta_1 x^k dp_{ik} = 0, \delta_2 x^k dp_{ik} = 0, \tag{4.65}$$

$\delta_1 x^k$ and $\delta_2 x^k$ denoting by the two components of linear integral demented that determine the considered two-dimensional integral element. These equations can be written as

$$\frac{dp_{i1}}{c_1} = \frac{dp_{i2}}{c_2} = \frac{dp_{i3}}{c_3}, \qquad (i = 1, 2, 3), \tag{4.66}$$

denoting by $c_i dx^i = 0$ the equation of the considered planar integral element. It follows that the dp_{ij} are proportional to the product $c_i c_j$. The integral element in two dimensions will be singular if one has

$$A^{ij} c_i c_j = 0. \tag{4.67}$$

This equation expresses that the three-dimensional space formed by the integral manifold considered, the elements derived from singular tangent planes of a point are tangent to a cone having second class for this vertex.

There are therefore two-dimensional characteristics manifold: what are the solutions in the range reported to the coordinates x^1, x^2, x^3, a partial differential equation of first order ordinary. The bi-characteristics of J. Hadamard are the characteristics of this equation, it is not properly speaking the characteristics of the differential system (4.59), unless the equation (4.67) consists of two linear equations, in which case the partial differential equation of the characteristics consists of two linear equations whose surfaces are the integrals of surfaces generated by characteristic lines of the first family or of the characteristic lines of the second family.

Theorem. *A partial differential equation of second order has an unknown function z of three independent variables can not admit singular solutions than those, if any, that annihilate the partial derivatives of the first member relative to the second order partial derivatives of z. The solutions always admit non-singular two-dimensional characteristics manifolds given by the integration of a partial differential equation of first order, the cone envelope of tangent planes at a given point with characteristic manifolds passing through that point are the second class. If the cone is divided into two straight lines Δ_1, Δ_2,* *there are two families of two-dimensional characteristic manifolds: one constituted by the surfaces of the trajectories of locations lines Δ_1, and the other by the surfaces of the trajectories of locations Δ_2 lines, and these trajectories are the characteristic lines of general integral manifolds of the given equation. Conversely, if the cone is not degenerate, there are no characteristic lines, that is to say, lines in which all elements are singular tangent; bi-characteristics of the J. Hadamard, that is to say the characteristics the partial differential equation of first order which gives the characteristic surfaces, are not strictly characteristic lines, in the sense that the varieties of two-dimensional integrals of the system (30) passing through a bi-characteristic are provided by the theorem of Cauchy-Kowdewski.*

Chapter 5
Differential system in involution

5.1 General facts. Systems in involution

78. In many applications the differential systems we have to regard of the independent variables given $x^1, x^2, \cdots, x^p$. The differential system being set in the form (4.1) of No. 52, it is only interested in $p-$dimensional integral manifolds and, among these, those which not introduce any relation between the variables $x^1, x^2, \cdots, x^p$.

Definition. *A differential system Σ with $n-p$ unknown functions z^λ of p independent variables x^i is said it is in involution if its genus is greater than or equal to p and if the equations defining the generic $p-$dimensional ordinary integral element do not introduce no any linear relation between $dx^1, dx^2, \cdots, dx^p$.*

It is clear that the ordinary $p-$dimensional integral manifolds, with independent variables $x^1, x^2, \cdots, x^p$, can be obtained by applying the existence theorems stated and proved in the preceding chapter.

79. *Systems of partial differential equations*
Any system of differential equations with external independent variables imposed can obviously be written as a system of partial differential equations with $n-p$ unknown functions of p independent variables. The converse is true. Let us suppose, to fix our ideas, a system consisting of a number of relations between the partial derivatives of the three first orders of q unknown functions z^λ. Denoting by $t_i^\lambda, t_{ij}^\lambda, t_{ijk}^\lambda$ this partial derivative, the system will consist of data relationships between variables

$$x^i, \; z^\lambda, \; t_i^\lambda, \; t_{ij}^\lambda, \; t_{ijk}^\lambda, \qquad (i, j, k = 1, 2, \cdots, p; \; \lambda = 1, 2, \cdots, q), \tag{5.1}$$

which we shall associate the Pfaffian equations.

$$dz^\lambda - t_i^\lambda \, dx^i = 0, \quad dt_i^\lambda - t_{ij}^\lambda \, dx^j = 0, \quad dt_{ij}^\lambda - t_{ijk}^\lambda \, dx^k. \tag{5.2}$$

We then add the equations which are deduced from the previous by exterior differentiation. The Σ system thus obtained will not contain differential equation actually exterior of degree greater than 2. We will have that to look for the $p-$dimensional integral manifolds of this system, and among them, those with no relationship between $x^1, x^2, \cdots, x^p$.

We know, moreover, that in the theory of partial differential equations of first order, S. Lie showed that there was interest to remove such latter restriction.

80. We will immediately indicate the condition for a differential system Σ is in involution.

Theorem. *For a closed differential system with $n-p$ unknown functions z^λ of p independent variables $x^1, x^2, \cdots, x^p$, is in involution, it is necessary and sufficient for polar systems of a generic integral point, of a generic integral element with dimension $q \leq p-1$, involve no linear relation between $dx^1, dx^2, \cdots, dx^p$.*

Proof. The condition is clearly necessary. It is sufficient because if satisfied, the equations of ordinary integral element generic $p-$dimensional result in no relationship between $dx^1, dx^2, \cdots, dx^p$. The regular integral elements of $q < p$ dimensions contained in an ordinary $p-$dimensional integral can then lead to more than $p-q$ independent relations between $dx^1, dx^2, \cdots, dx^p$. □

5.2 Reduced characters

81. We will indicate test for involution based on the consideration of what we call *reduced characters*.

First determine the $p-$dimensional integral elements (no linear relation between dx^i, we always assume in the sequel) starting from a generic integral point. If we exclude the existence of such an integral element would require new relations between the dependent and independent variables, in which case the rest of the system would not be in involution. We will assume then the family $\mathscr{F}$ integral elements of dimensions $1, 2, \cdots, p-1$ that may be contained in an $p-$dimensional integral element.

Definition. *We will call polar system reduced by one point integral, or an integral element, the polar system of this or that element, in the equations which we suppress the terms in $dx^1, dx^2, \cdots, dx^p$, keeping only the terms $dz^1, dz^2, \cdots, dz^q$.*

We shall call

$$s'_0, \quad s'_0+s'_1, \quad s'_0+s'_1+s'_2, \quad \cdots, \quad s'_0+s'_1+s'_2+\cdots+s'_{p-1}, \tag{5.3}$$

the rank of the reduced polar system of a generic integral point, of an $1-$dimensional integral element of the family $\mathscr{F}$, of a $2-$dimensional integral element of family $\mathscr{F}$ and so on, respectively.

The non-negative integers $s'_0, s'_1, s'_2, \cdots, s'_{p-1}$, will be called the reduced characters of order $0, 1, \cdots, p-1$. It is clear that the equations are different systems reduced with only polar variables dz^λ, we have

$$s'_0+s'_1+s'_2+\cdots+s'_{p-1} \leq n-p. \tag{5.4}$$

Finally, we introduce the *reduced character s'_p* by the relation

$$s'_0+s'_1+s'_2+\cdots+s'_{p-1}+s'_p = n-p. \tag{5.5}$$

82. Remark. It may happen that the $p-$dimensional integral elements of a point from integral generic form several distinct continuous families. At each of these families is a reduced set of characters. The problem of whether the differential system is given in involution arises differently for these different families, because the $p-$dimensional integral manifolds sought are not the same in different cases, since their $p-$dimensional tangent elements vary from one family to another. It is therefore possible that the given system is in involution for a family of $p-$dimensional integral manifolds and is not in involution for another.

5.3 Necessary and sufficient test of involution

83. We will now give a test of involution.

Necessary and sufficient test of involution. *Let Σ be a closed differential system of $n-p$ unknown functions of p independent variables; Let $s'_0, s'_1, \cdots, s'_{p-1}$ are reduced characters of the system corresponding to the family [or one of the families] of $p-$dimensional integral elements of the system Σ. For this system to be in involution it is necessary and sufficient that the number of independent equations linking the parameters t^λ_i of a p-dimensional generic integral family is equal to*

$$sp'_0+(p-1)s'_1+\cdots+s'_{p-1}. \tag{5.6}$$

If the system is not in involution, the number of these equations is higher.

84. Let's start with a remark, however important. We can assume, if necessary by a linear transformation with constant coefficients performed on $x^1, x^2, \cdots, x^p$, the reduced rank of the polar system of generic linear integral element of the family $\mathscr{F}$ for which $dx^1 = dx^2 d \cdots = dx^p = 0$ is equal to the number normal $s'_0 + s'_1$, that the rank of the system polar reduced the integral element and two dimensions of the generic family $\mathscr{F}$ for which $dx^1 = dx^2 d \cdots = dx^p = 0$ is equal to $s'_0 + s'_1 + s'_2$, and so on.

If we denote then by

$$dz^\lambda = t_1^\lambda \, dx^1 + t_2^\lambda \, dx^2 + \cdots + t_p^\lambda \, dx^p, \tag{5.7}$$

the equations of the generic element of p dimensions of the given family, we see that t_1^λ are subject to satisfy in s_0 linear independent equations, namely s_0 reduced polar equations of generic integral point where dz^λ is replaced by t_1^λ therein. The t_1^λ satisfy these equations, t_2^λ is subject to satisfy $s'_0 + s'_1$ independent linear equations, namely the $s'_0 + s'_1$ polar equations reduced to linear integral element ($\delta x^1 = 1, \delta x^2 = \cdots = \delta x^p = 0, \delta z^\lambda = t_1^\lambda$) where dz^λ is replaced by t_2^λ.

And so on. This shows that there are at least

$$ps'_0 + (p-1)s'_1 + \cdots + s'_{p-1}, \tag{5.8}$$

independent equations to be satisfied by the parameters t_i^λ. This justifies the last part of the statement of the test. We say that these equations, which are well defined, the system is in involution or not, are the equations to the parameters t_i^λ usually satisfied.

85. *Proof of the test.*
Come now to the proof of the main part of the test. Suppose first system in involution. Then all the elements for which the linear integral dx^i are not all zero belong to the family $\mathscr{F}$ and they are subject only to satisfy s'_0 equations involving the remains of any relation between the two; consequently $s_0 = s'_0$.[1] The polar system of regular linear integral element generic is then defined by $s'_0 + s'_1$ linearly independent equations is causing no relation between dx^i; consequently $s'_0 + s'_1 = s_0 + s_1$, where $s'_1 = s_1$, and all $2-$dimensional ordinary integral elements which exist only $p-2$ relations between dx^i belong to the family $\mathscr{F}$. We can continue the argument to the extreme and show that $s'_h = s_h$ for $h = 0, 1, 2, \cdots, p-1$. But then we know (No. 70) that the number of independent equations which satisfied the parameters t_i^λ an ordinary $p-$dimensional integral element is equal to

$$ps_0 + (p-1)s_1 + \cdots + s_{p-1} = ps'_0 + (p-1)s'_1 + \cdots + s'_{p-1} \tag{5.9}$$

The condition of involution set is required.

Conversely, suppose the system not in involution. The parameters s'_i satisfy in $ps'_0 + (p-1)s'_1 + \cdots + s'_{p-1}$ normal equations. But these are not the only ones. Indeed, if all linear integral for which there are only $p-1$ relations between dx^i belong to the family $\mathscr{F}$ is $s'_0 = s_0$, and if all integral elements in two dimensions for which exists $p-2$ relations between dx^i belong to the family $\mathscr{F}$, we have $s'_1 = s_1$, but we

[1] Remember that if s_i are the characters defined in No. 56 and 57, the s'_i are reduced characters.

can not continue this hypotheses to the end, otherwise the system would be in involution. Therefore assume for example that the three-dimensional integrals elements for which there are only $p-3$ relations between dx^i are not all the family {, this means that t_1^λ, t_2^λ and t_3^λ satisfy with other equations that $3s_0'+2s_1'+s_2'$, the normal equations that connect them. As a result the number of independent equations which satisfied the parameters t_i^λ is greater than $ps_0'+(p-1)s_1'+\cdots+s_{p-1}'$. □

Remark. Introducing the highest character as s_p' we can say that the necessary and sufficient condition of involution is the $p-$dimensional integral element in the most general family considered depends on

$$s_1'+2s_1'+\cdots+(p-1)s_{p-1}'+ps_p', \tag{5.10}$$

independent parameters.

86. Example I. Consider the system with two independent variables and three unknown functions, defined by the four equations

$$dx^1\wedge dz^1=0,\quad dx^2\wedge dz^1=0,\quad dx^1\wedge dz^2=0,\quad dx^1\wedge dz^3=0. \tag{5.11}$$

The two-dimensional integral elements form a single irreducible family defined by equations

$$dz^1=0,\quad dz^2=a\,dx^1+b\,dx^2,\quad dz^3=a'\,dx^1+b'\,dx^2. \tag{5.12}$$

with four arbitrary parameters: there are two relations between the six quantities t_1^λ, t_2^λ. Here we have $s_0=0$, and secondly we can take for generic linear element family $\mathscr{F}$ member

$$\delta x^1=\alpha,\quad \delta x^2=\beta,\quad \delta z^1=a\alpha+b\beta,\quad \delta z^2=a'\alpha+b'\beta; \tag{5.13}$$

its polar system is reduced

$$\alpha\,dz^1=0,\quad \beta\,dz^1=0,\quad (a\alpha+b\beta)\,dz^1=0,\quad (a'\alpha+b'\beta)\,dz^1=0; \tag{5.14}$$

we therefore $s_1'=1$, where $s_2'=2$ (for $s_1'+s_2'$ is equal to 3, number unknown functions). But the number of four independent parameters which depends the two dimensional generic integral element is less than of $s_1'+2s_2'=5$. The system is not in involution. one would arrive at the same conclusion if we removed the last equation of the system, would contain more then two unknown functions.

87. Example II. Let the following system, with four unknown functions z^λ of two independent variables x^1,x^2,

$$dx^1\wedge dz^1+dx^2\wedge dz^2=0,\ dx^2\wedge dz^1=0,\ dx^2\wedge dz^3=0,\ dx^3\wedge dz^4=0. \tag{5.15}$$

The two-dimensional integral elements are given by

$$dz^1 = a\,dx^2, \quad dz^2 = a\,dx^1 + b\,dx^2, \quad dz^3 = c\,dx^2, \quad c\,dx^2 \wedge dz^4 = 0. \qquad (5.16)$$

Two cases are distinguished:

1) $c \neq 0$. We then have the family irreducible

$$dz^1 = a\,dx^2, \quad dz^2 = a\,dx^1 + b\,dx^2, \quad dz^3 = c\,dx^2, \quad dz^4 = h\,dx^2, \qquad (5.17)$$

with four parameters a, b, c, h. The polar system reduces the linear element element ($\delta x^1 = \alpha, \delta x^2 = \beta, \delta z^1 = a\beta, \delta z^2 = a\alpha + b\beta, \delta z^3 = c\beta, \delta z^4 = h\beta$) is

$$\alpha\,dz^1 + \beta\,dz^2 = 0, \quad \beta\,dz^1 = 0, \quad \beta\,dz^3 = 0, \quad c\beta\,dz^4 - h\beta\,dz^3 = 0; \quad (5.18)$$

therefore we have $s'_1 = 4$, $s'_2 = 0$. The 4 number of independent parameters of the two-dimensional integral element is equal to $s'_1 + 2s'_2$, and the system is in involution.

2) $c = 0$. We then have the family irreducible

$$dz^1 = a\,dx^2, \quad dz^2 = a\,dx^1 + b\,dx^2, \quad dz^3 = 0, \quad dz^4 = h\,dx^1 + k\,dx^2, \quad (5.19)$$

with 4 independent parameters a, b, h, k. The polar system reduces the linear element ($\delta x^1 = \alpha, \delta x^2 = \beta, \delta z^1 = a\beta, \delta z^2 = a\alpha + b\beta, \delta z^3 = 0, \delta z^4 = h\alpha + k\beta$) is

$$\alpha\,dz^1 + \beta\,dz^2 = 0, \quad \beta\,dz^1 = 0, \quad \beta\,dz^3 = 0, \quad (h\alpha + k\beta)\,dz^3 = 0; \qquad (5.20)$$

we have $s'_1 = 3$, $s'_2 = 1$, as $4 < s'_1 + 2s'_2 = 5$, the system is not year involution.

One can notice that if we did not impose the choice of independent variables, we would have had a system in involution with one family of $2-$dimensional ordinary integral elements, namely the first family considered.

Remark. One might be tempted to extend the test if the s'_i would be defined by means of polar systems reduced integral elements of generic successive dimensions belonging or not belonging to the family $\mathscr{F}$. But the test could fall into default.

This can be seen by the example of No. 86. If we take into fact the system

$$dx^1 \wedge dz^1 = 0, \quad dx^2 \wedge dz^1 = 0, \; dz^1 \wedge dz^2 = 0, \; dz^1 \wedge dz^3 = 0, \qquad (5.21)$$

and they form the polar system reduces the linear integral element element ($\delta x^1 = \alpha, \delta x^2 = \beta, \delta z^1 = t^1, \delta z^2 = t^2, \delta z^3 = t^3$), we obtain

$$\alpha\,dz^1 = 0, \quad \beta\,dz^1 = 0, \quad t^1\,dz^2 - t^2\,dz^1 = 0, \quad t^1\,dz^3 - t^3\,dz^1 = 0, \qquad (5.22)$$

whose rank is $s'_1 = 3$. On the other hand the number of equations to be satisfied by two-dimensional integral elements was seen equal to 2, yet this number is less than with $2s_0 + s'_1 = 3$, one arrives at a result in contradiction with the last part of the test.

If we now take the system in involution

$$dx^1 \wedge dz^1 = 0, \quad dx^2 \wedge dz^1 = 0, \; dz^1 \wedge dz^2 = 0, \tag{5.23}$$

the non-polar system reduces the linear integral ($\delta x^1 = \alpha, \delta x^2 = \beta, \delta z^1 = t^1, \delta z^2 = t^2$) is

$$\alpha\, dz^1 = 0, \quad \beta\, dz^1 = 0, \quad t^1\, dz^2 - t^2\, dz^1 = 0; \tag{5.24}$$

Its rank is $s'_1 = 2$. On the other hand the number of equations to be satisfied settings for the integral two-dimensional is still equal to 2, this time in equal number $2s'_1 + s'_1$, nevertheless the system is not in involution.

5.4 A sufficient test of involution

89. We will now establish a second test just enough of involution, usually useful for applications.

Let a differential system closes at $n-p$ unknown functions of p independent variables. Let a family of irreducible $p-$dimensional integral elements and the family elements of three integrals corresponding to $q = 1, 2, \cdots, p-1$ dimensions.

Second sufficient test of involution. *Denote by $\sigma_0 = s'_0$ the rank of the system polar reduced by one point integral generic $\sigma_0 + \sigma_1$, the rank of the polar system of the linear element reduces generic family $\mathscr{F}$ to which $\delta x^2 = \delta x^3 = \cdots = \delta x^p = 0$, by $\sigma_0 + \sigma_1 + \sigma_2$, the rank of the polar system of the element reduces with two dimensions the most general the family $\mathscr{F}$ of for which $\delta x^3 = \cdots = \delta x^p = 0$, and so on. The system is in involution if the number of independent equations that govern the parameters of the considered family of $p-$dimensional integral elements is equal to $p\sigma_0 + (p-1)\sigma_1 + \cdots + \sigma_{p-1}$.*

Proof. We know by the first test that the total number of independent equations which satisfied the parameters of the element integral generic $p-$dimensional family is considered at least equal to $ps'_0 + (p-1)s'_1 + \cdots + s'_{p-1}$. We therefore

$$p\sigma_0 + (p-1)\sigma_1 + \cdots + \sigma_{p-1} \geq ps'_0 + (p-1)s'_1 + \cdots + s'_{p-1}. \tag{5.25}$$

Now we have the obvious inequality (which is actually the first equality)

$$\begin{aligned} \sigma_0 &\leq s'_0, \\ \sigma_0 + \sigma_1 &\leq s'_0 + s'_1, \\ \sigma_0 + \sigma_1 + \sigma_2 &\leq s'_0 + s'_1 + s'_2, \\ &\vdots \\ \sigma_0 + \sigma_1 + \cdots + \sigma_{p-2} + \sigma_{p-1} &\leq s'_0 + s'_1 + \cdots + s'_{p-2} + s'_{p-1}, \end{aligned} \tag{5.26}$$

that result by the addition

$$p\sigma_0 + (p-1)\sigma_1 + \cdots + \sigma_{p-1} \;\le\; ps'_0 + (p-1)s'_1 + \cdots + s'_{p-1}. \tag{5.27}$$

It follows that:

1) that the two members of the last inequality are equal and that consequently all the above inequalities reduce to equalities ($\sigma_i = s'_i$);
2) the system is in involution.

We may add that the elements integral to q dimensions of the family $\mathscr{F}$ for whom $\delta x^{q+1} = \delta x^{q+2} = \cdots = \delta x^p = 0$ are regular. □

91. *Additional remark.*
The second test would continue to be valid if the whole were calculated SIGMAI leaving aside one or more differential equations of the system gives, this could in effect lower the numerical value of integers σ_0, $\sigma_0 + \sigma_1$, $\sigma_0 + \sigma_1 + \sigma_2$ etc.., and given the proof of the test would not cease to be valid. It goes without saying that if we wish to take advantage of this remark, it is essential to take into account all; system of equations to determine the $p-$dimensional integral elements.

A particularly interesting case, because it occurs quite usually in applications, or dz^λ is not included in the first degree differential equations in the external tendering system gives firm. Indeed in this case the reduced characters can be computed without knowing previously the elements $p-$dimensional integrals. This is what the systems can be reduced polar forms directly without this prior knowledge.

Let for example

$$\begin{cases} f_\alpha(x,z) = 0 & (\alpha = 1,2,\cdots,r_0), \\ \theta_\alpha \equiv A_{\alpha i} dx^i + A_{\alpha\lambda}\, dz^\lambda = 0 & (\alpha = 1,2,\cdots,r_1), \\ \phi_\alpha \equiv \frac{1}{2} A_{\alpha ij}\, dx^i \wedge dx^j + A_{\alpha i\lambda}\, dx^i \wedge dz^\lambda = 0, & (\alpha = 1,2,\cdots,r_2) \\ \psi_\alpha \equiv \frac{1}{6} A_{\alpha ijk}\, dx^i \wedge dx^j \wedge dx^k & , \\ \qquad\qquad + \frac{1}{2} A_{\alpha ij\lambda}\, dx^i \wedge dx^j \wedge dz^\lambda = 0, & (\alpha = 1,2,\cdots,r_3) \\ \cdots\cdots\cdots\cdots \end{cases} \tag{5.28}$$

the equations of the system. The reduced character $s'_0 + s'_1$, is the rank of the system

$$A_{\alpha\lambda}\, dz^\lambda = 0, \tag{5.29}$$

if the sum s +, is the ranking system

$$A_{\alpha\lambda}\, dz^\lambda = 0, \qquad A_{\alpha i\lambda}\, \delta_1 x^i dz^\lambda = 0, \tag{5.30}$$

where $\delta_1 x^i$ are arbitrary parameters. The reduced character $s'_0 + s'_1 + s'_2$, is the rank of the system

$$A_{\alpha\lambda}\, dz^\lambda = 0, \qquad\qquad A_{\alpha i\lambda}\, \delta_1 x^i dz^\lambda = 0,$$
$$A_{\alpha i\lambda}\, \delta_2 x^i dz^\lambda = 0, \qquad\qquad A_{\alpha ij\lambda}\, \delta_1 x^i \delta_2 x^j dz^\lambda = 0, \tag{5.31}$$

where $\delta_1 x^i$ and $\delta_2 x^i$are arbitrary parameters, and so on.

Note, moreover, that in this case there is one family of irreducible $p-$dimensional integral elements, defined by relations

$$\begin{aligned} &A_{\alpha i}+A_{\alpha\lambda}\, t_i^\lambda = 0, \\ &\qquad (\alpha = 1,2,\cdots,r_1;\ i = 1,2,\cdots,p), \\ &A_{\alpha ij}+A_{\alpha i\lambda}\, t_j^\lambda - A_{\alpha j\lambda}\, t_i^\lambda = 0, \\ &\qquad (\alpha = 1,2,\cdots,r_2;\ i,j = 1,2,\cdots,p), \\ &A_{\alpha ijk}+A_{\alpha ij\lambda}\, t_k^\lambda - A_{\alpha ik\lambda}\, t_j^\lambda + A_{\alpha j\lambda}\, t_i^\lambda = 0, \\ &\qquad (\alpha = 1,2,\cdots,r_3;\ i,j,k = 1,2,\cdots,p). \end{aligned} \tag{5.32}$$

5.5 Case of two independent variables

93. If a closed differential system has only two independent variables, it will not appear in this system no exterior differential equation of degree greater than 2. It is therefore not necessary to deal with equations from the exterior differentiation of quadratic equations that the system may contain, for they are always satisfied by any two-dimensional plane element

We assume for simplicity, which essentially does not restrict the generality, the system has no finite equation. We will write s_0 independent linear equations as

$$\theta_\alpha = 0, \qquad (\alpha = 1,2,\cdots,s_0). \tag{5.33}$$

We then introduce with the differentials dx, dy independent variables, $n-s_0-2$ linear differential forms ($\lambda = 1,2,\cdots,n-s_0-2$) are mutually independent and independent of s_0+2 forms, θ_α, dx, dy. There. They thus form, with the θ_α, $n-2$ independent forms over differential unknown functions. Finally we can take, instead of dx, dy, two independent linear combinations of these ω^1, ω^2 differential, which can be convenient in applications.

This granted the given system will be in the form

$$\begin{cases} \theta_\alpha = 0, & (\alpha = 1,2,\cdots,s_0), \\ \phi_\alpha \equiv C_\alpha\, \omega^1\wedge\omega^2 + A_{\alpha\lambda}\, \omega^1\wedge\varpi^\lambda & \\ \qquad + B_{\alpha\lambda}\, \omega^2\wedge\varpi^\lambda + \dfrac{1}{2} D_{\alpha\lambda\mu}\, \varpi^\lambda\wedge\varpi^\mu, & (\alpha = 1,2,\cdots,r). \end{cases} \tag{5.34}$$

We only concern ourselves with the case or there are integral elements in two dimensions, which would, moreover, by replacing ω^λ less by ϖ^λ a linear combination of ω^1, ω^2, to assume the all coefficients C_α are zero. We therefore exclude systems that are not in involution.

If we know the general equations

$$\theta_\alpha = 0, \qquad \varpi^\lambda = t_1^\lambda\, \omega^2 + t_2^\lambda\, \omega^2, \tag{5.35}$$

integral components in two dimensions, we know the family of nine linear integral elements contained in an integral element in two dimensions. This granted, the polar system reduces an integral linear $(\omega_\delta^i, \varpi_\delta^i)$ of the family $\mathscr{F}$ consists of eight equations $\theta_\alpha = 0$ and equations

$$(A_{\alpha\lambda}\, \omega_\delta^1 + B_{\alpha\lambda}\, \omega_\delta^1)\varpi^\lambda + D_{\alpha\lambda\mu} \varpi_\delta^\lambda \varpi^\mu = 0. \tag{5.36}$$

The coefficient matrix of the polar system, or polar matrix, is none other than the matrix of partial derivatives $\partial\phi_\alpha / \partial\varpi_\delta^\lambda$, when we replace ω_δ^i, ϖ_δ^i by ω^i, ϖ^i, the matrix with r rows and $\nu = n - s_0$ columns

$$\left(\frac{\partial\phi_\alpha/\partial}{\varpi_\delta^\lambda} \right). \tag{5.37}$$

The reduced character s_1', we now write s_1, so that there will be no confusion to worry about, is the rank of the matrix polar linear elements not regular, or singular, are those who annihilate all determinants with s_1 row and $s-1$ columns of this matrix.

94. *Test of involution.*
We obtain immediately a sufficient test of involution by noting that if the reduced s_1 character, is equal to the number r of linearly independent forms ϕ_α the conditions with which the coefficients t_1^λ, t_2^λ general equations (5.36) of the two-dimensional elements integral are reduced to s_1 conditions

$$(A_{\alpha\lambda} t_2^1 - B_{\alpha\lambda} t_1^\lambda + D_{\alpha\lambda\mu} t_1^\lambda t_2^\mu = 0, (\alpha = 1, 2, \cdots, s_1). \tag{5.38}$$

These conditions are necessarily independent, since the number of independent relations between t_1^λ and t_2^λ is at least equal to s_1.

On the other hand it is a case where this sufficient test is also necessary is the one where ϖ^λ come linearly in the forms ϕ_α. Indeed in this case the equations to be satisfied by t_1^λ and t_2^λ are

$$C_\alpha + A_{\alpha\lambda} t_2^1 - B_{\alpha\lambda} t_1^\lambda = 0, \tag{5.39}$$

and it is clear that there are as many equations of this system there are linearly independent forms ϕ_α linearly independent (taking into account the assumption made once and for all that these equations are compatible).

We thus arrive to the following theorem.

Theorem. *The sufficient condition for a differential system closed two independent variables is in involution is reduced as the character s, equals the number of quadratic forms p linearly independent. This condition is also necessary if the forms ϕ_α contain the first level forms ϖ^λ (if $D_{\alpha\lambda\mu}$ coefficients are all zero).*

The following example shows that the condition is not always necessary. Consider the system of three exterior differential equations quadratic form

$$\varpi^2 \wedge \varpi^3 = 0, \qquad \varpi^3 \wedge \varpi^1 = 0, \qquad \varpi^1 \wedge \varpi^2 = 0. \tag{5.40}$$

The integral elements in two dimensions are given by the equations

$$\varpi^1 = a_1\,\omega^1 + b_1\,\omega^2, \qquad \varpi^2 = a_2\,\omega^1 + b_2\,\omega^2, \qquad \varpi^3 = a_3\,\omega^1 + b_3\,\omega^2, \tag{5.41}$$

with

$$a_2 b_3 - b_2 a_3 = 0, \qquad a_3 b_1 - b_3 a_1 = 0, \qquad a_1 b_2 - b_1 a_2 = 0; \tag{5.42}$$

the number of independent parameters on which they depend is 4 (a_1, a_2, a_3 are arbitrary and b_1, b_2, b_3 they are proportional). On the other hand the matrix is polar

$$\begin{pmatrix} 0 & \varpi^3 & -\varpi^2 \\ -\varpi^3 & 0 & \varpi^1 \\ \varpi^2 & -\varpi^1 & 0 \end{pmatrix}; \tag{5.43}$$

its rank is equal to 2: $s_1 = 2$ and $s_2 = 1$. It has $s_1 + 2s_2 = 2 + 2 = 4$, number of parameters independent of the integral element generic two-dimensional. However the number of linearly independent forms ϕ_α is $3 > s_1$.

Case $s_2 = 0$. Characteristics. - In the case $s_2 = 0$, that is, say $s_0 = n - s_1 - 2$, the number of forms ϖ^λ is equal to s_1 if the system is in involution. For any one-dimensional integral manifold uncharacteristic he spent a two-dimensional integral manifold and a single. The characteristic lines of a variety ordinary integral cancel all determinants s_1 row and s_1 columns of the matrix polar. In the case where s is equal to the number of linearly independent ϕ_α forms, the polar matrix has exactly s_1 row and s_1 columns, so that the characteristic lines given of an integral manifold is provided by an homogeneous equation of degree s_1 of ω^1, ω^2, that is to say, dx, dy.

In particular, take a case where the coefficients $D_{\alpha\lambda\mu}$ equations (5.36) are zero, that is to say, ϖ^λ come linearly in the ϕ_α. In this case lea matrix elements are polar $A_{\alpha\lambda}\,\omega^1 + B_{\alpha\lambda}\,\omega^2$. You can put quadratic equations $\phi_\alpha = 0$ in a form highlighting the remarkable s_1 family of characteristics, at least when these families are distinct.

Let indeed $= 0$ equation of one of these families. The coefficient m is the root of the equation

$$|A_{\alpha\lambda} + mB_{\alpha\omega^2 - m\omega^\lambda}| = 0, \qquad (\alpha, \lambda = 1, 2, \cdots, s_1). \tag{5.44}$$

Looking for a linear combination of equations $\phi_\alpha = 0$ which, the first member contains factor $\omega^2 - m\omega^1$. If $k^\alpha \phi_\alpha = 0$ is such a combination is that we will, whatever $\lambda = 1, 2, \cdots, s_1$,

$$k^\alpha \left(A_{\alpha\lambda} + mB_{\alpha\omega^2 - m\omega^\lambda}\right) = 0. \tag{5.45}$$

It is possible to find values for k^α not all zero satisfying in these s_1 homogeneous equations as the determinant of the coefficients of the unknowns is zero. A quadratic equations of the exterior differential system will be given in the form

$$(\omega^2 - m\omega^1) \wedge (c_\lambda \varpi^\lambda) = 0. \tag{5.46}$$

Result if s_1 families of characteristics are distinct and are given by equations $\omega^2 - m_i\omega^1 = 0$ $(i = 1, 2, \cdots, s_1)$, quadratic equations of the system may be placed under a form[2]

$$(\omega^2 - m\omega^1) \wedge (c_{i\lambda} \varpi^\lambda) = 0, \qquad (i = 1, 2, \cdots, s_1). \tag{5.47}$$

These equations show a very interesting fact. When one gives to determine an integral manifold ordinary solution a characteristic dimension, the problem is usually impossible. This result is obvious from equations (5.47), because if one $\omega^2 = m_i\omega^1$ bones along the curve given, it is necessary that along this curve we have also

$$c_{i\lambda} \varpi^\lambda = 0, \tag{5.48}$$

since the two-dimensional integral manifold of unknown form $\omega^2 - m\omega^1$ must be a multiple of $\omega^2 - m_i\omega^1$. The question of whether this necessary condition is also sufficient remains outstanding; may the rest if it is sufficient, the problem has infinite solutions. This is a point of the theory which has been little studied and on which we know little.

5.6 Systems in involution whose general solution depends only on an arbitrary function of one variable

We can assume, without loss of generality, that the system has no finite equations. If it is p independent variables, we denote by $\omega^1, \omega^2, \cdots, \omega^p$ a system of independent linear combinations of p their differential. are respectively

[2] We assumed the coefficients C_α zero, which is always feasible to ϖ^λ adding suitable linear combinations of ω^1, ω^2.

$$\begin{cases} \theta_\alpha = 0, & (\alpha = 1, 2, \cdots, s_0), \\ \phi_\alpha \equiv A_{\alpha i \lambda}\, \omega^i \wedge \varpi^\lambda = 0, & (\alpha = 1, 2, \cdots, r). \end{cases} \tag{5.49}$$

those of the system of equations which are the first and second degree. We are doing the hypothesis that in the forms ϕ_α the ϖ^λ not included in the first degree. We have not written in terms of $\omega^i \wedge \omega^j$ because the system is in involution, admits integral components of p dimensions, and that consequently, by adding them ϖ^λ linear combinations of ω^i can arrange to cancel coefficients of the products $\omega^i \wedge \omega^j$.

The system may contain equations of degree greater than 2, but there is no need to write them.

We intend to indicate a remarkable form of the equations $\phi_\alpha = 0$ and deduce important consequences in relation to characteristics of given differential system.

$$A_{\alpha i \lambda}\, \varpi^\lambda = 0, \tag{5.50}$$

Let s_1 is the character of the reduced first order and following characters are all zero by hypothesis. The number of forms ϖ^λ independent of each other and independent of θ_α and ω^i is equal to s_1. It may be presumed, if necessary by a suitable linear transformation of the half, the rank of the system

is equal to s_1 and even, by a linear transformation on the forms ϕ_α, that we have

$$\begin{cases} A_{\alpha 1 \lambda}\, \varpi^\lambda \equiv \varpi^\alpha, & (\alpha = 1, 2, \cdots, s_1), \\ A_{\beta 1 \lambda}\, \varpi^\lambda = 0, & (\alpha = s_1 + 1, \cdots, r). \end{cases} \tag{5.51}$$

This granted, saying that the system is in involution with $s_3 = s_2 = \cdots = s_p = 0$, this means that there are exactly $(p-1)s_1$ relations between the coefficients of equations

$$\varpi^\alpha = t_i^\alpha\, \omega^i, \tag{5.52}$$

that give the $p-$dimensional generic integral element. These relations are necessarily

$$t_i^\alpha = A_{\alpha i} t_i^\alpha, \qquad (\alpha = 1, 2, \cdots, s_1;\, i = 2, 3, \cdots, p). \tag{5.53}$$

This results in particular there's no more than s_1 forms ϕ^α linearly indenpedent $(r = s_1)$, because the consideration of the form ϕ_{s_1+1}, in which ω^1 not listed, give

$$A_{s_1+1, i, \lambda}\, t_1^\lambda = 0, \qquad (i = 2, 2, \cdots, p), \tag{5.54}$$

which would introduce relations between t_1^λ that can not be deduced from (5.53).

Now form the determinant of the matrix polar linear element (ω_δ^i), which is a homogeneous form of degree s_1 of $\omega^1, \omega^2, \cdots, \omega^p$. Assume, to stay in the general case, as for $\omega^3 = \cdots = \omega^p = 0$, the determinant decomposes into a product of s_1 distinct linear forms in ω^1, ω^2. On the reasoning after the No. 96, we see that we can, if necessary by a suitable linear substitution on the ϕ_α and ϖ^α, assuming

$$\phi_\alpha \equiv (\omega^1 - m_\alpha \omega^\alpha) \wedge \varpi^\alpha + A_{\alpha i\beta}\, \omega^i \wedge \varpi^\beta),$$
$$(\alpha = 1, 2, \cdots, s_1;\ i = 1, 2, \cdots, p). \tag{5.55}$$

Is deduced by expressing that the two forms $\varpi^\alpha = t_i^\alpha \omega^i$ annihilate the forms ϕ_α especially in the forms ϕ_α of coefficient the $\omega^2 \wedge \omega^i$ is zero,

$$m_\alpha t_i^\alpha = A_{\alpha i\beta}\, t_2^\beta, \tag{5.56}$$

whence, taking into account the values (5.53) of t_3^β and t_i^α,

$$m_\alpha A_{\alpha i\beta}\, t_1^\beta = A_{\alpha i\beta}\, m_\beta\, t_2^\beta, \tag{5.57}$$

and at last, we have

$$A_{\alpha i\beta} = 0 \quad \text{for} \quad \alpha \neq \beta. \tag{5.58}$$

By asking $A_{\alpha i\alpha} = m_{i\alpha}$, and, for reasons of symmetry, $m_\alpha = m_{2\alpha}$, we have finally

$$\phi_\alpha \equiv (\omega^1 - m_{i\alpha}\omega^i) \wedge \varpi^\alpha = 0. \tag{5.59}$$

This is the remarkable form which are capable of reaching the exterior quadratic equations given differential system.

99. The characteristic lines of integral manifolds are those which in each of their points are tangential to one of the s_1 $(p-1)-$dimensional planar elements defined by the equations

$$\omega^1 - m_{i\alpha}\omega^i = 0, \qquad (\alpha = 1, 2, \cdots, s_1). \tag{5.60}$$

These $(p-1)-$dimensional elements to have a remarkable property, according the following theorem:

Theorem. *Considered a manifold on integral given, set of all equations*

$$\omega^1 - m_{i\alpha}\omega^i = 0, \tag{5.61}$$

is completely integrable.

Proof. Before to the proof, we will put for short writing,

$$\omega^1 - m_{i\alpha}\omega^i = \bar{\omega}^\alpha, \qquad (\alpha = 1, 2, \cdots, s_1); \tag{5.62}$$

$\bar{\omega}^\alpha$ are s_1 the distinct forms of $\omega^1, \omega^2, \cdots, \omega^p$ that can naturally not be linearly independent.

This note first that put on an integral manifold we have relations of the form

$$\varpi^\alpha = t^\alpha \bar{\omega}^\alpha, \tag{5.63}$$

consequences of equations (5.59). On the other hand the equation $d\phi_\alpha = 0$ is a consequence differential equations of the given system, which is closed, the exterior differential $d\phi^alpha$ is identically zero when accounting equations $\theta_\beta = 0$ and when we replace ϖ^λ by $t^\lambda \bar{\omega}^\lambda$ (the system of equations which are of degree greater than 2 are indeed identically verified under the above conditions).

Now fix the index α. We have, by $\phi_\alpha = \bar{\omega}^\alpha \wedge \varpi^\alpha$,

$$d\phi_\alpha = d\bar{\omega}^\alpha \wedge \varpi^\alpha - \bar{\omega}^\alpha \wedge d\varpi^\alpha. \tag{5.64}$$

Suppose we have, taking into account the equations $\theta + \beta = 0$,[3]

$$\begin{cases} d\bar{\omega}^\alpha = \frac{1}{2} a_{ij}\, \omega^i \wedge \omega^j + a_{i\lambda}\, \omega^i \wedge \omega^\lambda, \\ d\varpi^\alpha = \frac{1}{2} c_{ij}\, \omega^i \wedge \omega^j + c_{i\lambda}\, \omega^i \wedge \varpi^\lambda + \frac{1}{2} c_{\lambda\mu}\, \varpi^\lambda \wedge \varpi^\mu; \end{cases} \tag{5.65}$$

the exterior differential $d\bar{\omega}^\alpha$ does not contain the term in the equations $\varpi^\lambda \wedge \varpi^\mu$ because $\omega^1 = \omega^2 = \cdots = \omega^p = 0$ form a completely integrable system.[4]

We have, according to (5.64) and (5.65),

$$\begin{aligned} d\varpi^\alpha = \frac{1}{2} a_{ij}\, \omega^i \wedge \omega^j \wedge \varpi^\alpha + a_{i\lambda}\, \omega^i \wedge \varpi^\lambda \wedge \varpi^\alpha - \frac{1}{2} c_{ij}\, \omega^i \wedge \omega^j \wedge \bar{\omega}^\alpha \\ + c_{i\lambda}\, \omega^i \wedge \bar{\omega}^\alpha \wedge \varpi^\lambda - \frac{1}{2} c_{\lambda\mu}\, \bar{\omega}^\alpha \wedge \varpi^\lambda \wedge \varpi^\mu; \end{aligned} \tag{5.66}$$

we must have, whatever arbitrary parameters t^λ,

$$\begin{aligned} \frac{1}{2} t^\alpha a_{ij}\, \omega^i \wedge \omega^j \wedge \bar{\omega}^\alpha + t^\alpha t^\lambda a_{i\lambda}\, \omega^i \wedge \bar{\omega}^\lambda \wedge \bar{\omega}^\alpha \\ - \frac{1}{2} c_{ij}\, \omega^i \wedge \omega^j \wedge \bar{\omega}^\alpha - \frac{1}{2} t^\lambda t^\mu c_{\lambda\mu}\, \bar{\omega}^\alpha \wedge \bar{\omega}^\lambda \wedge \bar{\omega}^\mu = 0. \end{aligned} \tag{5.67}$$

Equating to zero the coefficient of t^α and also $t^\alpha t^\beta$, where we assume $\beta \neq \alpha$, we obtain

$$a_{ij}\, \omega^i \wedge \omega^j \wedge \bar{\omega}^\alpha = 0, \qquad a_{i\beta}\, \omega^i \wedge \bar{\omega}^\beta \wedge \bar{\omega}^\alpha = 0, \tag{5.68}$$

equations in the second of which should not summon from β. These equations express that any integral manifold on the form vanishes $d\bar{\omega}^\alpha$ considering $\bar{\omega}^\alpha = 0$. This equation is completely integrable. □

100. The previous theorem can be stated as follows:

[3] Of course the coefficients $a_{ij}, a_{i\lambda}, \cdots$, equations (5.65) vary with the fixed index α.

[4] The forms ω^i are independent linear combinations of $dx^1, dx^2, \cdots, dx^p$

Theorem. *Any differential system in involution with p independent variables whose general solution depends only on s_1 arbitrary functions of one variable s_1 is generally accepted families of characteristics varieties with $p-1$ dimensions such that it passes one and only one variety of each family through a point of an integral manifold, all the curves plotted on a variety of these characteristics are themselves characteristics.*

Remark. If s_1 families of linear characteristics were not distinct, the reduced form (5.59) quadratic equations, exterior differential of the system would be less easy. One can show that the polar matrix with s_1 columns and s_1 rows could be reduced so that all elements above the main diagonal are zero, this means that the determinant of the matrix is composed of a polar product of s_1 forms linear $\omega^1, \omega^2, \cdots, \omega^p$.

We will finish this chapter with an application of theory of systems in involution in properties of Pfaffian systems.[5]

5.7 A theorem of J. A. Schouten and W. Van der Kulk

101. This theorem relates a generic system of linear equations to total differential (Pfaffian system). Let

$$\theta_\alpha = 0, \qquad (\alpha = 0, 1, 2, \cdots, q) \tag{5.69}$$

a system of $q+1$ linearly independent equations of n variables $x^1, x^2, \cdots, x^n$ as dependent than independent. F. Engle has focused attention on a new invariant of a digital Pfaffian system, namely, in the present case, the largest integer m such that the exterior form

$$\theta_0 \wedge \theta_1 \wedge \cdots \wedge \theta_q \wedge (\lambda_0\,\theta_0 + \lambda_1\,\theta_1 + \cdots + \lambda_q\,\theta_q). \tag{5.70}$$

of degree $2m+q+1$, is not identically zero; where $\lambda_0, \lambda_1, \cdots, \lambda_q$, are arbitrary parameters.

The theorem in question is stated in the following manner:

[5] Once we returned the forms $A_{\alpha 1\lambda}\varpi^\lambda$ to ϖ^α, we can look at the s_1^2 coefficients $A_{\alpha 1\lambda}$, where i is set greater than 1, as elements of a matrix S_i The condition of involution of the system, which is expressed by the fact that relations $A_{\alpha i\lambda}\,t_i^\lambda = A_{\alpha j\lambda}\,t_i^\lambda$, are consequences of equations (5.53), where t_1^λ, are bound by any relation, is simply equivalent to the property of $p-1$ matrices $S_2, \cdots, S_p$, to be exchangeable between them. It follows from matrix theory Exchangeable that the characteristic equation of matrix $u^1S_1 + u^2S_2 + \cdots + u^pS_p$, where u^i are parameters, has all its roots linear in $u^2, u^3, \cdots, u^p$. As the matrix S_1 is unit matrix, this means that the determinant of the matrix $u^2S_2 + u^3S_3 + \cdots + u^pS_p$ is the product of linear forms in $u^1, u^2, \cdots, u^p$. By replacing the u^i by ω^i, we get the result of the text.

Theorem. *If m is the invariant of the system of E. Engel (5.69), it is possible to find an algebraically equivalent system whose members are all first class* $2m+1$.

It is clear that if $2m+q$ is at least equal to n, the form (5.70) is identically zero, we therefore surely $2m < n-q$.

102. To equations the problem of determining the system algebraically equivalent to the given system possessing the property specified, assume it is always possible that the form

$$\theta_0 \wedge \theta_1 \wedge \cdots \wedge \theta_q \wedge (d\theta_0)^m, \tag{5.71}$$

is not identically zero, and let

$$\Theta = \theta_0 + u^1\,\theta_1 + u^2\,\theta_2 + \cdots + u^q\,\lambda_q, \tag{5.72}$$

where $u^1, u^2, \cdots, u^q$ denote unknown functions of the variables $x^1, x^2, \cdots, x^n$.

We will determine these unknown functions so that the form Θ of class or $2m+1$. This condition is expressed by the equation of degree $2m+3$

$$\theta \wedge (d\theta_0)^{m+1} = 0, \tag{5.73}$$

which should be added that thus obtained by exterior differentiation, i.e.

$$(d\theta_0)^{m+2} = 0. \tag{5.74}$$

Hypothetically, if you look $u^1, u^2, \cdots, u^p$ as constants, the exterior differential $d\Theta$ is reducible ($\mod \theta_0, \theta_1, \cdots, \theta_q$) to

$$\omega^1 \wedge \omega^2 + \omega^3 \wedge \omega^4 + \cdots + \omega^{2m-1} \wedge \omega^{2m}, \tag{5.75}$$

where ω^i are $2m$ linearly independent differential forms built with x^i variables and their differential coefficients may depend naturally $u^1, u^2, \cdots, u^q$.

We consequently looking now u^i as variables,

$$d\Theta \equiv \omega^1 \wedge \omega^2 + \cdots + \omega^{2m-1} \wedge \omega^{2m} + \theta_1 \wedge \varpi^1 + \cdots + \theta_q \wedge \varpi^q \quad \mod \Theta, \tag{5.76}$$

where $\varpi^i + du^i$ is a linear form in $dx^1, dx^2, \cdots, dx^n$.

103. We will first look for the integral elements of the generic n-dimensional system (5.73), (5.74). Equation (5.73) expresses that is due ($\mod \Theta$) reducible to a quadratic form built with $2m$ independent linear exterior forms, i.e. that the second member of the congruence (5.76) is, when we replace ω^i by their values , reducible ($\mod \Theta$) to

$$\varpi^1 \wedge \varpi^2 + \varpi^3 \wedge \varpi^4 + \cdots + \varpi^{2m-1} \wedge \varpi^{2m}, \tag{5.77}$$

where ϖ^i is the sum of and a linear combination of forms $\theta_0, \theta_1, \cdots, \theta_q$ and $n-2m-q-1$ other independent forms ω^i and θ_α, or $\omega^{2m+1}, \cdots, \omega^{n-q-1}$. But we immediately see that second member of (5.76) is incompatible with the presence of these $n-q-1$ recent forms. It was therefore finally congruence

$$\begin{aligned} d\Theta \equiv (\omega^1 + a^{1\alpha}\,\theta_\alpha) \wedge (\omega^2 + a^{2\alpha}\,\theta_\alpha) + \cdots \\ \cdots + (\omega^{2m-1} + a^{2m-1,\alpha}\,\theta_\alpha) \wedge (\omega^{2m} + a^{2m,\alpha}\,\theta_\alpha) \quad \mathrm{mod}\, \Theta, \end{aligned} \tag{5.78}$$

The result, taking the coefficients of $theta_0, \theta_1, \cdots, \theta_q$, developed in the second member,

$$\begin{aligned} \varpi^\alpha = a^{1\alpha}\,\omega^2 - a^{2\alpha}\,\omega^1 + \cdots + a^{2m-1,\alpha}\,\omega^{2m} \\ - a^{2m,\alpha}\,\omega^{2m-1} + b^{\alpha\lambda}\,\theta_\lambda + c^\alpha\,\Theta \qquad (\alpha = 1, 2, \cdots, q). \end{aligned} \tag{5.79}$$

Identification with (5.76) gives

$$\begin{aligned} b^{\alpha\beta} - b^{\beta\alpha} = & a^{1\alpha} a^{2\beta} - a^{1\beta} a^{2\alpha} + \cdots \\ & \cdots + a^{2m-1,\alpha} a^{2m,\beta} - a^{2m,\alpha} a^{2m-1,\beta} \qquad (\alpha, \beta = 1, 2, \cdot s, q). \end{aligned} \tag{5.80}$$

The element of n dimensions defined by the equations (5.79) where the coefficients satisfy the relations (5.80), satisfies the equation (5.73). It is automatically satisfied with the equation (5.74), because the shape $d\Theta$ being, according to (5.73), the sum of $m+1$ independent quadratic monomials at most power $(m+2)$th is identically zero.

The number of independent parameters on which the full element most general n-dimensional system of the form (5.73), (5.74) is then equal to the number of parameters has $2mq$, increases the number $q(q+1)/2$ of independent parameters $b^{\alpha\beta}$ [q^2 parameters with lids the $q(q11)/2$ relations (5.80)] increased the number finally q parameters c^α, giving a total number of independent parameters equal to

$$2mq + \frac{q(q+3)}{2}. \tag{5.81}$$

104 Now come to the determination of the reduced system of differential characters. Equation (5.73) is of degree $2m+3$, all elements to $2m+1$ dimensions are integral. We therefore have

$$s_0 = s_1 = s_2 = \cdots = s_{2m+1} = 0. \tag{5.82}$$

We now consider the following string of elements integral $E_{2m+2}, \cdots, E_{n-1}$, which introduce successively the following relations between the differentials of n independent variables:

$$
\begin{array}{lcll}
E_{2m+2} & : & \omega^{2m+1} = \cdots = \omega^{n-q-1} = 0, & \theta_2 = \theta_3 = \cdots = \theta_q = 0; \\
E_{2m+3} & : & \omega^{2m+1} = \cdots = \omega^{n-q-1} = 0, & \theta_3 = \cdots = \theta_q = 0; \\
\vdots & & \vdots & \vdots \\
E_{2m+q} & : & \omega^{2m+1} = \cdots = \omega^{n-q-1} = 0, & \theta_q = 0; \\
E_{2m+q+1} & : & \omega^{2m+1} = \cdots = \omega^{n-q-1} = 0, & \\
\vdots & & \vdots & \\
E_{2m+q+1} & : & \omega^{n-q-1} = 0, &
\end{array}
\tag{5.83}
$$

To form the polar system reduces each of these integrals, we'll use that from equation (5.73), we then obtain the rank $\sigma_{2m+2}, \sigma_{2m+2} + \sigma_{2m+3}, \cdots$. Which will be at most to the ranks $\sigma_{2m+2}, \sigma_{2m+2} + \sigma_{2m+3}, \cdots$. According to the sufficient test of No. 29, the differential system in involution will certainly be given if the number (5.80) independent parameters of the generic integral element is equal to n dimensions

$$
(2m+2)\sigma_{2m+2} + (2m+3)\sigma_{2m+3} + \cdots + n\sigma_n. \tag{5.84}
$$

105. To form the polar system of reduced E_{2m+2} for which we

$$
\omega^{2m+1} = \omega^{2m+2} = \cdots = \omega^{n-q-1} = 0, \qquad \theta_2 = \theta_3 = \cdots = \theta_q = 0, \tag{5.85}
$$

simply, in the calculation of $(d\Theta)^{m+1}$ to take into account only in terms of ω^1, $\omega^2, \cdots, \omega^{2m}$, θ_1, which reduces to $(d\Theta)^{m+1}$ a numerical factor meadows $\omega^1 \wedge \omega^2 \wedge \cdots \wedge \omega^{2m} \wedge \theta_1 \mod \Theta$ and hence $\Theta(d\Theta)^{m+1}$ to $\omega^1 \wedge \omega^2 \wedge \cdots \wedge \omega^{2m} \wedge \theta_1 \mod \Theta$, and we have $\sigma_{2m+2} = 1$.

The calculation of the polar system is reduced by reducing $d\Theta \mod \Theta$ to

$$
\omega^1 \wedge \omega^2 + \cdots + \omega^{2m-1} \wedge \omega^{2m} + \cdots + \theta_1 \wedge \varpi^1 + \theta_2 \wedge \varpi^2, \tag{5.86}
$$

from which equations

$$
\varpi^1 = 0, \qquad \varpi^2 = 0; \tag{5.87}
$$

was therefore $\sigma_{2m+2} + \sigma_{2m+3} = 2$, thus $\sigma_{2m+3} = 1$.

We will continue and step by step and you will find

$$
\sigma_{2m+4} = \cdots = \sigma_{2m+q+1} = 1. \tag{5.88}
$$

As the sum of σ has already calculated is equal to q, the number of unknown functions, all σ are zero.

An immediate calculation gives now

$$(2m+2)\sigma_{2m+2}+\cdots+n\sigma_n = 2mq+(2+3+\cdots+q+1)$$
$$= 2mq+\frac{q(q+3)}{2}. \qquad (5.89)$$

This result shows that *the system is in involution and its general solution depends on an arbitrary function of* $2m+q+1$ *variables.*

As there is always a solution (and even infinite) for which the unknown functions $u^1, u^2, \cdots, u^q$ are, for numerical data $(x^i)_0$ of variables x^i arbitrarily given numerical values, we can find $q+1$ particular solutions such that for $x^i = (x^i)_0$, of $q+1$ the corresponding theta forms are linearly independent $dx^1, dx^2, \cdots, dx^n$. The theorem is proved. □

Chapter 6
Prolongation of a differential system

6.1 A fundamental problem

106. We have shown in the preceding chapters the existence theorems for certain differential systems imposed on independent variables, which we called systems in involution. Solutions of these systems we demonstrated by applying the Cauchy-Kowalewski theorem are those which constitute the general solution of the considered system. But we know it can exist in any other, it is the singular solutions, given by new differential systems each of which is obtained by adding to the equations of the given system of new relations between the dependent and independent variables, and these new systems are not generally in involution. A fundamental problem is to ascertain what information we have on the solutions of a system that is not in involution, especially *given a particular solution of a differential system given, Can be obtained as the solution non-singular involution of a system can be deduced from the system given by a method regular?*

This is the answer to that question what the focus of this Chapter. The regular process which is alluded to based on the notion of *prolongation* of a differential system, we introduce in the next section.

107. *Reduction in the case of linear Pfaffian system.* We can always, for the convenience of exposition, assume that the given system contains only finite equations between the independent and dependent variables and equations linear in the differential variables, both dependent or independent. Indeed (No. 79) while differential system can be regarded as a system of partial differential equations of first order in $n-p$ unknown functions z^λ of p independent variables $x^1, x^2, \cdots, x^p$. From this point of view it can be written

$$\begin{cases} F_\alpha(x^i, z^\lambda, t_i^\lambda) = 0, & (i = 1, \cdots, p;\ \lambda = 1, \cdots, n-p;\ \alpha = 1, \cdots, r_0), \\ dz^\lambda - t_i^\lambda\, dx^i = 0, & (\lambda = 1, \cdots, n-p); \end{cases} \tag{6.1}$$

should be added those and these equations are deduced by exterior differentiation, i.e.

$$\begin{cases} \dfrac{\partial F_\alpha}{\partial x^i}\,dx^i + \dfrac{\partial F_\alpha}{\partial z^\lambda}\,dz^\lambda + \dfrac{\partial F_\alpha}{\partial t_i^\lambda}\,dt_i^\lambda = 0, & (\alpha = 1,2,\cdots,r_0), \\ dx^i \wedge dx^j = 0, & (\lambda = 1,2,\cdots,n-p); \end{cases} \tag{6.2}$$

Change of notations: denote by ν the total number of dependent variables z^λ, t_i^λ involved in equations (6.1) and (6.2), the closed system (6.1), (6.2) with ν unknown functions, we denote now all the general notation z^λ $(\lambda = 1,2,\cdots,\nu)$ can describe the form

$$\begin{cases} F_\alpha(x,z) = 0, & (\alpha = 1,2,\cdots,r_0), \\ \theta_\alpha = 0, & (\alpha = 1,2,\cdots,r_1), \\ \phi_\alpha \equiv C_{\alpha ij}\,dx^i \wedge dx^j + A_{\alpha i\lambda}\,dx^i \wedge \varpi^\lambda = 0, & (\lambda = 1,2,\cdots,r_2). \end{cases} \tag{6.3}$$

The $\theta_a lpha$ are linear forms in $dx^1, dx^2, \cdots, dx^p, dz^1, dz^2, \cdots, dz^\lambda$, consisting of the first members of the last equations (6.1) and first equations (6.2) or by independent linear combinations of these first members. The ϕ_α are the first members of last equations (6.2) or independent linear combinations of these first members. The ϖ^λ are linear differential forms with θ_α constituting a system of Pfaffian forms ν independent from $dz^1, dz^2, \cdots, dz^\lambda$. There may be remains of interest to replace the differential $dx^1, dx^2, \cdots, dx^p$ a system of p independent linear combinations of these differentials, then we denote by $\omega^1, \omega^2, \cdots, \omega^p$, of the coefficients will depend of both dependent variables as independent variables.

108. *Remark.*
We can assume the r_1 form θ_α independent, that is to say that the rank of the linear system $\theta_\alpha = 0$ equal to r_1. Of course this assumes that fits in a generic point (x,z), the x^i and z^λ satisfy the equations of the finite system (6.3); such a point is a regular integral point. We will care as integral manifolds of system (6.3) whose generic points are regular. Other solutions would be another system that they deduce the system (6.3) adding the this finite relations that express the rank of the system $\theta_\alpha = 0$ has a given value less than r_1.

6.2 Prolongation of a differential system

The operation of the extension of a differential system is basically identical to that which is, given a system (the partial differential equations, in addition to those equations of this system we deduce by deriving all or in part, with respect to one or more of independent variables. Consider the system (6.3) and form the general equations which give the $p-$dimensional integral elements

$$\omega^\lambda = t_i^\lambda\,dx^i, \qquad (\lambda = 1,2,\cdots,\nu - r_1); \tag{6.4}$$

these equations are

$$H_{\alpha ij} \equiv C_{\alpha ij} + A_{\alpha i\lambda} t_j^\lambda - A_{\alpha j\lambda} t_i^\lambda = 0, \quad (\alpha = 1, 2, \cdots, r_2;\ i, j = 1, 2, \cdots, p). \quad (6.5)$$

We can look at t_i^λ as new unknown functions subject to satisfy equations (6.5). There will be a prolongation of the system (6.3) by adding the finite equations (6.5), the linear equations (6.4) and the equations are deduced from (6.5) and (6.4) by exterior differentiation. Note that in the new system thus obtained can be removed outside quadratic equations ϕ_α i.e. contained in the original system (6.3), since they are algebraic consequences of equations (6.4).

We can also perform a partial extension without adding that some of the equations (6.4).

It can be shown that if the system (6.3) is in involution, it is (the same system fully extended following the previous scheme (E. Cartan [4], Chapter I, pp. 154-175, especially pp. 166-171, No. 7-9), but we will not need this theorem. It could be the remains in default if it was only a partial extension.

110. Suppose that the given system (6.3) is not in involution. Several cases are possible.

First case. *The equations (6.5) that provide the $p-$dimensional integral elements of a point from full credits are incompatible.* In this case, the compatibility equations (6.5) leads to relations between the coordinates x^i, z^α of the origin point of the integral element.

If these equations lead to relations between the independent variables, or if they entail the consequence that the ranking system $\theta_a lpha = 0$ is less than r_1, the problem proposed admits no solution.

If neither of these cases is impossible happens, it will add to the equations (6.3) the finite relations between dependent and independent variables that express the compatibility of equations (6.5), and the equations which are deduced by exterior differentiation, the numbers r_0 and r_1 are thereby increased, quadratic equations $\theta_a lpha = 0$ does not change. This will provide a new system with the same dependent and independent variables as the first, with increasing integers r_0 and r_1: in particular the integer $\nu - r_1$ decreased.

Second case. *The equations (6.5) are compatible in all respects regular integral of space, but the system is not in involution.* In this case we extend the system as explained in No. 109, we thus obtain a new system with new dependent variables. Compared to the old system, there will be increased over r_0 because we will add to the finite relations (6.5) between the independent and dependent variables, relations that express the $p-$dimensional element (6.4) is integral, the entire r_1 will also be increased by the addition of equations (6.4) and equations that result from differentiation of equations (6.5). As for the quadratic equations of the new system, they no longer contain the equations $\phi_a lpha = 0$ of the former, if we did a complete extension of the system, but it will add equations resulting from the exterior differentiation of equations (6.4). If the prolongation is only partial, some of the equations $\phi_a lpha = 0$ should be maintained.

111. We see that after the above if the given system is not in involution, we have a regular means to deduce a result of new systems admit the same solutions as the given system. It can be shown, *under certain conditions it is in any case not easy to specify, they will eventually come to a system in involution.*

We will not stop the general case and we are just going to show how one can demonstrate in the simplest case, or that there are only two independent variables. The proof we give will not extend to the rest of the case of any number of independent variables.

6.3 Case of two independent variables

112. We denote by x and y the independent variables. We will retain the previous notations. We assume for simplicity that the presentation in quadratic equations $\phi_a lpha = 0$, which we write

$$\phi_\alpha \equiv C_\alpha \, dx \wedge dy + A_{\alpha\lambda} \, dx \wedge \varpi^\lambda + B_{\alpha\lambda} \, dy \wedge \varpi^\lambda = 0, \tag{6.6}$$

we have got rid the forms θ_α, if necessary by adding to $\theta_a lpha$, a quadratic form congruent to zero $\mod (\theta_1, \theta_2, \cdots, \theta_{r_1})$. We call ρ the difference $\nu - r_1$, so that the forms ϕ_α does appear that forms $\varpi^1, \varpi^2, \cdots, \varpi^\rho$ independent of each other and independent of θ_α.

We will state how we'll do the extension of the system, when the equations (6.5) will be compatible, the system is not in involution. As we indicated, we will only the partial prolongations.

113. Recall the test of involution statement at number 94 in the event by a generic point integral passes at least one two-dimensional integral element. The necessary and sufficient condition of involution is the reduced s character, equals the number of linearly independent quadratic forms ϕ_α. If we put ourselves and we can assume zero coefficients C_α formulas (6.6). Substituting the differentials dx, dy two independent linear combinations ω^1, ω^2 and then by writing

$$\phi_\alpha \equiv A_{\alpha\lambda} \, \omega^1 \wedge \varpi^\lambda + B_{\alpha\lambda} \, \omega^2 \wedge \varpi^\lambda, \tag{6.7}$$

we can assume that the linear integral $\omega^2 = 0$ is regular, so that s is the number the forms $A_{\alpha\lambda} \, \varpi^\lambda$ which are independent. One can, for a change of writing, we assume $A_{\alpha\lambda} \, \varpi^\lambda \equiv \varpi^\alpha$, so we will have

$$\phi_\alpha \equiv \omega^1 \wedge \varpi^\alpha + B_{\alpha\lambda} \, \omega^2 \wedge \varpi^\lambda, \qquad (\alpha = 1, 2, \cdots, s_1). \tag{6.8}$$

If the system is not in involution, there exist forms independent of s_1 previous forms ϕ_α, but there coefficients of these forms are linear combinations of $\varpi^1, \varpi^2, \cdots, \varpi^{s_1}$. We can therefore assume that the form ϕ_{s_1+1} is devoid of such term ω^1. We will show that the coefficient of bone is a linear combination the s_1 forms . $\varpi^1, \varpi^2, \cdots, \varpi^{s_1}$ Suppose that we have for example $\phi_{s_1+1} \equiv \omega^2 \wedge \varpi^{s_1+1}$; it is easy to see that then the first character would be reduced at higher s_1, because $s1+1$

first equations reduced to the pole piece of the linear integral ($\omega^1 = 1, \omega^2 = m$) are

$$\begin{aligned} \varpi^\alpha + mB_{\alpha\lambda}\,\varpi^\lambda &= 0, \qquad (\alpha = 1, 2, \cdots, s_1), \\ m\varpi^{s_1+1} &= 0; \end{aligned} \tag{6.9}$$

but if m is given a sufficiently small value the rank of this system is $s_1 + 1$, so that character would be the first year less than $s_1 + 1$.

114. This being established we can assume, if necessary by performing a linear substitution on s_1 forms $\phi_1, \phi_2, \cdots, \phi_{s_1}$ that is a non-zero multiple of ϕ_{s_1+1} so that we can write

$$\phi_{s_1+1} \equiv \omega^2 \wedge \varpi^1. \tag{6.10}$$

We will show that the coefficient of ω^1 in ϕ_1 depends only on $\varpi^1, \varpi^2, \cdots, \varpi^{s_1}$. Indeed, the polar system reduces the integral linear element ($\omega^1 = 1, \omega^2 = m$) contains the equations

$$\begin{aligned} \varpi^1 + mB_{1\lambda}\,\varpi^\lambda &= 0, \\ \varpi^\alpha + mB_{\alpha\lambda}\,\varpi^\lambda &= 0, \qquad (\alpha = 2, \cdots, s_1). \end{aligned} \tag{6.11}$$

this system, when m tends to zero, tends to the system

$$B_{1\lambda}\,\varpi^\lambda = 0, \quad \varpi^2 = 0, \quad \cdots, \quad \varpi^{s_1} = 0, \quad \varpi^1 = 0; \tag{6.12}$$

like its rank must be equal to s_1 is that in form $B_{\alpha\lambda}\,\varpi^\lambda$ may appear only $\varpi^1, \varpi^2, \cdots, \varpi^{s_1}$.

We can assume, if necessary by a linear transformation on my ϖ^α ($\alpha < s_1$) that the coefficient of bone is equal to ϖ^1:

$$\phi_1 \equiv \omega^1 \wedge \varpi^1 + \omega^2 \wedge \varpi^2. \tag{6.13}$$

We continue the reasoning. The coefficient of ω^2 in there must be a linear combination of s_1 forms $\varpi^1, \varpi^2, \cdots, \varpi^{s_1}$. If this combination is independent of ϖ^1 and ϖ^2 we can assume that it is equal to ϖ^3 and so on. There will come a time when the coefficient of ω^2 in the successive forms
$phi_1, \phi_2, \cdots$ we have to consider not only depend on forms previously encountered: for example we will

$$\begin{aligned} \phi_1 &\equiv \omega^1 \wedge \varpi^1 + \omega^2 \wedge \varpi^2, \\ \phi_2 &\equiv \omega^1 \wedge \varpi^2 + \omega^2 \wedge \varpi^3, \\ &\cdots\cdots\cdots\cdots\cdots \\ \phi_{h-1} &\equiv \omega^1 \wedge \varpi^{h-1} + \omega^2 \wedge \varpi^h, \\ \phi_h &\equiv \omega^1 \wedge \varpi^h + B_1\,\omega^2 \wedge \varpi^1 + B_2\,\omega^2 \wedge \varpi^2 + \cdots + B_h\,\omega^2 \wedge \varpi^h, \\ \phi_{s_1+1} &\equiv \omega^2 \wedge \varpi^1. \end{aligned} \tag{6.14}$$

This granted we deduce, for any two-dimensional integral element

$$\begin{cases} \varpi^1 = t_1\,\omega^2, \\ \varpi^2 = t_1\,\omega^1 + t_2\,\omega^2, \\ \varpi^3 = t_2\,\omega^2 + t_3\,\omega^2, \\ \cdots\cdots\cdots\cdots\cdots \\ \varpi^h = t_{h-1}\,\omega^1 + t_h\,\omega^2, \end{cases} \tag{6.15}$$

with the relation

$$t_h = B_1 t_1 + B_2 t_2 + \cdots + B_h t_h. \tag{6.16}$$

115. The first result obtained, perform a partial extension of the differential system given extension that will introduce $h-1$ new unknown functions $t_1, t_2, \cdots, t_{h-1}$. We will then add to linear differential equations $\theta_\alpha = 0$ system given to the new independent linear equations

$$\begin{cases} \varpi^1 - t_1\,\omega^2 = 0, \\ \varpi^2 - t_1\,\omega^1 - t_2\,\omega^2 = 0, \\ \varpi^3 - t_2\,\omega^2 - t_3\,\omega^2 = 0, \\ \cdots\cdots\cdots\cdots\cdots \\ \varpi^h - t_{h-1}\,\omega^1 - (B_1 t_1 + B_2 t_2 + \cdots + B_h t_h)\,\omega^2 = 0. \end{cases} \tag{6.17}$$

Quadratic equations to former, whose number will be reduced from the rest of h, will be added h the quadratic equations resulting from the exterior differentiation of equations (6.17). The fundamental result obtained by this extension is that the whole pa decreased: indeed the entire r_1 was increased by h units, while the number of dependent variable ν was increased only $h-1$ units, the integer $\rho = \nu - r_1$ has actually decreased by one.

116. Analysis of past issues resulting in a regular method to obtain, starting from a non-differential system in involution, a result of differential system admitting the same solutions as the initial system. If at some point the resulting system is inconsistent, it is the same initial system, if the resulting system does not allow two-dimensional integral elements originating from a generic point of the space of dependent and independent variables , we deduce a new system for which the integer r decreased, obtained if the system admits of two-dimensional integral elements originating from a generic point, but the system is in involution, we deduce a new system for which the integer ρ declined further. As the integer ρ can not decrease indefinitely, so we arrive at some point either to a system incompatible either with a system in involution

Definition. *Theorem. Any solution of a differential system closed two independent variables can be regarded as part of the general solution of a system* *in involution that can be formed on a regular basis after a finite number of operations.*

117. Remark. In reality we have limited ourselves to solutions of the initial system (6.3) for which the rank of the system formed by the differential equations of first degree has its maximum value, and the same restrictions were implicitly made in respect of successive differential systems obtained. Especially if the original differential system of singular solutions, these solutions were left to side. If we wanted to give the theorem stated in the previous issue any validity, it should therefore focus attention on a particular solution of the initial system and for each successive system, start by adding, where appropriate, the relations between the dependent and independent variables that express the ring system of linear differential equations has the value corresponding to the proposed solution. Unfortunately it is not clear that the rank of the new system of linear differential equations has increased, that is to say that the entire ρ has decreased, although the number of independent relations between dependent and independent variables has increased . Thus it can one, strictly speaking, say that the theorem is demonstrated. Nevertheless, considerations of this Section provide a convenient method to obtain all solutions of a given system, so that each is part of the general solution of a system in involution formed without prior integration.

Part II
Applications to differential geometry

Chapter 7
Differential system, Theory of surfaces

7.1 Summary of the principles of the theory of moving trihedrals

1. Consider in ordinary space a family $\mathscr{F}$ of orthogonal trihedral depending on any number of parameters. We denote by **A** the origin of one of those trihedrals and by $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, the unit vectors inclined on the axes. The infinitesimal displacement that brings the trihedral family $\mathscr{F}$ in an infinitely near trihedral $\mathscr{F}'$ is defined if we know the infinitesimal vectors $\mathbf{de}_1, \mathbf{de}_2, \mathbf{de}_3$. The decomposition of these vectors by projection on the axes of $\mathscr{F}$ leads the relations

$$\mathbf{dA} = \omega^i \mathbf{e}_i, \qquad d\mathbf{e}_i = \omega_i^j \mathbf{e}_j, \tag{7.1}$$

where ω^i and ω_i^j are linear differential forms with respect to differential parameters of the family $\mathscr{F}$. These are the *relative components* of infinitesimal displacement of the trihedral. They are, however, not independent, because the vectors $\mathbf{e}_i$ are subject to satisfy the relations

$$(\mathbf{e}_1)^2 = (\mathbf{e}_2)^2 = (\mathbf{e}_3)^2 = 1, \qquad \mathbf{e}_1 \cdot \mathbf{e}_2 = \mathbf{e}_2 \cdot \mathbf{e}_3 = \mathbf{e}_3 \cdot \mathbf{e}_1 = 0. \tag{7.2}$$

These relations gives the following differential relations

$$\omega_1^1 = \omega_2^2 = \omega_3^3 = 0, \qquad \omega_2^3 + \omega_3^2 = \omega_1^3 + \omega_3^1 = \omega_2^1 + \omega_1^2 = 0. \tag{7.3}$$

We write now either o: or ω_j^i or ω_{ij} outside the three forms $\omega^1, \omega^2, \omega^3$ that determine components the axes of the $\mathscr{F}$ of the translation that brings into coincidence **A** with $\mathbf{A} + \mathbf{dA}$, there are three other forms

$$\omega_{12} = -\omega_{21}, \qquad \omega_{13} = -\omega_{31}, \qquad \omega_{23} = -\omega_{32}, \tag{7.4}$$

that define the components of the rotation which brings the trihedral $\mathscr{F}$ to be equipollent to near infinitesimally trihedral.

2. The six forms $\omega^1, \omega^2, \omega^3, \omega_{23}, \omega_{31}, \omega_{12}$ satisfy relations G. Darboux used systematically in the theories that involve movements of two parameters. These relations result from the exterior differentiation equations (7.1). It has in fact

$$d\omega^i \mathbf{e}_i - \omega^i \wedge \mathbf{de}_i = 0, \qquad \text{or} \qquad d\omega^i - \omega^k \wedge \omega^i_k = 0,$$
$$d\omega^j_i \mathbf{e}_j - \omega^j_i \wedge \mathbf{de}_j = 0, \qquad \text{or} \qquad d\omega^j_i - \omega^k_i \wedge \omega^j_k = 0, \tag{7.5}$$

which leads *the structure equations*

$$\begin{cases} d\omega^i = \omega^k \wedge \omega_{ki} = \omega_{ik} \wedge \omega^k, \\ d\omega^j_i = \omega^k_i \wedge \omega^j_k. \end{cases} \tag{7.6}$$

The last equations (7.6) can be rewritten as

$$d\omega_{ij} = -\omega^k_i \wedge \omega_{jk}. \tag{7.7}$$

3. Conversely suppose we are given six differential forms ω^i, $\omega_{ij} = -\omega_{ji}$ constructed with q variables u^k and their differentials and satisfying equations (7.6). There is a family of orthogonal trihedral depending on q parameters $u^1, u^2, \cdots, u^k$, such that forms ω^i, ω_{ij} are related components of their infinitesimal displacement. Indeed, consider the family the most general possible orthogonal trihedral, depending on six parameters $v^1, v^2, \cdots, v^6$, and also $\bar{\omega}^i(v, dv)$, $\bar{\omega}_{ij}(v, dv)$ components of their corresponding relative infinitesimal displacement. The equations

$$\begin{cases} \bar{\omega}^i(v, dv) = \omega^i(u, du), \\ \bar{\omega}_{ij}(v, dv) = \bar{\omega}_{ij}(u, du), \end{cases} \tag{7.8}$$

where v^i are regarded as unknown functions of the variables $u^1, u^2, \cdots, u^q$, is a completely integrable system, because the exterior differentiation applied to these equations gives, by virtue relations (7.6), verified by hypothesis by the forms $\bar{\omega}^i$, $\bar{\omega}_{ij}$ as well as done by the forms, ω^i, ω_{ij}, relations which are consequences of (7.8). So we can match to each system of values u^i one and only one orthogonal trihedral with parameters v^i, if one imposes the condition that the given values $(u^i)_0$ corresponds to a given trihedral of parameters $(v^i)_0$. It is clear that all the families of trihedrals searched are deducible from one of them by an arbitrary displacement, with or without symmetry (a real displacement if we rely on the orientation of the trihedrals searched).

7.2 The fundamental theorems of the theory of surfaces

4. To any given surface S can be attached a family of positively oriented orthogonal trihedral which based on the surface and whose vector $\mathbf{e}_3$ is normal to the surface.

This family depends on two parameters, if one attaches to each point **A** of the surface a trihedral $\mathscr{F}$ as the following law, for example by taking $\mathbf{e}_1$ and $\mathbf{e}_2$ the unit vectors carried by the principal tangents at **A**,[1] still we assume that **A** remains in a region of the surface of privately umbilics. But the family may also depend on three parameters, if one attaches to each point **A** all rectangular frames whose vector $\mathbf{e}_3$ is normal to the surface **A**.

In each case, the vector $\mathbf{d}A$ is tangent to the surface, thus, we have

$$\omega^3 = 0, \tag{7.9}$$

where, by virtue of structure of equations (7.6) and the expression of $d\omega^3$, we have

$$\omega^1 \wedge \omega_{13} + \omega^2 \wedge \omega_{23} = 0. \tag{7.10}$$

Conversely, whenever we have a family of rectangular trihedral such as the form ω^3 is identically zero, the origin of these trihedral **A** will describe a surface which the vectors $\mathbf{e}_3$ will be normal to it. Indeed, the equations of $\mathbf{e}_1 = \mathbf{e}_2 = 0$ form a completely integrable system, by virtue of structure equations

$$d\omega^1 = \omega^2 \wedge \omega_{21}, \qquad \omega^2 = \omega^1 \wedge \omega_{12}. \tag{7.11}$$

Let u and v be two independent first integrals of this system, the equation

$$\mathbf{dA} = \omega^1 \mathbf{e}_1 + \omega^2 \mathbf{e}_2, \tag{7.12}$$

then shows that the point **A** depends only on u and v. Therefore the point **A** describes a surface with a tangent plane contains each of the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ and as a result is normal to $\mathbf{e}_3$. We have implicitly assumed that ω^1 and ω^2 forms are linearly independent, otherwise the point **A** would describe a line and not a surface.

5. Consider a curve on the surface of this curve define a positive direction and denote by **T**, **N**, **B** the unit vectors of the Frenet trihedral attached to a point **A** of the curve. Let θ be the angle $(\mathbf{e}_1, \mathbf{T})$, the positive direction of rotation in the tangent plane, which brings $\mathbf{e}_1$ to $\mathbf{e}_2$ by a rotation of $+\pi/2$. Let ε be the unit vector which is derived from **T** by a rotation of $+\pi/2$ in the tangent plane. Finally, let ϖ be the angle $(\mathbf{N}, \mathbf{e}_3)$, the positive direction of rotation in the plane perpendicular to **T** which brings ε to $\mathbf{e}_3$ by a rotation of $+\pi/2$. We have

$$\begin{cases} \mathbf{T} = \cos\theta\, \mathbf{e}_1 + \sin\theta\, \mathbf{e}_2, \\ \varepsilon = -\sin\theta\, \mathbf{e}_1 + \cos\theta\, \mathbf{e}_2, \\ \mathbf{N} = \cos\theta\, \mathbf{e}_3 + \sin\theta\, \varepsilon, \\ \mathbf{B} = \sin\theta\, \mathbf{e}_3 - \cos\theta\, \varepsilon. \end{cases} \tag{7.13}$$

Finally recall the formulas of Frenet

[1] We refer to these trihedral under the name *Darboux trihedral*

$$\begin{cases} \mathbf{T} = & \dfrac{ds}{\rho}\mathbf{N}, \\ \varepsilon = -\dfrac{ds}{\rho}\mathbf{T} & +\dfrac{ds}{\tau}\mathbf{B}, \\ \mathbf{N} = & -\dfrac{ds}{\tau}\mathbf{N}, \end{cases} \tag{7.14}$$

where the element of arclength denoted by ds, the curvature and torsion by $1/\rho$ and $1/\tau$.

Differentiating the first equation (7.13) leads

$$\frac{ds}{\rho}\mathbf{N} = (d\theta + \omega_{12})\,\varepsilon + (\omega_{13}\cos\theta + \omega_{12}\sin\theta)\,\mathbf{e}_3, \tag{7.15}$$

the coefficient of ε is the projection onto the tangent plane of the vector ds/ρ carried by the principal normal; the coefficient $\mathbf{e}_3$ is its projection on the normal to the surface. We can deduce

$$\begin{aligned} d\theta + \omega_{12} &= \frac{\sin\varpi\, ds}{\rho} = \frac{ds}{R_g}, \\ \omega_{13}\cos\theta + \omega_{23}\sin\theta &= \frac{\cos\varpi\, ds}{\rho} = \frac{ds}{R_n}, \end{aligned} \tag{7.16}$$

$1/R_g$ and $1/R_n$ are the *geodesic curvature* and *normal curvature*, respectively. We have

$$\begin{aligned} \frac{ds^2}{R_n} &= \omega_{13}\cos\theta\, ds + \omega_{23}\sin\theta\, ds \\ &= \omega^1.\omega_{13} + \omega^2.\omega_{23}, \end{aligned} \tag{7.17}$$

the form $\omega^1 \odot \omega_{13} + \omega^2 \odot \omega_{23}$ is the *second fundamental form* Φ of F. Gauss, it is also equal to

$$\begin{aligned} -\mathbf{de}_3 \cdot \mathbf{dA} &= -(\omega_{31}\,\mathbf{e}_1 + \omega_{32}\,\mathbf{e}_2)\cdot(\omega^1\,\mathbf{e}_1 + \omega^2\,\mathbf{e}_2) \\ &= (\omega_{13}\,\mathbf{e}_1 + \omega_{23}\,\mathbf{e}_2)\cdot(\omega^1\,\mathbf{e}_1 + \omega^2\,\mathbf{e}_2). \end{aligned} \tag{7.18}$$

Thus, we have

$$\Phi = \omega^1.\omega_{13} + \omega^2.\omega_{23} = \frac{ds^2}{R_n}. \tag{7.19}$$

The *first fundamental form* F of the surface is ds^2, because

$$F = (\omega^1)^2 + (\omega^2)^2 = ds^2. \tag{7.20}$$

A third form also plays an important role. Differentiating the equation

$$\mathbf{e}_3 = \cos\varpi\,\mathbf{N} + \sin\varpi\,\mathbf{B}, \tag{7.21}$$

which is easily deduced from equations (7.13). It was, only taking into account (7.13) and (7.14),

$$\begin{aligned} \mathbf{de}_3 &= \omega_3^1\,\mathbf{e}_1 + \omega_3^2\,\mathbf{e}_2 \\ &= -d\varpi\,\varepsilon - ds\,\frac{\cos\varpi}{\rho}\,\mathbf{T} - \frac{ds}{\tau}\,\varepsilon \\ &= -\frac{ds}{R_n}\,\mathbf{T} + \left(d\varpi + \frac{ds}{\tau}\right)\varepsilon, \end{aligned} \tag{7.22}$$

hence, by projecting on $\mathbf{T}$ and ε, we have

$$\begin{aligned} -\mathbf{T}\cdot\mathbf{de}_3 &= \frac{ds}{R_n} &&= \omega_{13}\cos\theta + \omega_{23}\sin\theta, \\ -\varepsilon\cdot\mathbf{de}_3 &= d\varpi + \frac{ds}{\tau} &&= -\omega_{13}\sin\theta + \omega_{23}\cos\theta. \end{aligned} \tag{7.23}$$

We find the expression of ds/R_n previously provided by the second formula (7.16). Regarding the quantity $d\varpi/ds + 1/\tau$ is the geodesic torsion $1/T_g$ and we have, by replacing $\cos\theta$ by ω^1/ds and $\sin\theta$ by ω^2/ds, the relation

$$\frac{ds^2}{T_g} = \left(\frac{d\varpi}{ds} + \frac{1}{\tau}\right) ds = \omega^1.\omega_{23} - \omega^2.\omega_{13}. \tag{7.24}$$

Formula

$$\Psi = \omega^1 \odot \omega_{23} - \omega^2 \odot \omega_{13}, \tag{7.25}$$

is the *third fundamental form* of the surface.

Note that the second fundamental form depends on the chosen positive direction normal to the surface, but not the orientation of trihedral. The third fundamental form on the contrary changes sign with the orientation of the trihedral, but not dependent on the chosen positive direction on normal.

6. The formulas of the preceding section are valid even if one attaches to each point on the surface of an infinite rectangular trihedral, provided you have the same vector $\mathbf{e}_3$ normal to the surface. Now suppose we attach to each point $\mathbf{A}$ of S a determined trihedral. The relation (7.10) allows to write

$$\begin{cases} \omega_{13} = a\,\omega^1 + b\,\omega^2, \\ \omega_{13} = b\,\omega^1 + c\,\omega^2. \end{cases} \tag{7.26}$$

The three fundamental forms can be written as

$$
\begin{cases} F = (\omega^1)^2 + (\omega^2)^2, \\ \Phi = a\,(\omega^1)^2 + 2b\,\omega^1\omega^2 + c\,(\omega^2)^2, \\ \Psi = b\,(\omega^1)^2 + (c-a)\,\omega^1\omega^2 - b\,(\omega^2)^2. \end{cases} \tag{7.27}
$$

Note that the form Ψ is the Jacobian forms F and Φ, that is to say the determinant of the half-partial derivatives of these forms from ω^1 and ω^2.

The lines of curvature are given by the equation $\Psi = 0$, the asymptotic lines by the equation $\Phi = 0$. As for the principal curvatures, they are given by the system of equations

$$
\frac{a\,\omega^1 + b\,\omega^2}{\omega^1} = \frac{b\,\omega^1 + c\,\omega^2}{\omega^2} = \frac{1}{R}, \tag{7.28}
$$

where one draws the equation of the second degree in $1/R$

$$
\left(a - \frac{1}{R}\right)\left(c - \frac{1}{R}\right) - b^2 = 0, \tag{7.29}
$$

then, we have

$$
\frac{1}{R_1} + \frac{1}{R_2} = a + c, \qquad \frac{1}{R_1 R_2} = ac - b^2. \tag{7.30}
$$

In the case where the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ are carried by the tangents principal we have $1/R_1 = a$, $1/R_2 = c$, $b = 0$ and

$$
\begin{cases} \Phi = \dfrac{1}{R_1}\,(\omega^1)^2 + \dfrac{1}{R_2}\,(\omega^2)^2, \\ \Psi = \left(\dfrac{1}{R_2} - \dfrac{1}{R_1}\right)\omega^1.\omega^2. \end{cases} \tag{7.31}
$$

If θ is the angle made with the first principal tangent to the positive tangent of an oriented curve, then for this curve, we have

$$
\begin{cases} \dfrac{1}{R_n} = \dfrac{\cos^2\theta}{R_1} + \dfrac{\sin^2\theta}{R_2}, \\ \dfrac{1}{T_g} = \left(\dfrac{1}{R_2} - \dfrac{1}{R_1}\right)\sin^2\theta\cos^2\theta, \\ \dfrac{1}{R_g} = \dfrac{d\theta + \omega_{12}}{ds}. \end{cases} \tag{7.32}
$$

Finally, note that if we take vector $\mathbf{e}_1$ the unit tangent vector to a given curve C drawn on the surface. we have for each point in this curve

$$
\frac{1}{R_n} = a, \qquad \frac{1}{T_g} = b, \qquad \frac{1}{R_g} = \frac{d\theta + \omega_{12}}{ds}. \tag{7.33}
$$

It follows in particular that if C is an asymptotic line of the surface, then at each such point we have $a = 0$, whence, from equation (7.30),

$$b^2 = \frac{-1}{R_1 R_2}, \tag{7.34}$$

it leads that the torsion of the curve is equal to $\pm\sqrt{-1/R_1 R_2}$ (*Theorem of Enneper*).

7. We will now discuss various problems related to the classical theory of surfaces. These problems have mainly focused on research areas of certain properties or enjoying the search for pairs of surfaces admitting a point correspondence enjoying the given properties. We will bring the problems of the first category in search of a family of trihdrals rectangular attached to different points on the surface sought and whose vector $\mathbf{e}_3$ is normal to the surface. In many cases it will be shown to attach to each point of the triad Darboux whose vectors $\mathbf{e}_1$, $\mathbf{e}_2$ are carried by the principal tangents, which, indeed, the drawback of being restricted to a region of the surface where only is no umbilic, an umbilical since the Darboux trihedral is not determined. But often it may be desirable to attach to each point on the surface any rectangular trihedral whose vector $\mathbf{e}_3$ is normal to the surface, removing the restriction in question just been. It is true that the latter way to proceed seems to introduce unknown parasites, but, as we shall see, this is only an appearance. In the second category of problems, we bring back from even looking for a family of rectangular trihedral attached to both surfaces searched, each trihdral attaches to a point $\mathbf{A}$ of the first surface corresponding to a specific attachment to the trihedral corresponding point $\mathbf{A}'$ of the second surface, the point correspondence between two surfaces in fact establishes a correspondence determined between tangents $\mathbf{AT}$ and $\mathbf{AT}'$ from two corresponding points $\mathbf{A}$ and $\mathbf{A}'$.

Problem 1. Surfaces which has all the points are umbilics

8. The second fundamental form being proportional to the first, thus, we have, in equations (7.26) (No. 6),

$$a = c, \qquad b = 0. \tag{7.35}$$

The family of rectangular trihedral attached to different points on the surface can therefore be regarded as constituting an integral manifold of the equations

$$\omega^3 = 0, \qquad \omega_{13} = a\,\omega^1, \qquad \omega_{23} = a\,\omega^2, \tag{7.36}$$

which, closed by exterior differentiation, provide the system

$$\omega^3 = 0, \quad \omega_{13} = a\,\omega^1, \quad \omega_{23} = a\,\omega^2, \quad \omega^1 \wedge da = 0, \quad \omega^2 \wedge da = 0. \tag{7.37}$$

This system is not in involution, it leads $da = 0$, where the new system

$$\omega^3 = 0, \qquad \omega_{13} = a\,\omega^1, \qquad \omega_{23} = a\,\omega^2, \qquad da = 0, \tag{7.38}$$

one seen easily be completely integrable, the exterior differentiation involving no new equation.

If the constant a is zero, we see that $d\mathbf{e}_3 = 0$, the surface normal is fixed direction: the surface is a plane. If the constant a is not zero, the point $\mathbf{P} = \mathbf{A} + (1/a)\,\mathbf{e}_3$, is fixed as

$$d\mathbf{P} = \left(\omega^1 + \frac{1}{a}\,\omega_{31}\right)\mathbf{e}_1 + \left(\omega^2 + \frac{1}{a}\,\omega_{32}\right)\mathbf{e}_2 = 0. \tag{7.39}$$

We have has a sphere of center $\mathbf{P}$ and radius $1/a$. All surfaces are then searched and obtained, which depend on four arbitrary constants.

But the family of trihedrals that we have taken as unknown auxiliary only not only depend on arbitrary constants, because at each point of the spheres that constitute the family of surfaces searched, one can arbitrarily choose the rectangular trihedrals attached to it under the only on condition that its vector $\mathbf{e}_3$ is normal to the surface, these trihedrals therefore depend on an arbitrary function of two variables. But this function is parasitic and does not involve the initial problem. It is notified by the differential system itself (17.38) that states the conditions of the problem. Indeed this system does not involve the six forms $\omega^1, \omega^2, \omega^3, \omega_{23}, \omega_{31}, \omega_{12}$; it does intervene only five $\omega^1, \omega^2, \omega^3, \omega_{23}, \omega_{31}$. Equating to zero we obtain the five forms a completely integrable system [that is characteristic of the system (7.38)] whose solution depends on five arbitrary constants $u_1, u_2, \cdots, u_5$, each defining a particular solution one parameter family of rectangular trihedral. The geometrical meaning of such a family is easy to obtain, because if we stay within the family, forms, $\omega^1, \omega^2, \omega^3, \omega_{23}, \omega_{31}$ remain at zero; originally of the trihedral remains fixed ($\omega^1 = 0$, $\omega^2 = 0$, $\omega^3 = 0$) and the vector $\mathbf{e}_3$ also remains fixed ($\omega_{23} = 0$, $\omega_{31} = 0$). The family is formed of a trihedrals with origin $\mathbf{A}$ and given a vector $\mathbf{e}_3$ given and is geometrically equivalent to an element of contact Lie (point and plane through this point). The differential system closes (17.38) which involves the independent and dependent variables $u_1, u_2, \cdots, u_5, a$, has simply expresses a property of elements of contact with the surface sought, precisely that which characterizes the elements of contact with a surface, every point are umbilics, and in the differential system to which we got involved only the components ω^i, ω_{ij} that play a role in the actual problem proposed.

Problem 2. Establish between two given surfaces conformally point correspondence

9. Let S and $\bar{S}$ are both given surfaces: it there between these two surfaces a point correspondence such that a relation of the form

$$d\bar{s}^2 = u^2\,ds^2, \tag{7.40}$$

is between their linear elements ds^2 and $d\bar{s}^2$?

Attach to every point of each of the two surfaces the most general right-handed rectangular trihedron having this point as origin, and which the vector $\mathbf{e}_3$ is normal to the surface. We have the relation

$$(\bar{\omega}^1)^2 + (\bar{\omega}^2)^2 = u^2 \left((\omega^1)^2 + (\omega^2)^2 \right), \tag{7.41}$$

therefore

$$\begin{aligned} \bar{\omega}^1 &= u \left(\omega^1 \cos\theta + \omega^2 \sin\theta \right), \\ \bar{\omega}^2 &= u \left(-\omega^1 \sin\theta + \omega^2 \cos\theta \right), \end{aligned} \tag{7.42}$$

or

$$\begin{aligned} \bar{\omega}^1 &= u \left(\omega^1 \cos\theta + \omega^2 \sin\theta \right), \\ \bar{\omega}^2 &= u \left(\omega^1 \sin\theta - \omega^2 \cos\theta \right), \end{aligned} \tag{7.43}$$

One can always reduce to the first case, because if one is in the second case, we replace the vectors $\mathbf{e}_2$ and $\mathbf{e}_3$ of the trihedral attached to the second surface by $-\mathbf{e}_2$ and $-\mathbf{e}_3$, which will give a new direct trihdral lead to the first formulas.

By then turn this trihedral angle θ about its third axis, there will be finally maps each trihedral attached to the first surface attached to a trihedral determined the second surface so that one has the relation

$$\bar{\omega}^1 = u\,\omega^1, \qquad \bar{\omega}^2 = u\,\omega^2. \tag{7.44}$$

This system is closed by exterior differentiation and gives the new system

$$\begin{cases} \bar{\omega}^1 = u\,\omega^1, \\ \bar{\omega}^2 = u\,\omega^2, \\ \omega^1 \wedge du - u\,\omega^2 \wedge (\bar{\omega}_{12} - \omega_{12}) = 0, \\ \omega^2 \wedge du + u\,\omega^1 \wedge (\bar{\omega}_{12} - \omega_{12}) = 0. \end{cases} \tag{7.45}$$

Forms ω^1 and ω^2 are linear combinations of the differentials of parameters needed for the current position of a point of the surface S, the number of unknown functions is equal to four, namely the two parameters needed for the position of the point $\mathbf{A}$ of S which corresponds to point $\mathbf{A}$ to S, the ratio of local similarity of he tangent plane at $\mathbf{A}$ to the tangent of S corresponding to a given tangent to S at point $\mathbf{A}$. In the system (7.45) are effectively four distinct forms of ω^1, ω^2, namely $\bar{\omega}^1$, $\bar{\omega}^2$, du, and $\bar{\omega}_{12} - \omega_{12}$. One can notice than annihilate these four forms, is to let the fixed point $\bar{\mathbf{A}}$ and the function u, and also express the corresponding trihedral of origin $\mathbf{A}$ and $\bar{\mathbf{A}}$ turning by the same angle around the third axis, that is to say, the vectors $\mathbf{e}_1$ of the two trihedrals correspond consistently.

The polar matrix (No. 93) of the system (7.45) is, assuming that the columns correspond to the forms du and $\bar{\omega}_{12} - \omega_{12}$ is

$$\begin{bmatrix} \omega^1 & -u\,\omega^2 \\ \omega^2 & u\,\omega^1 \end{bmatrix} = u\Big((\omega^1)^2 + (\omega^2)^2\Big). \tag{7.46}$$

Its rank $s_1 = 2$ is equal to the number of quadratic forms appearing in the equations (7.45). The system is in involution (No. 94) and its general solution depends on two arbitrary functions of one argument. The function u is essentially zero, there is no singular solution, there is also no real characteristics.

10. *Cauchy problem.*
Any solution is completely determined by a one-dimensional solution of the equations

$$\bar{\omega}^1 = u\,\omega^1, \qquad \bar{\omega}^2 = u\,\omega^2. \tag{7.47}$$

We will give this to an arbitrary curve C on the surface S and an arbitrary curve C on the surface S. Given the arbitrary choice of the trihedral attached to each point of S, we will take any point C of the vector $\mathbf{e}_1$ tangent to C, in order to have $\omega^2 =$; we will therefore also the vector $\mathbf{e}_1$ tangent at each point of C; but the vector $\mathbf{e}_2$ can be chosen in two ways different to each of which corresponds a definite choice of $\mathbf{e}_3$. This fact will establish a point correspondence (analytical) between arbitrary C and $\bar{C}$, which provide the function $u = d\bar{s}/ds$. Each of these solutions one-dimensional of the system (7.45) provide, therefore, according to the general theory, a Fixed-line representation of $\bar{S}$ on S.

Analytically, the problem is easily solved. The two surfaces being supposed analytic, can be found on each of them a system of curvilinear coordinates: x, y in S, and $\bar{x}$, $\bar{y}$ for $\bar{S}$, so that we have

$$ds^2 = A^2\,(dx^2 + dy^2), \qquad d\bar{s}^2 = \bar{A}^2\,(d\bar{x}^2 + d\bar{y}^2). \tag{7.48}$$

Any conformal representation is obtained by taking for $\bar{x} + i\bar{y}$ an analytic function of $x + iy$, or an analytic function of $x - iy$. The analytical curve C is moreover defined by taking $x + iy$ is an analytic function $f(t)$ of a real parameter t. If we assign to each point of C the same parameter at the point corresponding to t, we will have

$$\bar{x} + i\bar{y} = \bar{f}(t), \tag{7.49}$$

$f(t)$ being a certain analytic function of t. By eliminating t we will have an analytic relationship between the two complex variables $x + iy$, $\bar{x} + i\bar{y}$ and this will define the analytic relationship a consistent correspondence searched. The other will be obtained by determination t from the equation $\bar{x} + i\bar{y} = \bar{f}(t)$ and the equation which expresses $x - iy$ as a function of t.

Problem 3. Weingarten surfaces

11. A surface with a given relation between the its principal curvatures is called *Weingarten surfaces*. More symmetrically they can define by a relation (analytical) between the mean curvature $1/R_1 + 1/R_2$, and the total curvature $1/R_1R_2$. If we attach to each point on a surface S searched an arbitrary right-handed rectangular trihedral subject to the sole condition that the vector $\mathbf{e}_3$ is normal to the surface, the differential system that puts the problem into an equation is provided by the relations

$$\begin{cases} \omega^3 = 0, \\ \omega_{13} = a\,\omega^1 + b\,\omega^2, \\ \omega_{23} = b\,\omega^1 + c\,\omega^2, \\ F(a+c, ac-b^2) = 0, \\ F_a\,da + F_b\,db + F_c\,dc = 0. \end{cases} \tag{7.50}$$

This system is then closed by adding the exterior quadratic equations

$$\begin{cases} \omega^1 \wedge (da - 2b\,\omega_{12}) + \omega^2 \wedge (db + \overline{(a-c)}\,\omega_{12}) = 0, \\ \omega^1 \wedge (db + \overline{(a-c)}\,\omega_{12}) + \omega^2 \wedge (dc + 2b\,\omega_{12}) = 0. \end{cases} \tag{7.51}$$

One checks easily that the last equation of (7.50) can be written as

$$F_a\,(da - 2b\,\omega_{12}) + F_b\,(db + \overline{(a-c)}\,\omega_{12}) + F_c\,(dc - 2b\,\omega_{12}) = 0. \tag{7.52}$$

The equations of the system thus are involved forms ω^1, ω^2, independent linear combinations of the differentials of curvilinear coordinates of a point on the surface S, and six independent forms ω^3, ω_{13}, ω_{23}, $da - 2b\,\omega_{12}$, $db + \overline{(a-c)}\,\omega_{12}$, $dc + 2b\,\omega_{12}$, between the last three of which are relation (7.52); there are indeed five unknown functions to define the contact element of S corresponding to a system of given values the curvilinear coordinates.

Here the polar matrix, whose columns correspond to the three forms $da - 2b\,\omega_{12}$, $db + \overline{(a-c)}\,\omega_{12}$, $dc + 2b\,\omega_{12}$ is

$$\begin{pmatrix} \omega^1 & \omega^2 & 0 \\ 0 & \omega^1 & \omega^2 \\ F_a & F_b & F_c \end{pmatrix}; \tag{7.53}$$

we have $s_1 = 2$, caracteriatiques being defined by the equation

$$F_a\,(\omega^1)^2 - F_b\,\omega^1.\omega^2 + F_c\,(\omega^2)^2 = 0, \tag{7.54}$$

or again, by asking $a + c = u$, and $ac - b^2 = v$, we have

$$F_u\left((\omega^1)^2 + (\omega^2)^2\right) + F_v\left(a\,(\omega^1)^2 + 2b\,\omega^1.\omega^2 + c\,(\omega^2)^2\right) = 0. \tag{7.55}$$

We see that the characteristics tangents on each integral surface belong to the involution defined by the asymptotic tangents and minima tangents.

The generic two-dimensional integral element is given by the relations

$$\begin{cases} da - 2b\,\omega_{12} = \alpha\,\omega^1 + \beta\,\omega^2, \\ db + (a-c)\,\omega_{12} = \beta\,\omega^1 + \gamma\omega^2, \\ dc + 2b\,\omega_{12} = \gamma\omega^1 + \delta\,\omega^2, \end{cases} \tag{7.56}$$

with

$$\begin{aligned} \alpha F_a + \beta F_b + \gamma F_c = 0, \\ \beta F_a + \gamma F_b + \delta F_c = 0. \end{aligned} \tag{7.57}$$

12. *Cauchy problem.*
Any one-dimensional solution, non-characteristic of equations (7.50) provides an unambiguously Weingarten surface of a given class. Such a solution is defined by a one-parameter family of rectangular trihedral to each of which are attached three numbers a, b, c. We will get by taking a curve C at each point of which we seek an analytical law following an arbitrary rectangular trihedral directly with the vector $\mathbf{e}_1$ is tangent to C (this is possible because of the indeterminacy the trihedrals fasteners each point on the surface searched). We will have then, according to formulas (7.33) of No. 6,

$$a = \frac{1}{R_n} = \frac{\cos\varpi}{\rho}, \qquad b = \frac{1}{T_g} = \frac{d\varpi}{ds} + \frac{1}{\tau}, \tag{7.58}$$

ϖ has the meaning given above, (No. 5). As for c, is given by the equation

$$F(a+c, ac-b^2) = 0; \tag{7.59}$$

Each solution of this equation a solution correspond to a dimension of equations (7.1) and consequently a Weingarten surface containing the curve C and normal vector $\mathbf{e}_3$ attached to each point on this curve.

The Cauchy problem may be impossible or indeterminate if one starts with a solution to a characteristic dimension, that is to say, from (7.55), if $F_u + aF_v = F_c = 0$ or if the value chosen for c at each point of C is a double root of equation $F = 0$ who gives it. It is easy to see that if the curve C is taken arbitrarily, the problem is impossible. Indeed we have by (7.52) along C,

$$F_a\left(da - 2b\,\omega_{12}\right) + F_b\left(db - (\overline{a-c})\,\omega_{12}\right) = 0, \tag{7.60}$$

where, according to (7.56),

$$\alpha F_a + \beta F_b = 0, \qquad \beta F_a + \gamma F_b = 0, \tag{7.61}$$

and consequently, along C (since $\omega^2 = 0$), we have

$$F_a\left(db + (\overline{a-c})\,\omega_{12}\right) + F_b\left(dc + 2b\,\omega_{12}\right) = 0, \tag{7.62}$$

It is seen that a, b, c are known according to the curvature $1/\rho$, torsion $1/\tau$ of the curve and the angle ϖ fact that the normal to the principal normal with S, the two equations $F = 0$, $F_c = 0$ and equation (7.62) provide three relations between $1/\rho$, $1/\tau$ and ϖ, that their derivatives with respect to s. The elimination of c and ϖ will therefore provide a relation between the curvature $1/\rho$, the torsion $1/\tau$ and their derivatives with respect to arc length, which restricts the possible choice of the curve C.

We will study some special cases.

13. First particular case. *Surface with a given constant value α for its principal curvature.* Here, by the formula (7.29), the relation $F = 0$ is

$$F \equiv (\alpha - a)(\alpha - c) - b^2 = 0. \tag{7.63}$$

Characteristics are given by

$$a(\omega^1)^2 + 2b\,\omega^1.\omega^2 + c(\omega^2)^2 = a\big((\omega^1)^2 + (\omega^2)^2\big). \tag{7.64}$$

The relations between the elements of a characteristic curve are

$$a = \alpha, \qquad b = 0, \qquad (c - \alpha)^2\,\omega_{12} = 0. \tag{7.65}$$

Two cases are possible. If the value of c along C is not constantly equal to α, we must have $\omega_{12} = 0$; line C must be a geodesic of the surface, with $\varpi = 0$, where

$$\alpha = \frac{1}{\rho}, \qquad \frac{1}{\tau} = 0, \tag{7.66}$$

the characteristic of C is a circle with radius $1/\alpha$, the clclic developable of the surface along the C is a cylinder of revolution with the same radius.

If instead $c = \alpha$ at all points of C, the line C is a line of umbilics; is a line traced on a sphere of radius α, because the equations

$$\frac{\cos\varpi}{\rho} = \alpha, \qquad \frac{d\varpi}{ds} + \frac{1}{\tau} = 0, \tag{7.67}$$

this leads

$$\rho^2 + \tau^2\left(\frac{d\rho}{ds}\right)^2 = \frac{1}{\alpha^2}, \tag{7.68}$$

this sphere is still a surface of answer to the question.

Weingarten surfaces of the relevant class are none other than the cylinderical-surfaces, envelope of a family of spheres of radius $1/\alpha$ depending on a parameter. Now we see easily that if we are given a curve C drawn on a sphere Σ of radius $1/\alpha$ and that is not a large circle of this sphere, the only cylinderical-surfaces that can contain C and is tangent to C along the sphere Σ is the sphere itself. the Cauchy problem in this case contains one and only one solution. If we are given instead to a circle of radius C, there exists an infinity of cylinderical-surfaces containing the same geodesic; So, the Cauchy problem is indeterminate.[2]

14. Second particular case. *Surfaces whose difference of principal curvatures has a given value* 2α. The relationship between a, b, c is here

$$F \equiv (a-c)^2 + 4(b^2 - \alpha^2) = 0. \tag{7.69}$$

The characteristics are given by

$$(a-c)\,(\omega^1)^2 + 4b\,\omega^1.\omega^2 - (a-c)\,(\omega^2)^2 = 0. \tag{7.70}$$

Which satisfies the three relations is a characteristic.

$$a - c = 0, \qquad b = \pm\alpha, \qquad dc + 2b\,\omega_{12} = 0, \tag{7.71}$$

where, assuming $b = \alpha$, we have

$$\begin{aligned} a = c = \frac{\cos\varpi}{\rho}, \qquad \frac{d\varpi}{ds} + \frac{1}{\tau} = \alpha, \\ \cos\varpi.\frac{d(1/\rho)}{ds} + \frac{\sin\varpi}{\rho}\left(\frac{1}{\rho} + \alpha\right) = 0. \end{aligned} \tag{7.72}$$

If $\frac{d(1/\rho)}{ds} = 0$ with $\frac{1}{\rho} \neq 0$, we have either $\frac{1}{\tau} = \alpha$, with $\varpi = 0$, or $\frac{1}{\tau} = -\alpha$, with $\frac{d\varpi}{ds} = 2\alpha$; in both cases the curve C is a circular helix. Otherwise the curve C is subject to the relation

$$\begin{aligned} \frac{d(1/\rho)}{ds}\frac{d(1/\tau)}{ds} - \left(\frac{1}{\tau} + \alpha\right)\frac{d^2(1/\rho)}{ds^2} \\ + 2\frac{1}{\rho}\left(\frac{d(1/\rho)}{ds}\right)^2 + \left(\frac{1}{\tau^2} - \alpha^2\right)\left(\frac{1}{\tau} + \alpha\right) = 0. \end{aligned} \tag{7.73}$$

If, this relation is verified, then the angle ϖ is determined at each point to a multiple near it.

The problem has no singular solution, because the equation of characteristics can not be reduced to an identity that if $a = c$, $b = \theta$, which contradicts the equation $F = 0$.

15. Third particular case. *Surface with constant mean curvature.*
The relationship between a, b, c is here

$$F \equiv a + c - \alpha = 0; \tag{7.74}$$

characteristics, imaginary, are minimal lines of the integral surface. The Cauchy problem has a one and only one solution if one gives a curve C at each point and a

[2] The problem here has singular solutions, the equation of characteristics becomes an identity if $a = c = \alpha$, $b = 0$; these are the spheres of radius $1/\alpha$.

trihedral whose vector $\mathbf{e}_1$ is tangent to the curve: we have then along C,

$$a = \frac{\cos\varpi}{\rho}, \qquad c = \alpha - \frac{\cos\varpi}{\rho}, \qquad b = \frac{d\varpi}{ds} + \frac{1}{\tau}. \tag{7.75}$$

16. Fourth particular case. *Surfaces with constant curvature K.* In this case

$$F \equiv ac - b^2 - K = 0. \tag{7.76}$$

The characteristics are given by

$$a\,(\omega^1)^2 + 2b\,\omega^1.\omega^2 + c\,(\omega^2)^2 = 0. \tag{7.77}$$

it is the asymptotic surface integrals, real if $K < 0$. The three relations to be satisfied by a characteristic, assumed at each point tangent to the vector $\mathbf{e}_1$, are

$$a = 0, \qquad b^2 = -K, \qquad c^2\,\omega_{12} + 2b(dc + 2b\omega_{12}) = 0; \tag{7.78}$$

or

$$\frac{\cos\varpi}{\rho} = 0, \qquad \frac{d\varpi}{ds} + \frac{1}{\tau} = \sqrt{-K},$$
$$\frac{dc}{ds} + \frac{c^2 - 4K}{2\sqrt{-K}}\,\frac{\sin\varpi}{\rho} = 0. \tag{7.79}$$

If C is not a straight line, it was $\cos\varpi = 0$, $\sin\varpi = \pm 1$, $1/\tau = \sqrt{-K}$; torsion of the curve is constant and equal to $\sqrt{-K}$ (Enneper's theorem), and further,

$$c = \pm\sqrt{-K}\tan\int\frac{ds}{\rho}. \tag{7.80}$$

If C is a right, the speed of rotation of the normal to the surface along this line is equal to $\sqrt{-K}$ and c must be taken to a constant value along the right.

As in the previous problems we leave aside the question of whether the data satisfy the necessary conditions above, the Cauchy problem has a solution or infinitely many solutions. There are certainly cases where it has an infinite number of solutions, otherwise the surfaces searched could depend on more than an arbitrary function of one variable, such as torsion curves given constant.

Problem 4. Isothermal surfaces

17. Isothermal surface is defined by the property that its ds^2 is reducible to the form $A(d\xi^2 + d\eta^2)$, where ξ and η are the parameters of lines of curvature. Attach to each point of such a surface a Darboux trihedral whose vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ are carried by the principal tangents, which forces us to restrict ourselves to the consideration of portions of the surface without umbilicus.

Denoting by a and c principal curvatures, we first equations

$$\omega^3 = 0, \qquad \omega_{13} = a\,\omega^1, \qquad \omega_{23} = c\,\omega^2. \tag{7.81}$$

We must then express that there is a function u such that the two forms ω^1 and ω^2 are exact differentials, giving

$$\begin{cases} \omega^1 \wedge \dfrac{du}{u} + \omega^2 \wedge \omega_{12} = 0, \\ \omega^2 \wedge \dfrac{du}{u} - \omega^2 \wedge \omega_{12} = 0. \end{cases} \tag{7.82}$$

Finally externally by differentiating equations (7.81) and taking into account the structure equations, we obtain

$$\begin{cases} \omega^1 \wedge dc + (a-c)\,\omega^2 \wedge \omega_{12} = 0, \\ \omega^2 \wedge dc - (a-c)\,\omega^2 \wedge \omega_{12} = 0. \end{cases} \tag{7.83}$$

The equations (7.81, 7.82, and 7.83) are closed differential system from of problem. The generic two-dimensional integral element is given by the equations

$$\begin{cases} \omega_{12} = h\,\omega^1 + k\,\omega^2, \\ \dfrac{du}{u} = -k\,\omega^1 + h\,\omega^2, \\ da = a_1\,\omega^1 + (a-c)h\,\omega^2, \\ dc = (a-c)k\,\omega^1 + c_2\,\omega^2, \end{cases} \tag{7.84}$$

it depends on four arbitrary parameters h, k, a_1, c_2, as there are four linearly independent quadratic equations (7.82) and (7.83), the system is in involution and its general solution depends on four arbitrary functions of a variable. The determinant of the polar system is $\omega^1.\omega^2.\big((\omega^1)^2 + (\omega^2)^2\big)$, the real characteristics are lines of curvature of the integral surfaces.

Any solution to a dimension of equations (7.81) can be obtained by a curve C at each point of which was attached a rectangular trihedral whose vector $\mathbf{e}_3$ is normal to C; by calling the angle through which must be rotated $\mathbf{e}_1$ around in the forward direction for the tangent to C positive, we will

$$\begin{aligned} a\cos^2\theta + c\sin^2\theta &= \frac{\cos\varpi}{\rho}, \\ (c-a)\cos\theta\sin\theta &= \frac{d\varpi}{ds} + \frac{1}{\tau}, \end{aligned} \tag{7.85}$$

where ϖ denotes the angle S with the normal to the principal normal C. It may be arbitrarily according to the angle ϖ and the angle θ and the two preceding equations provide a and c, however, if $\sin\theta$ and $\cos\theta$ is not zero. Finally the function u will be chosen arbitrarily along C. A data correspond a surface isothermal and one containing C and for admitting Darboux tribhedral at each point of the trihedral C we attach to this point.

The data depend on five arbitrary functions of one variable, it agrees with what we got, because you can determine the surface S by taking the C section of this surface by a given plane, which reduces to four arbitrary functions the data corresponding to the Cauchy problem.

Now suppose we take $\theta = 0$, the curve C becomes line of curvature of the surface integral unknown. Equations (7.85) reduce to

$$a = \frac{\cos\varpi}{\rho}, \qquad \frac{d\varpi}{ds} + \frac{1}{\tau} = 0, \tag{7.86}$$

but there is an additional condition relating to the equations (7.82) and 7.83) result

$$\omega^2 \wedge \left(\frac{u}{u} + \frac{dc}{a-c}\right) = 0; \tag{7.87}$$

we should have along C

$$\frac{u}{u} + \frac{dc}{a-c} = 0. \tag{7.88}$$

It may give the curve C and the function c, the angle ϖ is given by a constant expression $(-\int ds/\tau)$ then we will have $a = \cos\varpi/\rho$, and finally the function u will be known up to a factor (*which does not play any role in the rest the question*). This time the data involve only three arbitrary functions of one variable. We leave aside the question of whether there are solutions compatible with these data and with what degree of indeterminacy. There are certainly cases where it is an infinity of integral surfaces.

Particular case. The minimal surfaces are particular isothermal surfaces, because for such a surface ($c = -a$), equations (7.82) and (7.83) show that $du/u = da/a$, which allows us to delete the equations (7.82). More generally, constant mean curvature surfaces are isothermal $du/u = da/(a-c)$ since that is an exact differential.

Problem 5. Pairs of surfaces isometric

19. Let S and $\bar{S}$ two isometric surfaces, that is to say, have a same ds^2. Attach to each point of S the right-handed rectangular trihedral the most general of which the vector $\mathbf{e}_3$ is normal to S. By hypothesis there exists a point correspondence between S and preserving the ds^2. According to an argument made in the resolution of Problem 2, we can associate each direct trihedral S attached to a direct trihedral attached to $\bar{S}$ so that we have, by this correspondence, relations $\bar{\omega}^1 = \omega^1$, $\bar{\omega}^2 = \omega^2$. These two relations give, by exterior differentiation,

$$\omega^1 \wedge (\bar{\omega}_{12} = \omega_{12}) = 0, \qquad \omega^2 \wedge (\bar{\omega}_{12} = \omega_{12}) = 0, \tag{7.89}$$

from which we deduce $\bar{\omega}_{12} = \omega_{12}$. Finally the differential system on which the solution is formed by the linear equations

$$\begin{aligned} &\omega^3 = 0, &&\bar{\omega}^3 = 0, &&\omega^1 = 0, \\ &\bar{\omega}^1 = \omega^1, &&\bar{\omega}^2 = \omega^2, &&\bar{\omega}_{12} = \omega_{12}. \end{aligned} \tag{7.90}$$

It is supplemented by exterior differentiation of the above equations, giving the three quadratic exterior equations

$$\begin{cases} \omega^1 \wedge \omega_{13} + \omega^2 \wedge \omega_{23} = 0, \\ \omega^1 \wedge \bar{\omega}_{13} + \omega^2 \wedge \bar{\omega}_{23} = 0, \\ \bar{\omega}_{13} \wedge \bar{\omega}_{23} - \omega_{13} \wedge \omega_{23} = 0. \end{cases} \tag{7.91}$$

All these equations involve the eleven forms ω^1, ω^2, ω^3, $\bar{\omega}^1$, $\bar{\omega}^2$, $\bar{\omega}^3$, ω_{13}, ω_{23}, $\bar{\omega}_{13}$, $\bar{\omega}_{23}$, $\bar{\omega}_{12} - \omega_{12}$, there are indeed eleven variables as dependent than independent: ten elements determine the contact of the two surfaces, its eleventh determines the correspondence between the tangents to the two surfaces in two corresponding points.

The integral element two-dimensional generic is given by the relations

$$\begin{cases} \omega_{13} = a\,\omega^1 + b\,\omega^2, \\ \omega_{13} = b\,\omega^1 + c\,\omega^2, \\ \bar{\omega}_{13} = \bar{a}\,\omega^1 + \bar{b}\,\omega^2, \\ \bar{\omega}_{13} = \bar{b}\,\omega^1 + \bar{c}\,\omega^2, \end{cases} \tag{7.92}$$

with

$$\bar{a}\bar{c} - \bar{b}^2 = ac - b^2. \tag{7.93}$$

Note immediately that the relation (7.93) expresses the equality of total curvature two corresponding points on both surfaces, and the equation $\bar{\omega}_{12} = \omega_{12}$, expresses the equality of the geodesic curvatures of two corresponding curves (Gauss theorem).

Polar matrix, whose columns correspond to the forms ω_{13}, ω_{23}, $\bar{\omega}_{13}$, $\bar{\omega}_{23}$ is

$$\begin{pmatrix} \omega^1 & \omega^2 & 0 & 0 \\ 0 & 0 & \omega^1 & \omega^2 \\ \omega_{23} & -\omega_{13} & -\bar{\omega}_{23} & \bar{\omega}_{13} \end{pmatrix}; \tag{7.94}$$

its rank is equal to 3, number of quadratic equations (7.91), the system is in involution and its general solution depends on $s_2 = 1$ arbitrary function of two variables.

The one-dimensional solutions features, which reduce to two the rank of the diepolar are given by equations

$$\omega^1.\omega_{13} + \omega^2.\omega_{23} = 0, \qquad \omega^1.\bar{\omega}_{13} + \omega^2.\bar{\omega}_{23} = 0; \tag{7.95}$$

Singular solutions of the system are those for which the two previous equations are identities, and are formed by a pair of planes whatsoever; they are trivial. they exist only when the two surfaces S and $\bar{S}$ admit two asymptotic lines that match.

20. Search surfaces $\bar{S}$ isometric to a given surface. The surface S being given, the closed differential system which gives $\bar{S}$ is reduced to the equations

$$\begin{cases} \bar{\omega}^1 = \omega^1, & \bar{\omega}^2 = \omega^2, \\ \bar{\omega}^3 = 0, & \bar{\omega}_{12} = \omega_{12}, \\ \omega^1 \wedge \bar{\omega}_{13} & +\omega^2 \wedge \bar{\omega}_{23} = 0, \\ \bar{\omega}_{13} \wedge \bar{\omega}_{23} & -K\,\omega^1 \wedge \omega^2, \end{cases} \tag{7.96}$$

K denotes the total curvature $ac - b^2$ of S.

The integral element two-dimensional generic is given by

$$\begin{cases} \bar{\omega}_{13} = \bar{a}\omega^1 + \bar{b}\,\omega^2, \\ \bar{\omega}_{23} = \bar{b}\omega^1 + \bar{c}\,\omega^2, \end{cases} \tag{7.97}$$

with

$$\bar{a}\bar{c} - \bar{b}^2 = K. \tag{7.98}$$

The matrix for determining polar

$$\left| \frac{\omega^1}{-\bar{\omega}_{23}} \cdot \frac{\omega^2}{\omega_{13}} \right| = \omega^1 \bar{\omega}_{13} + \omega^2 . \bar{\omega}_{23}; \tag{7.99}$$

rank 2 equals the number of quadratic equations (7.10), the system is in involution and its general solution depends on two arbitrary functions of one variable.

21. *Cauchy problem.*

Give us a curve C on the surface S, one can enjoy the indeterminacy of trihedral attached to different points of S and just keep at each point of a C trihedral whose vector $\mathbf{e}_1$ is tangent to C in one direction chosen as positive on this curve. Let us now a curve $\bar{C}$ and trying to attach to each point of $\bar{C}$ a rectangular trihedral to obtain a solution of one-dimensional equations (7.96), there will be between C and $\bar{C}$ correspond with conservation of arcs elements $d\bar{s} = ds$; the vector $\mathbf{e}_1$ the trihedral will be tangent to $\bar{C}$ in the positive direction, the vector $\mathbf{e}_3$ is normal to $\bar{C}$ and finally condition $\bar{\omega}_{12} = \omega_{12}$, gives

$$\frac{\sin\bar{\varpi}}{\bar{\rho}} = \frac{\sin\varpi}{\rho}; \tag{7.100}$$

the curve C being given, this relations gives $\bar{\varpi} = \sin\varpi.(\rho/\bar{\rho})$; and it must, for the problem is possible that the curvature $1/\bar{\rho}$ of $\bar{C}$ is less than the geodesic curvature of C on the surface S. If this condition is satisfied, there are two values for ϖ additional one another, which gives for the vector $\mathbf{e}_3$ of the trihedral attaches to $\bar{C}$ in two possible positions. One of these positions being chosen, there will be a surface $\bar{S}$ and one containing the curve $\bar{C}$, which carries the normal vector $\mathbf{e}_3$ and which is isometric to S.

The exception is if $\bar{a} = 0$, that is to say, if $\cos\bar{\boldsymbol{\varpi}} = 0$, the curve $\bar{C}$ would be, if the problem is possible, asymptotic line of $\mathscr{S}$. This case would occur have at each point of the curve $\bar{C}$ was equal in absolute value than the geodesic curvature of C at the corresponding point. In this case, there would generally not possible. The relation (V7.98) indeed gives $\bar{b}^2 = -K$, or $\bar{b}$ is the geodesic torsion, that is to say by twisting ordinary, $\bar{C}$, so it is necessary that the torsion of $\bar{C}$ is equal to absolute value to the square root of the total curvature change of sign of the surface S at the corresponding point of C (Enneper theorem). The characteristics of the surface integral $\bar{S}$ are not arbitrary curves.

One can observe that the remaining quadratic equations (7.96) can be written

$$\begin{aligned} \left(\bar{\omega}_{13} + \sqrt{-K}\,\omega^2\right) \wedge \left(\bar{\omega}_{13} - \sqrt{-K}\,\omega^2\right) = 0, \\ \left(\bar{\omega}_{13} - \sqrt{-K}\,\omega^2\right) \wedge \left(\bar{\omega}_{13} + \sqrt{-K}\,\omega^2\right) = 0; \end{aligned} \tag{7.101}$$

they show the two families of characteristics

$$\begin{aligned} \bar{\omega}_{13} + \sqrt{-K}\,\omega^2 = 0, \qquad \omega_{13} - \sqrt{-K}\,\omega^2 = 0, \\ \bar{\omega}_{13} - \sqrt{-K}\,\omega^2 = 0, \qquad \bar{\omega}_{13} + \sqrt{-K}\,\omega^2 = 0; \end{aligned} \tag{7.102}$$

curves of each family satisfy the equation asymptotic on the other hand we have for the first family

$$\omega^1.\bar{\omega}_{23} - \omega^2.\bar{\omega}_{13} = \sqrt{-K}\left((\omega^1)^2 + (\omega^2)^2\right), \qquad \text{or} \qquad \frac{1}{\bar{\tau}} = \sqrt{-K}, \tag{7.103}$$

and for the second

$$\omega^1.\bar{\omega}_{23} - \omega^2.\bar{\omega}_{13} = -\sqrt{-K}\left((\omega^1)^2 + (\omega^2)^2\right), \qquad \text{or} \qquad \frac{1}{\bar{\tau}} = -\sqrt{-K}. \tag{7.104}$$

Problem 6. Pairs of surfaces isometric with conservation of a family of asymptotic lines

22. Will focus on a point of each of the two surfaces a rectangular trihedral whose directly vector $\mathbf{e}_1$ will be tangent to the asymptotic family correspondence stored by the vector e and which will be normal to the surface so that one has $\bar{\omega}^1 = \omega^1$, $\bar{\omega}^2 = \omega^2$ (the trihedral is chosen on S, one attached to S is determined). The consideration of the relations (7.92) and (7.93) the previous number, where $a = \bar{a} = 0$, leads to equations

$$\begin{cases} \bar{\omega}^1 = \omega^1, & \bar{\omega}^2 = \omega^2, \\ \omega^3 = 0, & \bar{\omega}^3 = 0, \\ \bar{\omega}_{12} = \omega_{12}, & \bar{\omega}_{13} = \varepsilon\,\omega_{13}, \qquad (\varepsilon = \pm 1), \\ \omega^1 \wedge \omega_{13} + \omega^2 \wedge \omega_{23} = 0, & \omega^1 \wedge \bar{\omega}_{13} + \omega^2 \wedge \bar{\omega}_{23} = 0, \\ \omega^2 \wedge \omega_{13} = 0, & \omega_{12} \wedge \left(\bar{\omega}_{23} - \varepsilon\,\omega_{23}\right) = 0; \end{cases} \tag{7.105}$$

the last equation arises from the exterior differentiation of the equation $\bar{\omega}_{13} = \varepsilon\omega_{13}$.

The generic two-dimensional integral element is given by the relations

$$\begin{aligned} \omega_{13} &= b\,\omega_{13}, & \omega_{23} &= b\,\omega^1 + c\,\omega^2, \\ \bar{\omega}_{13} &= \varepsilon b\,\omega_{13}, & \bar{\omega}_{23} &= b\,\omega^1 + \bar{c}\,\omega^2, \\ & & (\bar{c} - \varepsilon c)\,\omega_{12} \wedge \omega^2 = 0. \end{aligned} \tag{7.106}$$

Two cases are distinguished

1) $\bar{c} = \varepsilon c$. In this case $\bar{\omega}_{13} = \varepsilon\omega_{13}$, $\bar{\omega}_{23} = \varepsilon\omega_{23}$, $\bar{\omega}_{12} = \omega_{12}$, and if $\varepsilon = +1$ families of the two triples are equal, it is the same on both surfaces S and $\bar{S}$, if $\varepsilon = +1$ these two surfaces are symmetrical. This solution is trivial.
2) $\bar{c} - \varepsilon c \neq 0$. In this case we have $\omega_{12} = h\,\omega^2$; the full two-dimensional element depends on four arbitrary parameters $b, c, \bar{c}, h$.

The determinant of the matrix polar, whose columns correspond to the forms ω_{13}, ω_{23}, $\bar{\omega}_{23}$, ω_{12}, is

$$\begin{vmatrix} \omega^1 & \omega^2 & 0 & 0 \\ \varepsilon\omega^1 & 0 & \omega^2 & 0 \\ \omega^2 & 0 & 0 & 0 \\ 0 & -\varepsilon\omega_{12} & \omega_{12} & -\bar{\omega}_{23} + \varepsilon\omega_{23} \end{vmatrix} = (\varepsilon c - \bar{c})(\omega^2)^4. \tag{7.107}$$

Its rank 4 is equals to the number of quadratic equations (7.106), the system is in involution and its general solution depends on four arbitrary functions of one variable. The characteristics are the asymptotic lines which correspond to both surfaces.

The surfaces S and $\bar{S}$ have a simple geometric property, one to be resolved; indeed when one moves along the asymptotic line $\omega^2 = 0$, we have

$$d\mathbf{A} = ds.\mathbf{e}_1, \qquad d\mathbf{e}_1 = \omega_{12}.\mathbf{e}_2 + \omega_{13}.\mathbf{e}_3 = 0. \tag{7.108}$$

The point $\mathbf{A}$ describes a straight line, the vector $\mathbf{e}_1$ is constant.

23. *Cauchy problem.*
There will be a one-dimensional solution of linear equations (7.106) by giving two oriented curves $C, \bar{C}$ is corresponding with conservation of curvilinear abscissa. We focus on each point of a C right-handed trihedral rectangular directly, whose vector $\mathbf{e}_3$ is normal to C, or θ the angle between the vector $\mathbf{e}_1$ with the tangent positive C. The corresponding point of $\bar{C}$ must be attaching a directly trihedral whose vector $\mathbf{e}_3$ is normal to $\bar{C}$ and the vector $\mathbf{e}_1$ makes the same angle θ with the tangent positive

$\bar{C}$. To satisfy the last two linear equations (7.106) will require that we have[3]

$$\begin{cases} \dfrac{\sin\bar{\varpi}}{\bar{\rho}} = \dfrac{\sin\varpi}{\rho}, \\ \cos\bar{\theta}\,\dfrac{\cos\bar{\varpi}}{\bar{\rho}} - \sin\theta\left(\dfrac{d\bar{\varpi}}{d\bar{s}} + \dfrac{1}{\bar{\tau}}\right) = \varepsilon\cos\theta\,\dfrac{\cos\varpi}{\rho} - \sin\theta\left(\dfrac{d\varpi}{ds} + \dfrac{1}{\tau}\right). \end{cases} \tag{7.109}$$

The two curves oriented C, $\bar{C}$ being given, and the angle θ, the two preceding equations determine ϖ and $\bar{\varpi}$. If chooses for example $\theta = \pi/2$, then

$$\begin{cases} \dfrac{d(\bar{\varpi} - \varepsilon\varpi)}{ds} = \dfrac{\varepsilon}{\tau} - \dfrac{1}{\tau}, \\ \tan\dfrac{\bar{\varpi} + \varepsilon\varpi}{2} = \dfrac{\bar{\varpi} + \varepsilon\rho}{\bar{\varpi} - \varepsilon\rho}.\tan\dfrac{\bar{\varpi} + \varepsilon\,\varpi}{2}, \end{cases} \tag{7.110}$$

which gives $\bar{\varpi} + \varepsilon\varpi$ to an additive constant, then where $\tan((\bar{\varpi} + \varepsilon\varpi)/2)$ to a multiple of 2π around. The angles ϖ and $\bar{\varpi}$ given fixed, the two surfaces S and $\bar{S}$ are well defined, each point of C by the plane tangent to S the right carrying the vector $\mathbf{e}_1$, the right caused the surface S, the surface $\bar{S}$ is engendered a similar manner.

Otherwise we could have proceeded by giving arbitrary curves C and $\bar{C}$ and at each point of each curve the vector $\mathbf{e}_3$ the normal curve, under the only condition that we has $\sin\varpi/\rho = \sin\bar{\varpi}/\bar{\rho}$; the angle θ is then determined, after the equations of (7.109), for its tangent, provided that this angle is not zero is expressed by the inequality

$$\frac{\cos\bar{\varpi}}{\bar{\rho}} \neq \varepsilon\,\frac{\cos\varpi}{\rho}. \tag{7.111}$$

Cauchy-Kowdewski's theorem fall in default if the normal curvature of C and $\bar{C}$ are zero. In this case we know that these two lines should be straight lines such that the angle which rotates the vector $\mathbf{e}_3$ when moving on a certain segment of $\bar{C}$ is equal to the angle C multiplied by similar relative ε ($\bar{b} = \varepsilon\, b$). It is clear that then the problem has infinitely many solutions.

Remark I. If the surface S is given and the curve C, the corresponding curve $\bar{C}$, and the angle $\bar{\varpi}$, Are given by two equations $1/\bar{\rho}$, $1/\bar{\tau}$, $\bar{\varpi}$, $d\bar{\varpi}/d\bar{s}$; we see that the curve $\bar{C}$ must satisfy a certain relation between its curvature, its twist and their derivatives with respect to the arc.

Remark II. Two surfaces can be adjusted without their isometric generation correspond.

[3] In after equations (7.27) number 6, was the identity, $\omega_1\Phi - \omega_2\Psi = \omega_{13}F$, where F, Φ, Ψ denote the three basic forms, we deduce $\omega_{12} = (1/R_n)\,\omega_1 - (1/T_g)\,\omega_2$, the result is the second equation (7.109).

Problem 7. Two isometric surfaces with conservation of line of curvature

24. The equations

$$\begin{cases} \bar{\omega}^1 = \omega^1, & \bar{\omega}^2 = \omega^2, \\ \omega^3 = 0, & \bar{\omega}^3 = 0, \\ \bar{\omega}_{13} = \omega_{13}, & \end{cases} \tag{7.112}$$

between any two isometric surfaces, add a new equation of the form

$$\omega^1.\bar{\omega}_{23} - \omega^2.\bar{\omega}_{13} = u\left(\omega^1.\omega_{23} - \omega^2.\omega_{13}\right), \tag{7.113}$$

u being an unknown function essentially different from zero. This equation allows to write

$$\begin{cases} \bar{\omega}_{13} = u\,\omega_{13} + v\,\omega^1, \\ \bar{\omega}_{23} = u\,\omega_{23} + v\,\omega^2. \end{cases} \tag{7.114}$$

The equations (7.112, 7.114, and 7.112) are the closed differential system to solve. The integral element two-dimensional generic is given by

Finally the exterior differentiation of equation (7.112) and (7.114) provides the quadratic equations

$$\begin{cases} \omega^1 \wedge \omega_{13} + \omega^2 \wedge \omega_{23} = 0, \\ \bar{\omega}_{13} \wedge \bar{\omega}_{23} + \omega_{13} \wedge \omega_{23} = 0, \\ \omega_{13} \wedge du + \omega^1 \wedge dv = 0, \\ \omega_{23} \wedge du + \omega^2 \wedge dv = 0. \end{cases} \tag{7.115}$$

The equations (7.112, 7.114, and 7.115) are the closed differential system to solve.

The integral element two-dimensional generic is given by

$$\begin{cases} \omega_{13} = a\,\omega^1 + b\,\omega^2, \\ \omega_{23} = b\,\omega^1 + c\,\omega^2, \\ du = u_1\,\omega^1 + u_2\,\omega^2, \\ dv = (b u_2 - c u_1)\,\omega^1 + (b u_1 - a u_2)\,\omega^2, \end{cases} \tag{7.116}$$

with

$$(au+v)(cu+v) - b^2a^2 = ac - b^2; \tag{7.117}$$

it depends on four arbitrary parameters.

The determinant of the matrix polar, whose columns correspond to the forms ω_{13}, ω_{23}, du, dv, is

$$\begin{vmatrix} \omega^1 & \omega^2 & 0 & 0 \\ \omega_{23} - u\,\bar{\omega}_{23} & -\omega_{13} + u\,\bar{\omega}_{13} & 0 & 0 \\ -du & 0 & \omega_{13} & \omega^1 \\ 0 & -du & \omega_{23} & \omega^2 \end{vmatrix} = \tag{7.118}$$

$$= -\big(\omega^1.\omega_{23} - \omega^2.\omega_{13}\big).\Big(u.\big(\omega^1.\bar{\omega}_{13} - \omega^2.\bar{\omega}_{23}\big) - \big(\omega^1.\omega_{13} + \omega^2.\omega_{23}\big)\Big).$$

Its rank is equal to 4, number of quadratic equations (7.115), the system is in involution and its general solution depends on four arbitrary functions of one variable.

25. The singular solutions are those for which the determinant of the polar array is identically zero, which gives $u = \pm 1$, $v = 0$, then the two surfaces are equal to or symmetrical: trivial solutions. Must be added the singular solutions from the indeterminacy of lines of curvature pair of planes or spheres, are also trivial solutions.

It follows from what precedes that the surfaces S who has the property that there exists an isometric surface $\bar{S}$ of S with preservation of lines of curvature without being equal to S or symmetric with S, are extraordinary.

The characteristics are:

1) The two families of lines of curvature which correspond to both surfaces;
2) Defined by the two families

$$u.\big(\omega^1.\bar{\omega}_{13} - \omega^2.\bar{\omega}_{23}\big) = \omega^1.\omega_{13} + \omega^2.\omega_{23}; \tag{7.119}$$

these two families are not in general true. In fact the discriminant of the quadratic form

$$u.\big(\omega^1.\bar{\omega}_{13} - \omega^2.\bar{\omega}_{23}\big) - \omega^1.\omega_{13} + \omega^2.\omega_{23} = \tag{7.120}$$
$$= (u^2-1).\big(a\,(\omega^1)^2 + 2b\,\omega^1.\omega^2 + c\,(\omega^2)^2\big) + uv\,\big((\omega^1)^2 + (\omega^2)^2\big),$$

is

$$(u^2-1)^2 b^2 - ((u^2-1)a + uv)((u^2-1)c + uv) = \tag{7.121}$$
$$= (u^2-1)^2(b^2 - ac) - (u^2-1)uv(a+c) - u^2v^2;$$

taking into account the relation (7.117), is discriminant reduced to $-v^2 < 0$.

26. *Cauchy problem.*

Look for a solution to one dimension of system of equations (7.112) and (7.114). It can be assumed, because of the uncertainty attached to surfaces of the trihedral S and $\bar{S}$, which was $\omega_2 = 0$ to this solution. So we will take two oriented curves C and $\bar{C}$ preserving correspondence with the arcs ($d\bar{s} = ds$). At each point of each curve we attach a trihedral whose vector $\mathbf{e}_1$ is tangent to the curve in the positive direction. We will then

$$\frac{\sin\bar{\varpi}}{\bar{\rho}} = \frac{\sin\varpi}{\rho}, \qquad \frac{\cos\bar{\varpi}}{\bar{\rho}} = u\frac{\cos\varpi}{\rho} + v, \tag{7.122}$$
$$\frac{d\bar{\varpi}}{d\bar{s}} + \frac{1}{\bar{\tau}} = u\left(\frac{d\varpi}{ds} + \frac{1}{\tau}\right).$$

We see that the choice of vectors $\mathbf{e}_3$ is subject to one condition

$$\frac{\sin\bar{\varpi}}{\bar{\rho}} = \frac{\sin\varpi}{\rho}; \tag{7.123}$$

functions u and o are then determined along C and $\bar{C}$ by the other two equations which exclude essentially the lines of curvature. These data clearly determine the surfaces S and $\bar{S}$. We see that they involve five arbitrary functions of one variable (two arbitrary functions for each curve and an arbitrary function of the angle ϖ), but if you do restricted by take the curve C in a given plane, it only four arbitrary functions, according to the result obtained above.

Now consider the case where the Cauchy-Kowalewski theorem fall in default. In this case it was necessary conditions

$$\begin{aligned} \frac{\sin\bar{\varpi}}{\bar{\rho}} &= \frac{\sin\varpi}{\rho}, & \frac{d\bar{\varpi}}{d\bar{s}} + \frac{1}{\bar{\tau}} &= 0, \\ \frac{\cos\bar{\varpi}}{\bar{\rho}} &= u\frac{\cos\varpi}{\rho} + v, & \frac{d\varpi}{ds} + \frac{1}{\tau} &= 0. \end{aligned} \tag{7.124}$$

Equations (7.116) also show that in our case ($\omega_2 = 0$, $b = 0$) we must have $cdu + dv = 0$ along the lines C and $\bar{C}$. Now we

$$a = \frac{\cos\varpi}{\rho}, \qquad b = \frac{\cos\bar{\varpi}}{\bar{\rho}}, \qquad v = \bar{a} - ua, \tag{7.125}$$

then, according to (7.117), $\bar{a}(cu+v) = ac$; we deduce easily that

$$c = \frac{\bar{a}(\bar{a} - au)}{a - \bar{a}u}; \tag{7.126}$$

by replacing v and c by their values in the equation $dv + cdu = 0$, we obtain, to determine the function u, the Riccati equation

$$(\bar{a}^2 - a^2)\frac{du}{ds} + \bar{a}\frac{da}{ds}u^2 - \left(a\frac{da}{ds} + \bar{a}\frac{d\bar{a}}{d\bar{s}}\right)u + a\frac{d\bar{a}}{d\bar{s}} = 0. \tag{7.127}$$

Note that the two curves C and $\bar{C}$ can not be chosen arbitrarily, they depend in general only three arbitrary functions of one variable can be given arbitrarily depending on the angles ϖ and $\bar{\varpi}$ and the geodesic curvature common $1/R_g$; curvatures and torsions are then determined according to s.

27. Another method

The calculations become easier if we attach to each point of S the Darboux trihedral whose vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ are carried by the principle tangents, in limiting ourselves to naturally stay in areas without umbilicus. We may from equations

$$\begin{cases} \bar{\omega}^1 = \omega^1, & \bar{\omega}^2 = \omega^2, & \omega^3 = 0, \\ \bar{\omega}^3 = 0, & \bar{\omega}_{12} = \omega_{12}, & \omega_{13} = a\,\omega^1, \\ \bar{\omega}_{23} = c\,\omega^2, & \bar{\omega}_{13} = ta\,\omega^1, & \bar{\omega}_{23} = \dfrac{c}{t}\,\omega^2, \end{cases} \tag{7.128}$$

whose last two express equal total curvature two corresponding points on both surfaces.

The system was closed by exterior differentiation, hence the new equations are

$$\begin{cases} \omega^1 \wedge da + (a-c)\,\omega^2 \wedge \omega_{12} = 0, \\ \omega^2 \wedge dc + (a-c)\,\omega^1 \wedge \omega_{12} = 0, \\ \omega^1 \wedge d(ta) + \left(ta - \dfrac{c}{t}\right)\omega^2 \wedge \omega_{12} = 0, \\ \omega^2 \wedge d\left(\dfrac{c}{t}\right) + \left(ta - \dfrac{c}{t}\right)\omega^1 \wedge \omega_{12} = 0. \end{cases} \tag{7.129}$$

May be modified last two equations taking into account the first two, which gives the equivalent equations

$$\begin{cases} \omega^1 \wedge da + (a-c)\,\omega^2 \wedge \omega_{12} = 0, \\ \omega^2 \wedge da + (a-c)\,\omega^1 \wedge \omega_{12} = 0, \\ \omega^1 \wedge \left(\dfrac{t\,dt}{1-t^2} + \dfrac{c}{a}\dfrac{da}{a-c}\right) = 0, \\ \omega^2 \wedge \left(\dfrac{t\,dt}{t(1-t^2)} - \dfrac{a}{c}\dfrac{dc}{a-c}\right) = 0. \end{cases} \tag{7.130}$$

The characteristics are as shown in an easy calculation, data by the equation

$$\omega^1 \wedge \omega^2 \wedge \left((\omega^1)^2 + c^2\,(\omega^2)^2\right) = 0. \tag{7.131}$$

there are two families of lines of curvature and two imaginary families.

28. This method has the advantage of leading quickly to the system which gives differential $\bar{S}$ isometric surfaces of a given surface S, with matching lines of curvature, without that $\bar{S}$ is equal or symmetrical to S. The surface S is given, we have in fact simply to ascertain the unknown with the condition $t^2 \neq 1$. The last two equations (7.130) represent the differential sought system in which a and c are given. If we assume, for the surface S,

$$\omega_{12} = h\,\omega^1 + k\,\omega^2, \tag{7.132}$$

the first two equations (7.129) give us

$$a_2 = (a-c)\,h, \qquad c_1 = (a-c)\,k, \tag{7.133}$$

by appointing a_2 the second covariant derivative of a ($da = a_1\omega^1 + a_2\omega^2$) and c_1 the first covariant derivative c. The last two equations (7.130) we then give

$$\frac{t\,dt}{1-t^2} = t\frac{t\,a}{c}k\,\omega^1 - \frac{c}{a}\,\omega^2 = t^2\,\omega^1 - \omega^2, \tag{7.134}$$

asking to shorten

$$\frac{t\,a}{c}k\,\omega^1 = \varpi^1, \qquad \frac{c}{a}h\,\omega^2 = \varpi^2. \tag{7.135}$$

The exterior differentiation of (7.134) gives

$$t^2\left\{d\omega^1 - 2\,\varpi^1\wedge\varpi^2\right\} = d\varpi^2 - 2\,\varpi^1\wedge\varpi^2. \tag{7.136}$$

The coefficient of t^2 in the first member is known, represent it as $A\,\omega^1\wedge\omega^2$, the second member is also known, is $B\,\omega^1\wedge\omega^2$. That said, if A and B are zero or one nor the other, the differential system can not admit that the solution $t^2 = B/A$. In general it will not be a solution, because we know that the problem is possible only for a restricted class of surfaces S. It may happen that $t^2 = B/A$ is actually a solution and then it will be the unique solution of the problem, however if $B \neq A$.[4] If $A = B = 0$, equation (7.134) is completely integrable, in which case there is an infinity of surfaces $\bar{S}$ essentially dependent on an arbitrary constant (essentially means a displacement close).

The surfaces S for which this feature is present are characterized by the property that the two forms

$$\varpi^1 = \frac{a}{c}k\,\omega^1, \qquad \varpi^2 = \frac{c}{a}h\,\omega^2, \tag{7.137}$$

satisfy the two relations

$$d\varpi^1 = d\varpi^2 = 2\,\varpi^1\wedge\varpi^2. \tag{7.138}$$

especially the form $\varpi^1 - \varpi^2$ is an exact differential.

These surfaces can be determined. Using relations

$$d\omega^1 = h\,\omega^1\wedge\omega^2, \qquad d\omega^2 = k\,\omega^1\wedge\omega^2, \tag{7.139}$$

we find that

$$\left(\frac{h}{a}\right)_1 = \frac{hk}{c}, \qquad \left(\frac{k}{c}\right)_2 = -\frac{hk}{a}. \tag{7.140}$$

[4] In reality there will be two solutions corresponding to two symmetric surfaces S from each other.

Continuing the calculations[5], we find $hk = 0$.

For example, let $k = 0$. We have $d(h/a) = \lambda\,\omega^2$.

However, the exterior differentiation of the equation $\omega_{12} = h\,\omega^2$ gives $h_2 = ac + h^2$, and as $a_2 = (a-c)h$, it follows

$$\lambda = c\left(1+\frac{h^2}{a^2}\right). \tag{7.141}$$

The closed differential system which defines the class of surfaces S is regarded then

$$\begin{cases} \omega^3 = 0, & \omega_{13} = a\,\omega^1, \\ \omega_{23} = \omega^2, & \omega_{12} = h\,\omega^1, \\ d\left(\frac{h}{a}\right) + c\left(1+\frac{h^2}{a^2}\right)\omega^2 = 0, \\ \omega^1 \wedge da - h\,(a-c)\,\omega^1 \wedge \omega^2 = 0, \\ \omega^2 \wedge dc = 0. \end{cases} \tag{7.142}$$

It is in involution and its general solution depends on two arbitrary functions of one variable, and characteristics are the two families of lines of curvature.

29. We can geometrically characterize these surfaces. Note first that the lines of curvature of the second family ($\omega^1 = 0$) are geodesics ($\omega_{12} = 0$) and are plane because by moving along one of them was

$$\frac{d\mathbf{A}}{ds} = \mathbf{e}_2, \qquad \frac{d\mathbf{e}_2}{ds} = c\,\mathbf{e}_3, \qquad \frac{d\mathbf{e}_3}{ds} = -c\,\mathbf{e}_2. \tag{7.143}$$

c is its curvature; dc is a multiple of ω^2, all these planar geodesic are equal. The plans of these various lines of curvature envelop a developable surface, the surface is generated by a flat line Γ whose plane rolls without slipping on a developable sets (each point Γ moves normal to the plane of the valley). Since the vector normal to the plane e_1 of the curve for differential $(h\,\mathbf{e}_2 + a\,\mathbf{e}_2)\omega^1$, it follows that the characteristic plane Γ is in the plane $A\,\mathbf{e}_2\mathbf{e}_3$, the vector perpendicular to the $h\,\mathbf{e}_2 + a\,\mathbf{e}_3$; the generatrix of the envelope developable surface of the plane of Gamma is a straight line parallel to the vector $\mathbf{e}_2 - (h/a)\,\mathbf{e}_3$. As the differential of this vector is $c(h/a)(\mathbf{e}_2 - (h/a)\,\mathbf{e}_3)\omega^2$, it follows that the generator of the developable has a fixed direction. The plane Γ of a cylinder envelope so, so that the surfaces are sought-molding surfaces, which do depend on two arbitrary functions of a variable.

We therefore arrive at the following conclusion. The surfaces S such that there exists a surface $\bar{S}$ with conservation of isometric lines without curvature $\bar{S}$ is equal to S or are symmetric to S exceptional, forming a dependent class of four arbitrary functions of one variable. The corresponding surface $\bar{S}$ is generally unique to a dis-

[5] The differential system which gives the sought surfaces by simply using the results we just obtained is not in involution, that is, applying the method of extension specified in Chapter 6 we find the relation $hk = 0$.

placement or near symmetry, and there-except for molding-surfaces, for which the surfaces depends, has a displacement and a near symmetry, an arbitrary constant.

Problem 8. Two isometric surfaces with preservation of principal curvatures

30. It is sufficient that the mean curvature is preserved. If we use the same rectangular trihedral than in the previous problems, we have the equations

$$\bar{\omega}^1 = \omega^1, \quad \bar{\omega}^2 = \omega^2, \quad \omega^1 = 0, \quad \bar{\omega}^3 = 0, \quad \bar{\omega}_{12} = \omega_{12}. \tag{7.144}$$

As on the other hand

$$\begin{aligned} \omega^1 \wedge \omega_{23} - \omega^2 \wedge \omega_{13} &= (a+c)\, \omega^1 \wedge \omega^2 \\ &= \left(\frac{1}{R_1} + \frac{1}{R_1}\right) \omega^1 \wedge \omega^2, \end{aligned} \tag{7.145}$$

the differential system which expresses the closed conditions of the problem is obtained by adding to equations (7.144) quadratic equations

$$\begin{cases} \omega^1 \wedge \omega_{13} + \omega^2 \wedge \omega_{23} = 0, \\ \omega^1 \wedge \bar{\omega}_{13} + \omega^2 \wedge \bar{\omega}_{23} = 0, \\ \bar{\omega}_{13} \wedge \bar{\omega}_{23} - \bar{\omega}_{13} \wedge \omega_{23} = 0, \\ \omega^1 \wedge (\bar{\omega}_{23} - \varepsilon\, \omega_{23}) - \omega^2 \wedge (\bar{\omega}_{13} - \varepsilon\, \omega_{13}) = 0, \qquad (\varepsilon = \pm 1). \end{cases} \tag{7.146}$$

The double sign is that the principal radii of curvature of both surfaces can be worn in different directions on the vectors $\mathbf{e}_3$ attached to the surfaces S and $\bar{S}$. Can be reduced to the remainder $\varepsilon = 1$ case, the solutions of case $\varepsilon = -1$ being deduced if the solutions $\varepsilon = +1$ by symmetry of the surface $\bar{S}$ with respect to a plane.

The integral element two-dimensional generic is given by

$$\begin{cases} \omega_{13} = a\,\omega^1 + b\,\omega^2, \qquad \omega_{23} = b\,\omega^1 + c\,\omega^2, \\ \bar{\omega}_{13} = \bar{a}\,\omega^1 + \bar{b}\,\omega^2, \qquad \bar{\omega}_{23} = \bar{b}\,\omega^1 + \bar{c}\,\omega^2, \end{cases} \tag{7.147}$$

with

$$\bar{a}\bar{c} - \bar{b}^2 = ac - b^2, \qquad \bar{a} + c = a + c; \tag{7.148}$$

it depends on four arbitrary parameters.

The determinant of the matrix polar, whose columns correspond to the forms, ω_{13}, ω_{23}, $\bar{\omega}_{13}$, $\bar{\omega}_{23}$ is

$$\begin{vmatrix} \omega^1 & \omega^2 & 0 & 0 \\ 0 & 0 & \omega^1 & \omega^2 \\ \omega_{23} & -\omega_{13} & -\bar{\omega}_{23} & \bar{\omega}_{13} \\ \omega^2 & -\omega^1 & -\omega^2 & \omega^1 \end{vmatrix} = \tag{7.149}$$

$$= \Big((\omega^1)^2 + (\omega^2)^2\Big).\Big(\omega^1.\big(\omega_{13} - \bar{\omega}_{13}\big) + \omega^2.\big(\omega_{23} - \bar{\omega}_{23}\big)\Big).$$

The rank of the matrix being equal to 4, number of quadratic equations (7.6), the system is in involution and its general solution depends on four arbitrary functions of one variable.

Singular solutions. They correspond to $a = \bar{a}$, $b = \bar{b}$: $c = \bar{c}$, that is to say $\omega_{13} = \bar{\omega}_{13}$, $\omega_{23} = \bar{\omega}_{23}$. They are supplied by two equal or symmetrical surfaces ($\varepsilon = -1$). These are trivial solutions.

The *real characteristics* are provided by the pairs of corresponding curves have the same normal curvature.

31. Cauchy problem. Any solution of a dimension system (7.50) is obtained by taking two curves C and $\bar{C}$ oriented with conservation corresponding to arcs and by attaching to these curves of rectangular trihedral whose vector $\mathbf{e}_1$ is raised by the tangent b-positive the curve, the vectors $\mathbf{e}_3$ being chosen so as to satisfy the relation

$$\frac{\sin\bar{\varpi}}{\bar{\rho}} = \frac{\sin\varpi}{\rho}; \tag{7.150}$$

such a solution depends on five arbitrary functions of one variable. It is characteristic if it has at the same time

$$\frac{\cos\bar{\varpi}}{\bar{\rho}} = \frac{\cos\varpi}{\rho}; \tag{7.151}$$

that is to say if the curvature is the same function of the curvilinear abscissa for both curves. We will then take $\bar{\varpi} = \varpi$. Moreover, the equalities

$$a = \bar{a}, \qquad \bar{a} - \bar{c} = a + c, \qquad \bar{a}\bar{c} - \bar{b}^2 = ac - b^2, \tag{7.152}$$

leads

$$c = \bar{c}, \qquad \bar{b} = \pm b. \tag{7.153}$$

the two curves have therefore the torsions geodesic equal to or opposed. This is a new condition which must be satisfied that the two curves for the Cauchy problem is possible (See the problem VIII, who was asked by O. Bonnet, an article by E. Cartan [8]).

Problem 9. Pairs of surfaces point correspondence

to the lines of curvature and the principal curvatures

32. We will explain the problem by requesting that the correspondence between the two surfaces preserves the mean curvature and total curvature.

We will attach to each point on the surface S a directly Darboux trihedral[6], we can then attach in a one way and only one at the corresponding point of the surface S a Darboux trihedral as it has

$$\omega^3 = 0, \quad \bar{\omega}^3 = 0, \quad \bar{\omega}^1 = u\,\omega^1, \quad \bar{\omega}^2 = v\,\omega^2, \quad (u > 0,\ v > 0). \tag{7.154}$$

That being so, two essentially distinct cases are considered according to the principal curvatures are the same for lines of curvature which correspond to both surfaces, or after these principal curvatures are exchanged between the two families of lines of curvature when passing from one surface to another.

33. First problem. It will add to the equations (7.154) the equations

$$\omega_{13} = a\,\omega^1, \quad \omega_{23} = c\,\omega^2, \quad \bar{\omega}_{13} = \varepsilon a\,\bar{\omega}^1, \quad \bar{\omega}_{23} = \varepsilon c\,\bar{\omega}^2, \quad (\varepsilon = \pm 1), \tag{7.155}$$

then the equations are derived from (7.154) and (7.154) by exterior differentiation. These new equations are

$$\begin{cases} \omega^1 \wedge du + \omega^2 \wedge \left(u\,\omega_{12} - v\,\bar{\omega}_{12}\right) = 0, \\ \omega^2 \wedge dv + \omega^1 \wedge \left(u\,\bar{\omega}_{12} - v\,\omega_{12}\right) = 0, \\ \omega^1 \wedge da + (a-c)\,\omega^2 \wedge \omega_{12} = 0, \\ \omega^2 \wedge dc + (a-c)\,\omega^1 \wedge \omega_{12} = 0, \\ \bar{\omega}^1 \wedge da + (a-c)\,\bar{\omega}^2 \wedge \bar{\omega}_{12} = 0, \\ \bar{\omega}^2 \wedge dc + (a-c)\,\bar{\omega}^1 \wedge \bar{\omega}_{12} = 0. \end{cases} \tag{7.156}$$

Subtracting the fifth equation multiplied by u the third and sixth equation of the fourth multiplied by v, we obtain relations

$$\omega^2 \wedge \left(u\,\omega_{12} - v\,\bar{\omega}_{12}\right) = 0, \qquad \omega^1 \wedge \left(u\,\bar{\omega}_{12} - v\,\omega_{12}\right) = 0. \tag{7.157}$$

This allows to write the six quadratic equations in the form

$$\begin{cases} \omega^2 \wedge \left(u\,\omega_{12} - v\,\bar{\omega}_{12}\right) = 0, & \omega^1 \wedge du = 0, \\ \omega^1 \wedge \left(u\,\bar{\omega}_{12} - v\,\omega_{12}\right) = 0, & \omega^2 \wedge dv = 0, \\ \omega^1 \wedge da + (a-c)\,\omega^2 \wedge \omega_{12} = 0, & \\ \omega^2 \wedge dc + (a-c)\,\omega^1 \wedge \omega_{12} = 0. & \end{cases} \tag{7.158}$$

Equations (7.154, 7.155, and 7.159) are the differential system closed the first problem.

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[6] This implies that we only consider portions of the surface no umbilicus.

The determinant of a matrix whose columns are polar forms of du, dv, ω_{12}, $\bar{\omega}_{12}$ da, dc is, all calculations, given by

$$\Delta = (a-c)^2(u^2-v^2)(\omega^1)^3(\omega^2)^3. \tag{7.159}$$

Its rank is equal to 6, number of quadratic equations (7.159), the system is in involution and its general solution depends on six arbitrary functions of one variable.

Singular solutions. As $a-c$ is assumed to be essentially different from zero, the singular solutions are those for which $u^2 = v^2$ or $u-v$, since u and v are two functions essentially positive. The first two equations (7.159) then give the $du = 0$ and the next two give $\bar{\omega}_{12} = \omega_{12}$, hence, by exterior differentiation,

$$\bar{\omega}_{13} \wedge \bar{\omega}_{23} = \omega_{13} \wedge \omega_{23}, \tag{7.160}$$

or

$$(u^2-1)\,ac = 0. \tag{7.161}$$

If we leave the cete developable surfaces, we find $u = 1$, the equalities

$$\bar{\omega}^1 = \omega^1, \quad \bar{\omega}^2 = \omega^2, \quad \bar{\omega}_{12} = \omega_{12}, \quad \bar{\omega}_{13} = \varepsilon\,\omega_{13}, \quad \bar{\omega}_{23} = \omega_{23}, \tag{7.162}$$

then show that the two surfaces are equal or symmetrical: trivial solution.

Thus we see that the surfaces S such that there exists a surface $\bar{S}$ may, without being either symmetrical to S, or equal to S, to be matched with point S with preservation lines of curvature, mean curvature and the total curvature are exceptional: they are a class of six arbitrary functions depending on a variable.

Characteristics, according to (7.162), are lines of curvature.

34. Cauchy problem. Any solution to one dimension of equations (7.154) and (7.155) consists of two families of rectangular trihedral attached to two curves C and oriented $\bar{C}$ and four functions u, v, a, c, each trihedral is determined by the vector $\mathbf{e}_3$ normal to the corresponding curve and the angle θ is that the positive tangent to the curve with the vector $\mathbf{e}_1$. We then. from (7.27) and (7.31) of No. 6, relations

$$\begin{aligned} &\cos\theta\, d\bar{s} = u\cos\theta\, ds, && \sin\theta\, d\bar{s} = v\sin\theta\, ds, \\ &\frac{\cos\varpi}{\rho} = a\cos^2\theta + c\sin^2\theta, && \frac{d\varpi}{ds} + \frac{1}{\tau} = (c-a)\sin\theta\cos\theta, \\ &\frac{\cos\bar{\varpi}}{\bar{\rho}} = \varepsilon\left(a\cos^2\bar{\theta} + c\sin^2\bar{\theta}\right), && \frac{d\bar{\varpi}}{ds} + \frac{1}{\bar{\tau}} = \varepsilon(c-a)\sin\theta\cos\bar{\theta}. \end{aligned} \tag{7.163}$$

We arbitrarily give the law of point correspondence between C and $\bar{C}$, that is to say $d\bar{s}/ds = w$. We will then

$$u = \frac{w\cos\bar{\theta}}{\cos\theta}, \qquad v = \frac{w\sin\bar{\theta}}{\sin\theta}; \tag{7.164}$$

functions a, c, θ, $\bar{\theta}$ will be determined by the last four relations, where the angles ϖ and $\bar{\varpi}$ are known by the position vectors $\mathbf{e}_3$ attached to the trihedral of two curves. We see that the data depend on seven arbitrary functions of one variable (the two curves, the angles ϖ and $\bar{\varpi}$ and the function w). This agrees with the degree of arbitrariness of the general solution of the problem as a solution to the problem being given, we can take to curve section C of S given by a fixed plane.

The Cauchy-Kowalewski theorem falls into default if the curve C is line of curvature of S, that is to say, if $\theta = 0$ or $\pi/2$ (θ then has the same value). For example, assume $\theta = 0$, $\omega^2 = \bar{\omega}^2 = 0$. Equations (7.156) then show that one must have, for the possibility of the problem,

$$dv = 0, \qquad ,\omega_{12} - v\,\bar{\omega}_{12} = 0. \tag{7.165}$$

The data in this case are as follows. We must have

$$u = \frac{d\bar{s}}{ds}, \qquad a = \frac{\cos\varpi}{\rho} = \varepsilon\frac{\cos\bar{\theta}}{\bar{\rho}}, \qquad \frac{d\varpi}{ds} + \frac{1}{\tau} = 0,$$

$$dv = 0, \qquad \frac{\sin\varpi}{\rho} = v\frac{\sin\bar{\theta}}{\bar{\rho}}, \qquad \frac{d\bar{\varpi}}{d\bar{s}} + \frac{1}{\bar{\tau}} = 0. \tag{7.166}$$

Let-denote, for example, ϖ and $1/\rho$ depending on s, the torsion $1/\tau$ of C is given by $1/\tau = -d\varpi/ds$. Let us then u as a function of s and the constant value m of v. we will have

$$\tan\bar{\varpi} = \frac{\varepsilon}{m}\tan\varpi, \qquad \frac{1}{\bar{\rho}} = \varepsilon\frac{\cos\varpi}{\sin\bar{\varpi}}\frac{1}{\rho}, \qquad \frac{1}{\bar{\tau}} = \frac{-1}{u}\frac{d\bar{\varpi}}{ds}, \tag{7.167}$$

which gives $\bar{\varpi}$, $1/\bar{\rho}$ and $1/\bar{\tau}$ as a function of $\bar{s}$. Finally we give an arbitrary c as a function of s. The characteristic solution thus obtained depends on four arbitrary functions of s, namely ϖ, $1/\rho$, u and c. It is clear that there will be solutions to a characteristic dimension which correspond infinitely many solutions of the given problem.

35. Particular case: *Surfaces admitting a family of lines of curvature of geodesic forms.*

If the second family of lines of curvature of a surface S is formed of geodetic there will be a relation of the form

$$\omega_{12} = h\,\omega^1. \tag{7.168}$$

as $d\omega^2 = \omega^1 \wedge \omega_{12} = 0$, the form ω^2 is an exact differential $d\beta$. As we saw earlier (No. 29), the lines of curvature of this family are all flat and equally between them; the surface S is generated by a flat plane Γ whose plane rolls without sliding on a fixed developable.

If such a surface belongs to the class that we consider, the surface $\bar{S}$ will be given by the system

$$\begin{cases} \bar{\omega}^1 = u\,\omega^1, & \bar{\omega}_{13} = \varepsilon a\,\bar{\omega}^1, \qquad \bar{\omega}^3 = 0, \\ \bar{\omega}^2 = v\,\omega^2, & \bar{\omega}_{23} = \varepsilon c\,\bar{\omega}^2, \\ \omega^1 \wedge du = 0, & \omega^1 \wedge \left(u\,\bar{\omega}_{12} - v\,\omega_{12}\right) = 0, \\ \omega^2 \wedge dv = 0, & \omega^2 \wedge \left(u\,\omega_{12} - v\,\bar{\omega}_{12}\right) = 0. \end{cases} \tag{7.169}$$

In the second last of these equations and equation (7.168) it follows that $\bar{\omega}_{12}$ is proportional to ω^1 and the last equation (7.169) results

$$\bar{\omega}_{12} = \frac{u}{v}\,\omega_{12} = \frac{h}{v}\,\omega^1. \tag{7.170}$$

The surface S, if it exists, has also his second family of lines of curvature of geodesics formed. The functions u and v are given by the system

$$\omega^1 \wedge du = 0, \qquad \omega^2 \wedge dv = 0, \qquad \bar{\omega}_{12} = \frac{u}{v}\,\omega_{12} \tag{7.171}$$

hence, by exterior differentiation

$$h\,\omega^1 \wedge dv + acv(v^2-1)\,\omega^1 \wedge \omega^2 = 0. \tag{7.172}$$

We deduce from these equations

$$dv = ac\,\frac{v(1-v^2)}{h}\,\omega^2. \tag{7.173}$$

As $\omega^2 \wedge dc = 0$, we see that two cases are possible:

1) $\omega^2 \wedge d(a/h) = 0$. In this case equation (7.173) is completely integrable for v gives a function that depends on an arbitrary constant, the equation $\omega^1 \wedge du = 0$ then gives u for an arbitrary function of the parameter lines of curvature of the first family of the surface S. There is then an infinity of surfaces $\bar{S}$ with forming S a pair of surfaces satisfying the conditions of the problem.
2) $\omega^2 \wedge d(a/h) \neq 0$. In this case the equation (7.173) has the unique solution $v = 1$, and can still take for u an arbitrary function of the parameter lines of curvature of the first family of S, of the existence of yet oh an infinity of surfaces $\bar{S}$.

The first case is that of *surface-molding.*

36. Second problem. We start from equations (7.154) which one must add the equations

$$\bar{\omega}_{13} = a\,\omega^1, \quad \bar{\omega}_{23} = c\,\omega^2, \quad \bar{\omega}_{13} = \varepsilon c\,\omega^1, \quad \bar{\omega}_{23} = \varepsilon a\,\omega^2, \qquad (\varepsilon = \pm 1) \tag{7.174}$$

and quadratic equations

$$\begin{cases} \omega^1 \wedge du + \omega^2 \wedge \left(u\,\omega_{12} - v\,\bar{\omega}_{12}\right) = 0, \\ \omega^2 \wedge dv + \omega^1 \wedge \left(u\,\bar{\omega}_{12} - v\,\omega_{12}\right) = 0, \\ \omega^1 \wedge da + (a-c)\,\omega^2 \wedge \omega_{12} = 0, \\ \omega^2 \wedge dc + (a-c)\,\omega^1 \wedge \omega_{12} = 0, \\ \bar{\omega}^1 \wedge da + (a-c)\,\bar{\omega}^2 \wedge \bar{\omega}_{12} = 0, \\ \bar{\omega}^2 \wedge dc + (a-c)\,\bar{\omega}^1 \wedge \bar{\omega}_{12} = 0. \end{cases} \tag{7.175}$$

The determinant of the polar array whose columns correspond to the forms du, dv, ω_{12}, $\bar{\omega}_{12}$, da, dc, is

$$\begin{vmatrix} \omega^1 & 0 & u\,\omega^2 & -v\,\omega^2 & 0 & 0 \\ 0 & \omega^2 & -v\,\omega^1 & u\,\omega^1 70 & 0 & \\ 0 & 0 & (a-c)\,\omega^2 & 0 & \omega^1 & 0 \\ 0 & 0 & (a-c)\,\omega^1 & 0 & 0 & \omega^2 \\ 0 & 0 & 0 & -(a-c)\,\bar{\omega}^2 & 0 & \bar{\omega}^1 \\ 0 & 0 & 0 & -(a-c)\,\bar{\omega}^1 & \bar{\omega}^2 & 0 \end{vmatrix} =$$

$$= (a-c)^2\,\omega^1.\omega^2.\Big((\omega^1)^2.(\bar{\omega}^1)^2 - (\omega^2)^2.(\bar{\omega}^2)^2\Big). \tag{7.176}$$

The rank of the matrix being equal to 6, number of quadratic equations (7.175), the system is in involution and its general solution depends on six arbitrary functions of one variable. As $a-c$ is essentially zero, and lee functions u and v, there is no singular solution. The characteristics are lines of curvature and the two families give lineage by

$$\omega^1.\bar{\omega}^1 - \omega^2.\bar{\omega}^2 = 0 \qquad \text{or} \qquad u\,(\omega^1)^2 - v\,(\omega^2)^2 = 0, \tag{7.177}$$

there are two other characteristics of families, but they are imaginary.

37. Cauchy problem. The formulas in the first problem for the formulation of the Cauchy problem is here replaced with the following

$$\begin{aligned} &d\bar{s}\cos\bar{\theta} = u\,ds\,\cos\theta, && d\bar{s}\sin\bar{\theta} = v\,ds\,\sin\theta, \\ &\frac{d\varpi}{ds} + \frac{1}{\tau} = (c-a)\,\sin\theta\cos\theta, && \frac{\cos\varpi}{\rho} = a\cos^2\theta + c\sin^2\theta, \\ &\frac{d\bar{\varpi}}{d\bar{s}} + \frac{1}{\bar{\tau}} = \varepsilon(c-a)\,\sin\bar{\theta}\cos\bar{\theta}, && \frac{\cos\bar{\varpi}}{\bar{\rho}} = \varepsilon\left(c\cos^2\bar{\theta} + a\sin^2\bar{\theta}\right), \end{aligned} \tag{7.178}$$

We will arbitrarily two curves C and $\bar{C}$, and the angles ϖ and $\bar{\varpi}$ the law of point correspondence between them, that is to say the function $d\bar{s}/ds = w$. We will then

$$u = w\,\frac{\cos\bar{\theta}}{\cos\theta}, \qquad v = w\,\frac{\sin\bar{\theta}}{\sin\theta}, \tag{7.179}$$

functions a, c, θ, $\bar{\theta}$ will be determined by the last 4 relations.

The Cauchy-Kowalewski theorem falls in default or if the curve C is line of curvature of S, or if one has such $\sqrt{u}\cos\theta = \sqrt{v}\sin\theta$.

Let us look first the former case and assume, to fix the ideas, $\theta = 0$, where $\bar{\theta} = 0$. the equations

$$d\bar{s} = u\,ds, \quad a = \frac{\cos\varpi}{\rho}, \quad a = \varepsilon\,\frac{\cos\bar{\varpi}}{\bar{\rho}}, \quad \frac{d\varpi}{ds} + \frac{1}{\tau} = 0, \quad \frac{d\bar{\varpi}}{d\bar{s}} + \frac{1}{\bar{\tau}} = 0, \tag{7.180}$$

we should add new relations to ensure that the problem is possible. Relations (7.175) is deduced in effect

$$\omega^2 \wedge \left(da + dc + \overline{(a-c)\frac{dv}{v}}\right) = 0, \tag{7.181}$$

we should have along C

$$\frac{dv}{v} + \frac{da + dc}{a - c} = 0. \tag{7.182}$$

Let us, for example u, ϖ and $1/\rho$ as a function of s, the curve C will be determined by its curvature and torsion equal to $-d\varpi/ds$. The function $\bar{s}$ is known, let us ϖ and $1/\bar{\rho}$ as a function of $\bar{s}$; torsion $1/\bar{\tau}$ is determined. When the function u will be determined at a constant factor; the data dependent and five arbitrary functions of one variable.

Now consider the second case, or assumed $\sqrt{u}\cos\theta = \sqrt{v}\sin\theta$, or $\sqrt{u}\omega^1 = \sqrt{v}\omega^2$. Equations (7.175) is drawn equation

$$\left(\sqrt{u}\omega^1 - \sqrt{v}\omega^2\right) \wedge \left(v\,da + u\,dc + \overline{(a-c)}\sqrt{uv}(\bar{\omega}_{12} - \omega_{12})\right) = 0. \tag{7.183}$$

To equations (7.178) should therefore be added to equation

$$v\,da + u\,dc + \overline{(a-c)}\sqrt{uv}(\bar{\omega}_{12} - \omega_{12})\big) = 0. \tag{7.184}$$

Then we can give the curve C and the functions ϖ, u, v as a function of S. we will have

$$\tan\theta = \sqrt{\frac{u}{v}}, \qquad \tan\bar{\theta} = \sqrt{\frac{v}{u}}, \qquad d\bar{s} = \sqrt{uv}\,ds, \tag{7.185}$$

where θ, $\bar{\theta}$, $\bar{s}$; as to the functions a and c, they will be given by the equations

$$\frac{av + cu}{u + v} = \frac{\cos\varpi}{\rho}, \qquad (c - a)\,\frac{\sqrt{uv}}{u + v} = \frac{d\varpi}{ds} + \frac{1}{\tau}, \tag{7.186}$$

Finally, the curve $\bar{C}$ and the angle $\bar{\varpi}$ will be given by the equations

$$\frac{\cos\bar{\varpi}}{\bar{\rho}} = \varepsilon\frac{av+cu}{u+v}, \qquad \frac{d\varpi}{ds}+\frac{1}{\tau} = \varepsilon(a-c)\frac{\sqrt{uv}}{u+v} \tag{7.187}$$

and the value of $\sin\bar{\varpi}/\bar{\rho}$ will be derived from equation (7.184). The data dependent thus five arbitrary functions of one variable.

We leave aside the determination of the surface $\bar{S}$ when we know the surface S.

Problem 10. Couples of surfaces in point correspondence with preservation of geodesic torsion curves

38. We have excluded the trivial case of planes and spheres. The geodesic torsion being the ratio

$$\frac{\omega^1.\omega_{23}-\omega^2.\omega_{13}}{(\omega^1)^2+(\omega^2)^2}, \tag{7.188}$$

each form of the numerator and denominator must, from the first surface S to the second surface $\bar{S}$ to reproduce, even multiplied by a factor (the sign). the two surfaces are in line representation. Attach to different S pointed to the rectangular trihedral directly subject to the sole condition that the vector $\mathbf{e}_3$ is the surface normal; will strive to corresponding points of $\bar{S}$, without ambiguity, the corresponding trihedral so as to have

$$\bar{\omega}^1 = u\,\omega^1, \qquad \bar{\omega}^2 = u\,\omega^2, \qquad (u>0); \tag{7.189}$$

we must then have

$$\omega^1.\bar{\omega}_{23}-\omega^2.\bar{\omega}_{13} = \varepsilon u\left(\omega^1.\bar{\omega}_{23}-\omega^2.\bar{\omega}_{13}\right), \qquad (\varepsilon=\pm 1); \tag{7.190}$$

from which

$$\bar{\omega}_{13} = \varepsilon u\,(\omega_{13}+v\,\omega^1), \qquad \bar{\omega}_{23} = \varepsilon u\,(\omega_{23}+v\,\omega^2). \tag{7.191}$$

Finally the problem of closed differential system proposed will consist of the equations

$$\begin{aligned} &\omega^3 = 0, \qquad &&\bar{\omega}^1 = u\,\omega^1, \qquad &&\bar{\omega}_{13} = \varepsilon u\left(\omega_{13}+v\,\omega^1\right),\\ &\bar{\omega}^3 = 0, \qquad &&\bar{\omega}^2 = u\,\omega^2, \qquad &&\bar{\omega}_{23} = \varepsilon u\left(\omega_{23}+v\,\omega^2\right), \end{aligned} \tag{7.192}$$

and exterior quadratic equations

$$\begin{cases} \omega^1 \wedge \omega_{13} + \omega^2 \wedge \omega_{23} = 0, \\ \omega^1 \wedge \dfrac{du}{u} + \omega^2 \wedge \left(\omega_{12} - \bar{\omega}_{12}\right) = 0, \\ \omega^2 \wedge \dfrac{du}{u} - \omega^1 \wedge \left(\omega_{12} - \bar{\omega}_{12}\right) = 0, \\ \omega^1 \wedge dv + \omega_{13} \wedge \dfrac{du}{u} + \omega_{23} \wedge \left(\omega_{12} - \bar{\omega}_{12}\right) = 0, \\ \omega^2 \wedge dv + \omega_{23} \wedge \dfrac{du}{u} - \omega_{13} \wedge \left(\omega_{12} - \bar{\omega}_{12}\right) = 0. \end{cases} \tag{7.193}$$

The generic two dimensional integral element is given by the equations

$$\begin{cases} \omega_{13} = a\,\omega^1 + b\,\omega^2, \\ \omega_{23} = b\,\omega^1 + c\,\omega^2, \\ \dfrac{du}{u} = \alpha\,\omega^1 + \beta\,\omega^2, \\ \omega_{12} - \bar{\omega}_{12} = \beta\,\omega^1 - \alpha\,\omega^2, \\ dv = \left((a-c)\alpha + 2b\beta\right)\omega^1 + \left(2b\alpha + (c-a)\beta\right)\omega^2, \end{cases} \tag{7.194}$$

it depends on five arbitrary parameters a, b, c, α, β. The system is in involution and its general solution depends on five arbitrary functions of one variable if the determinant of the polar array is not identically zero. But this determinant, whose columns correspond to the forms ω_{13}, ω_{23}, du/u, $\omega_{12} - \bar{\omega}_{12}$, dv, is

$$\begin{vmatrix} \omega^1 & \omega^2 & 0 & 0 & 0 \\ 0 & 0 & \omega^1 & \omega^2 & 0 \\ 0 & 0 & \omega^2 & -\omega^1 & 0 \\ -\dfrac{du}{u} & \bar{\omega}_{12} - \omega_{12} & \omega_{13} & \omega_{23} & \omega^1 \\ \omega_{12} - \bar{\omega}_{12} & -\dfrac{du}{u} & \omega_{23} & -\omega_{13} & \omega^2 \end{vmatrix} = \frac{du}{u} . \left((\omega^1)^2 + (\omega^2)^2\right), \tag{7.195}$$

it is not identically zero. The actual characteristics of general solutions are one family characterized by the constancy of the function u.

39. *Cauchy problem.* Leaving aside for now the singular solutions of the problem proposed, let's get to the Cauchy problem on general solutions. Presumably for a one-dimensional solution (Cauchy data) system (7.192), was $\omega^2 = 0$. We then have for the couple of curves corresponding C, $\bar{C}$ and the couple of developable circles Σ, $\bar{S}igma$,

$$d\bar{s} = u\,ds, \qquad \frac{\cos\bar{\varpi}}{\bar{\rho}} = \varepsilon\left(\frac{\cos\varpi}{\rho} + v\right), \qquad \frac{d\bar{\varpi}}{d\bar{s}} + \frac{1}{\bar{\tau}} = \varepsilon\left(\frac{d\varpi}{ds} + \frac{1}{\tau}\right). \tag{7.196}$$

More simply we will arbitrarily two curves and two developable; the point correspondence between the two curves will be that which achieves equality, around with the sign s, the geodesic torsion; we will thus have the function $u = d\bar{s}/ds$ along curves, and the function v is the difference ($\varepsilon = 1$) or sum ($\varepsilon = -1$) of the normal curvatures in two corresponding points of two curves. The data depend on six arbitrary functions, which are reduced to five if we take the C section of the surface S by a given plane.

The Cauchy-Kowalewski Theorem falls in default if the data are characteristic, that is to say, if the function u is constant, equal to m for example. In this case we have an additional condition for the data. The coefficient alpha of the formulas (7.194) being zero, we must have along the curve C

$$dv = 2b\left(\omega_{12} - \bar{\omega}_{12}\right), \tag{7.197}$$

or

$$\frac{dv}{ds} = 2\left(\frac{d\varpi}{ds} + \frac{1}{\tau}\right).\left(\frac{\sin\varpi}{\rho} - m\frac{\sin\bar{\varpi}}{\bar{\rho}}\right). \tag{7.198}$$

The data are no longer dependent while four arbitrary functions of one variable, the curve C and being developable Σ data (three arbitrary functions), functions $1/\bar{\rho}$, $1/\bar{\tau}$, $\bar{\varpi}$ of $\bar{s}$ are linked by a finite relation and a differential equation.

If the surface S is given, the surface is determined by the $\bar{S}$ system formed the last five equations (7.192) and the last four equations (7.193). This system is not in involution, in the most favourable case the surface S depends, to a displacement around, five arbitrary constants; would still have to ensure that this case can actually occur.

39. *Singular solution.* Singular solutions of the system (7.192 and 7.193) are those for which the function u is a constant m. Equations (7.193

$$\bar{\omega}_{12} - \omega_{12} = 0, \qquad dv = 0, \ (v = n), \tag{7.199}$$

and, by exterior differentiation,

$$\bar{\omega}_{13} \wedge \bar{\omega}_{23} - \omega_{13} \wedge \omega_{23} = 0. \tag{7.200}$$

The closed differential system which gives singular solutions is then

$$\begin{aligned} &\omega^3 = 0, && \bar{\omega}^1 = m\,\omega^1, && \bar{\omega}_{12} = \omega_{12}, \\ &\bar{\omega}^3 = 0, && \bar{\omega}^2 = m\,\omega^2, && \\ &\bar{\omega}_{13} = \varepsilon m\left(\omega_{13} + n\,\omega^1\right), && \bar{\omega}_{23} = \varepsilon m\left(\omega_{23} + n\,\omega^2\right), && \\ &\omega^1 \wedge \omega_{13} + \omega^2 \wedge \omega_{23} = 0, && \bar{\omega}_{13} \wedge \bar{\omega}_{13} - \omega_{13} \wedge \omega_{23} = 0, && \end{aligned} \tag{7.201}$$

This system is in involution and its general solution depends on two arbitrary functions of one argument. Characteristics are given by the equation

$$\bar{\omega}^1 \wedge \bar{\omega}_{13} + \bar{\omega}^2 \wedge \bar{\omega}_{23} = \omega^1 \wedge \omega_{13} + \omega^2 \wedge \omega_{23}, \tag{7.202}$$

or further

$$(m^2-1)\left(\omega^1\wedge\bar{\omega}_{13}+\omega^2\wedge\omega_{23}\right)+m^2n\left((\omega^1)^2+(\omega^2)^2\right). \tag{7.203}$$

The last equation (X, 4) shows that the surfaces S are Weingarten surfaces satisfying the relation

$$\frac{m^2-1}{R_1R_2}+m^2n\left(\frac{1}{R_1}+\frac{1}{R_2}\right)+m^2n^2=0. \tag{7.204}$$

the corresponding surfaces $\bar{S}$ satisfy a relation similar where m is replaced $1/m$, and n by $-\varepsilon n$. If the surface S is given, the constants m and n are given, the corresponding surfaces $\bar{S}$ are given by the completely integrable system

$$\begin{aligned}
&\bar{\omega}^3=0, && \bar{\omega}^1=m\,\omega^1, && \bar{\omega}_{23}=\varepsilon m\left(\omega_{23}+n\,\omega^2\right),\\
&\bar{\omega}^2=m\,\omega^2, && \bar{\omega}_{12}=m\,\omega_{12}, && \bar{\omega}_{13}=\varepsilon m\left(\omega_{13}+n\,\omega^1\right),
\end{aligned} \tag{7.205}$$

these surfaces $\bar{S}$ are completely determined has a displacement or around a symmetry.

A singular solution of the system (7.192 7.193, and 7.194) is completely determined by the data of two curves C and $\bar{C}$ and two developable circles Σ, $\bar{\Sigma}$, satisfying the conditions

$$\begin{aligned}
d\bar{s}&=m\,ds, & \frac{d\varpi}{\bar{s}}+\frac{1}{\bar{\tau}}&=\varepsilon\left(\frac{dm}{ds}+\frac{1}{\tau}\right),\\
\frac{\sin\bar{\varpi}}{\bar{\rho}}&=\frac{\sin\varpi}{\rho} & \frac{\cos\bar{\varpi}}{\bar{\rho}}&=\varepsilon\left(\frac{\cos\varpi}{\rho}+n\right).
\end{aligned}$$

These data are characteristic if one has

$$\frac{\cos\varpi}{\rho}=\frac{m^2n}{1-m^2},\qquad \frac{\cos\bar{\varpi}}{\bar{\rho}}=\varepsilon\,\frac{n}{1-m^2}. \tag{7.206}$$

Problem 11. Surfaces with the same third fundamental form as a given surface

41. Lines of curvature corresponding to both surfaces, will focus on each point of a given surface S right-handed Darboux trihedral: it will correspond to the corresponding point of the researched surface $\bar{S}$ of a Darboux trihedral as it has

$$\bar{\omega}^3=0,\qquad \bar{\omega}^1=u\,\omega^1,\qquad \bar{\omega}^2=v\,\omega^2,\qquad (u>0,\ v>0), \tag{7.207}$$

the relation

$$\bar{\omega}^1 \wedge \bar{\omega}_{23} - \bar{\omega}^2 \wedge \bar{\omega}_{13} = \omega^1 \wedge \omega_{23} - \omega^2 \wedge \omega_{13}, \tag{7.208}$$

leads us to asking, by calling a and c principal curvatures of S,

$$\begin{aligned} \bar{\omega}_{13} &= \frac{1}{v}\left(\omega_{13} + w\,\omega^1\right) = \frac{a+w}{v}\,\omega^1, \\ \bar{\omega}_{23} &= \frac{1}{u}\left(\omega_{23} + w\,\omega^2\right) = \frac{c+w}{v}\,\omega^2. \end{aligned} \tag{7.209}$$

Exterior differentiation (7.207) and (7.209) gives the exterior quadratic equations

$$\begin{cases} \omega^1 \wedge du + \omega^2 \wedge \left(u\,\omega_{13} - v\,\bar{\omega}_{12}\right) = 0, \\ \omega^2 \wedge dv + \omega^1 \wedge \left(u\,\bar{\omega}_{13} - v\,\omega_{12}\right) = 0, \\ \omega^1 \wedge \left(dw - \dfrac{c+w}{u}\,du - \dfrac{a+w}{v}\,dv\right) = 0, \\ \omega^2 \wedge \left(dw - \dfrac{c+w}{u}\,du - \dfrac{a+w}{v}\,dv\right) = 0. \end{cases} \tag{7.210}$$

It follows the relation

$$dw = \frac{c+w}{u}\,du + \frac{a+w}{v}\,dv, \tag{7.211}$$

which, exterior differentiated, gives

$$dc \wedge \frac{du}{u} + da \wedge \frac{dv}{v} - (a-c)\,\frac{du}{u} \wedge \frac{dv}{v} = 0. \tag{7.212}$$

Finally, the closed differential system of problem is

$$\begin{aligned} &\bar{\omega}^3 = 0, \qquad \bar{\omega}^1 = u\,\omega^1, \qquad \bar{\omega}^2 = v\,\omega^2, \\ &\bar{\omega}_{13} = \frac{a+w}{v}\,\omega^1, \qquad \bar{\omega}_{23} = \frac{c+w}{u}\,\omega^2 \\ &dw = \frac{c+w}{u}\,du + \frac{a+w}{v}\,dv, \\ &\omega^1 \wedge du + \omega^2 \wedge \left(u\,\omega_{13} - v\,\bar{\omega}_{12}\right) = 0, \\ &\omega^2 \wedge dv + \omega^1 \wedge \left(u\,\bar{\omega}_{13} - v\,\omega_{12}\right) = 0, \\ &dc \wedge \frac{du}{u} + da \wedge \frac{dv}{v} - (a-c)\,\frac{du}{u} \wedge \frac{dv}{v} = 0. \end{aligned} \tag{7.213}$$

The generic element integral in two dimensions, by putting

$$\begin{aligned} da &= a_1\,\omega^1 + h(a-c)\,\omega^2, \\ dc &= k(a-c)\,\omega^1 + c_2\,\omega^2, \end{aligned} \qquad \omega_{12} = h\,\omega^1 + k\,\omega^2, \tag{7.214}$$

is given by

$$dv = \gamma\omega^1 + \delta\,\omega^2, \qquad \omega_{12} = \frac{uh-\beta}{v}\,\omega^1 + \frac{vk+\gamma}{u}\,\omega^2,$$
$$du = \alpha\,\omega^1 + \beta\,\omega^2, \tag{7.215}$$

where the coefficients α, β, γ, δ are related by the relations

$$-(a-c)(\alpha\delta - \beta\gamma) + (a-c)(vk\delta - uh\gamma) + ua_1\delta - vc_2\alpha = 0. \tag{7.216}$$

The two-dimensional integral element depends on three arbitrary parameters.

As the quadratic equations (7.225) are three in number, the system is in involution and *its general solution depends on three arbitrary functions of an variable*.

Characteristics annihilate the determinant of the polar matrix

$$\begin{vmatrix} u\,\omega^1 & 0 & -v\,\omega^2 \\ 0 & v\,\omega^2 & u\,\omega^2 \\ dc + (a-c)\,\dfrac{dv}{v} & da - (a-c)\,\dfrac{du}{u} & 0 \end{vmatrix} =$$
$$= u^2\left(da - \frac{a-c}{u}\,du\right).(\omega^2)^1 - v^2\left(dc + \frac{a-c}{v}\,dv\right).(\omega^2)^2. \tag{7.217}$$

42. *Singular solutions.*

They are given by the two additional equations

$$\frac{du}{u} = \frac{da}{a-c}, \qquad \frac{dv}{v} = \frac{dc}{c-a}, \tag{7.218}$$

which, by exterior differentiation, leads to the condition $da \wedge dc = 0$ to be satisfied by the given surface S. It expresses that S is a Weingarten surface. If so, we find, according to (7.214) and (7.215),

$$\frac{u_2}{u} = \frac{\beta}{u} = \frac{a_2}{a-c} = h, \qquad \text{where} \qquad uh = \beta,$$
$$\frac{v_2}{v} = \frac{\gamma}{v} = \frac{c_1}{c-a} = -k, \qquad \text{where} \qquad vk + \gamma = 0. \tag{7.219}$$

It was as a result $\omega_{12} = 0$, where $\omega_{13} \wedge \omega_{23} = 0$. This relation expresses that the surface is developable $\bar{S}$; expressions of $\bar{\omega}_{13}$ and $\bar{\omega}_{23}$ given by equations (7.225) then show that $(w+a)(w+c) = 0$, for example

$$w = -a, \qquad \omega_{13} = 0. \tag{7.220}$$

Was to determine the surface $\bar{S}$, the system

$$\frac{du}{u} = \frac{da}{a-c}, \qquad \frac{dv}{v} = \frac{dc}{c-a},$$
$$\bar{\omega}_{13} = 0, \qquad \bar{\omega}_{23} = \frac{c-a}{u}\,\omega^2 \qquad \bar{\omega}_{12} = 0. \tag{7.221}$$

system is completely integrable. So *if S is a Weingarten surface of the system (7.225) admits singular solutions, consisting of a family of developable surfaces (cylinders of revolution) depending on arbitrary constants.*

43. *Generar solution; Cauchy problem.*
Give us on the surface S and a curve C trihedral attached thereto. We have a solution of one-dimensional. system (7.225) giving us a curve $\bar{C}$ must correspond to C, and the developable circumscribed $\bar{\Sigma}$ of $\bar{S}$ along to $\bar{C}$. We have, denoting by θ angle with the first main tangent S and $\bar{\theta}$ the angle relative to similar $\bar{C}$

$$\begin{cases} \cos\bar{\theta}\,d\bar{s} = u\cos\theta\,ds, \qquad \sin\bar{\theta}\,d\bar{s} = v\sin\theta\,ds, \\ \dfrac{1}{\bar{R}_n} = \dfrac{a+w}{uv}\cos^2\bar{\theta} + \dfrac{c+w}{uv}\sin^2\bar{\theta} \\ \qquad = \dfrac{ds^2}{d\bar{s}^2}\left(\dfrac{u}{v}(a+w)\cos^2\theta + \dfrac{v}{u}(c+w)\sin^2\theta\right), \\ \dfrac{1}{\bar{T}_g} = \dfrac{c-a}{uv}\sin\bar{\theta}\cos\bar{\theta} = \dfrac{ds^2}{d\bar{s}^2}\dfrac{1}{T_g}, \\ \dfrac{dw}{ds} = \dfrac{c+w}{u}\dfrac{du}{ds} + \dfrac{a+w}{v}\dfrac{dv}{ds}. \end{cases} \tag{7.222}$$

In these equations $1/R_n$ $1/T_g$, θ, a, c, are known functions of an s. Moreover we know in terms of $\bar{s}$ the curvature $1/\bar{\rho}$, torsion $1/\bar{\tau}$, and the angle $\bar{\varpi}$. It remains to determine the functions u, v, w at different points of $\bar{C}$. We first establish the point correspondence between C and $\bar{C}$ by the equation

$$\frac{d\bar{s}^2}{\bar{T}_g} = \frac{ds^2}{T_g}. \tag{7.223}$$

We then have

$$\begin{aligned} u^2\cos^2\theta + v^2\sin^2\theta &= \frac{d\bar{s}^2}{ds}, \\ u^2(a+w)\cos^2\theta + v^2(c+w)\sin^2\theta &= \frac{uv}{\bar{R}_n}\frac{d\bar{s}^2}{ds}. \end{aligned} \tag{7.224}$$

Finally we have the differential equation given by the last equation (7.222) to determine who will complete u, v, w to an arbitrary constant. The data therefore involve three arbitrary functions, which is consistent with the result obtained above.

We leave aside the determination of characteristic data.

Problem 12. Pairs of surfaces with conservation of conformal representation asymptotic lines

44. It is evident that the correspondence line, keeping the asymptotic lines, also keeps the lines of curvature. By attaching to each point of the first surface S the cube corner live the most general of which the vector $\mathbf{e}_3$ is normal to S, it will match unambiguously trihedral those attached to $\bar{S}$ and we shall relations

$$\omega^3 = 0, \qquad \bar{\omega}^3 = 0, \qquad \bar{\omega}^1 = u\,\omega^1, \qquad \bar{\omega}^2 = u\,\omega^2, \qquad (u > 0). \tag{7.225}$$

thus, we have

$$\bar{\omega}^1 \wedge \bar{\omega}_{13} + \bar{\omega}^2 \wedge \bar{\omega}_{23} = uv\big(\omega^1 \wedge \omega_{13} + \omega^2 \wedge \omega_{23}\big), \tag{7.226}$$

where

$$\bar{\omega}_{13} = v\,\omega_{13} + w\,\omega^2, \qquad \bar{\omega}_{23} = v\,\omega_{23} - w\,\omega^1, \tag{7.227}$$

but the relation $\bar{\omega}^1 \wedge \bar{\omega}_{13} + \bar{\omega}^2 \wedge \bar{\omega}_{23}$ requires $w = 0$.

The closed system of differential problem is

$$\begin{cases} \omega^3 = 0, \qquad \bar{\omega}^2 = u\,\omega^2, \qquad \bar{\omega}_{13} = v\,\omega_{13}, \\ \bar{\omega}^3 = 0, \qquad \bar{\omega}^1 = u\,\omega^1, \qquad \bar{\omega}_{23} = v\,\omega_{23}, \\ \omega^1 \wedge \dfrac{du}{u} + \omega^2 \wedge \big(\omega_{12} - \bar{\omega}_{12}\big) = 0, \\ \omega^2 \wedge \dfrac{du}{u} - \omega^1 \wedge \big(\omega_{12} - \bar{\omega}_{12}\big) = 0, \\ \omega_{13} \wedge \dfrac{dv}{v} + \omega_{23} \wedge \big(\omega_{12} - \bar{\omega}_{12}\big) = 0, \\ \omega_{23} \wedge \dfrac{dv}{v} - \omega_{13} \wedge \big(\omega_{12} - \bar{\omega}_{12}\big) = 0. \end{cases} \tag{7.228}$$

The most general two-dimensional integral element is given by

$$\begin{cases} \omega_{13} = a\,\omega^1 + b\,\omega^2, \\ \omega_{23} = b\,\omega^1 + c\,\omega^2, \\ \omega_{12} - \bar{\omega}_{12} = \beta\,\omega^1 - \alpha\,\omega^2, \\ \dfrac{du}{u} = \alpha\,\omega^1 + \beta\,\omega^2, \\ \dfrac{dv}{v} = \lambda\,\omega^1 + \mu\,\omega^2. \end{cases} \tag{7.229}$$

with the relations

$$\begin{cases} a\alpha + b\beta - c\lambda + b\mu = 0, \\ b\alpha + c\beta + b\lambda - a\mu = 0, \end{cases} \tag{7.230}$$

it depends on five arbitrary parameters, as 5 is the number of independent quadratic equations system (7.244), the system is in involution and *its general solution depends on five arbitrary functions of one variable.*

The characteristics are obtained by annihilating the determinant of the polar matrix. This, by matching the shapes ml columns ω_{13}, ω_{23}, $\omega_{12} - \bar{\omega}_{12}$, du/u, dv/v, can be written

$$\begin{vmatrix} \omega^1 & \omega^2 & 0 & 0 & 0 \\ 0 & 0 & \omega^2 & \omega^1 & 0 \\ 0 & 0 & -\omega^1 & \omega^2 & 0 \\ -\dfrac{dv}{v} & \bar{\omega}_{12}-\omega_{12} & \omega_{23} & 0 & \omega_{13} \\ \bar{\omega}_{12}-\omega_{12} & \dfrac{dv}{v} & -\omega_{13} & 0 & \omega_{23} \end{vmatrix} = \big((\omega^2)^1+(\omega^2)^2\big)\times$$
$$\times\left(\frac{dv}{v}\left(\omega^1.\omega_{13}+\omega^2.\omega_{23}\right)\left(\omega_{12}-\bar{\omega}_{12}\right).\left(\omega^1.\omega_{13}-\omega^2.\omega_{23}\right)\right). \qquad (7.231)$$

Taking into account equations (7.229), the determinant of the polar matrix is equal to

$$\big((\omega^1)^2+(\omega^2)^2\big).\Big((a\lambda-b\beta)\,(\omega^1)^3+(2b\lambda+a\mu+b\alpha+\overline{a-c}\beta).(\omega^1)^2.\omega^2$$
$$+(2\lambda+2b\mu+\overline{c-a}\alpha+b\beta).\omega^1.(\omega^2)^2+(c\mu-b\alpha)\,(\omega^2)^3\Big). \qquad (7.232)$$

These are obtained by adding four linear equations in α, β, λ, μ to the system of equations

$$\begin{cases} a\lambda-b\beta=0, \\ 2b\lambda+a\mu+b\alpha+(a-c)\beta=0 \\ c\lambda+2b\mu+(c-a)\alpha+b\beta=0 \\ c\mu-b\alpha=0, \end{cases} \qquad (7.233)$$

the determinant of the coefficients of these equations is

$$(b^2-ac)(4b^2+(a-c)^2). \qquad (7.234)$$

Several cases may be distinguished.

1) *The determinant is not zero.* - We then have $\alpha=\beta=\lambda=\mu=0$, it follows from (7.229),

$$\frac{du}{u}=0, \qquad \frac{dv}{v}=0, \qquad \omega_{12}=\bar{\omega}_{12}, \qquad (7.235)$$

where, by exterior differentiation,

$$\bar{\omega}_{13}\wedge\bar{\omega}_{23},-\omega_{13}\wedge\omega_{23}=0, \qquad (v^2-1)\omega_{13}\wedge\omega_{23}=0. \qquad (7.236)$$

As b^2-ac was assumed zero, we see that $v^2=1$; because the two surfaces are directly or inversely similar: it is a trivial solution.

2) *The determinant is zero and $b^2-ac\neq 0$.* - Then we have $a=c$, $b=0$, the two surfaces are planes ($a=0$) or spheres. In the first case the solution is trivial in

the second case ($a \neq 0$) is also, according to (7.233), $\lambda = \mu = 0$, where $dv = 0$. These surfaces are any two spheres is one obvious solution of the problem.

3) *The determinant is zero and $b^2 - ac = 0$.* - Both surfaces are developable. The surface S is given, we can assume the trihedral chosen to be $b = c = 0$, $a \neq 0$, then we have from (7.230) and (7.233),

$$\alpha = \beta = \lambda = \mu = 0, \tag{7.237}$$

functions u, v are constants and $\bar{\omega}_{12} = \omega_{12}$. The surface is then given by the completely integrable system

$$\begin{aligned} &\bar{\omega}^3 = 0, && \bar{\omega}^1 = m\,\omega^1, && \bar{\omega}^2 = m\,\omega^2, \\ &\bar{\omega}_{23} = 0, && \bar{\omega}_{12} = \omega_{12}, && \bar{\omega}_{13} = n\,\omega_{13}, \end{aligned} \tag{7.238}$$

where m and n are constants. There exists among the curvilinear abscissa cusp edges of Γ, $\bar{=}\Gamma$ of the two surfaces S, $\bar{S}$, the relationship $\bar{s} = ms$. If the curvature $1/\rho$ of Γ is equal to $\varphi(s)$, the curvature $1/\bar{\rho}$ of $\bar{\Gamma}$ the corresponding point is equal to $(1/m)\varphi(\bar{s}/m)$; torsion of $\bar{\gamma}$ results from torsion of Γ at the corresponding point in the multiplied by n.

46. *Singular solution; Cauchy problem.*

It can be assumed for any solution to one dimension of system (7.244) which has been $\omega^2 = 0$. Keeping the notations, we will arbitrarily corresponding to two given curves C and $\bar{C}$ and the two circumscribed developable Σ and $\bar{\Sigma}$,

$$d\bar{s} = u\,ds, \quad \frac{\cos\bar{\varpi}}{\bar{\rho}} = v\frac{\cos\varpi}{\rho}ds, \quad \left(\frac{d\bar{\varpi}}{d\bar{s}} + \frac{1}{\bar{\tau}}\right)d\bar{s} = v\left(\frac{d\varpi}{ds} + \frac{1}{\tau}\right)ds. \tag{7.239}$$

We know $1/\rho$, $1/\tau$ and ϖ in terms of s; $1/\bar{\rho}$, $1/\bar{\tau}$ and $\bar{\varpi}$ according to $\bar{s}$. We will then have the relation between $\bar{s}$ and s by the equation

$$\frac{1}{\bar{R}_n}\frac{1}{\bar{T}_g} = \frac{1}{R_n}\frac{1}{T_g}, \tag{7.240}$$

$\bar{s}$ being known as a function of s, we have $u = d\bar{s}/ds$ and function in v, deduce immediately. The solution will be considered one-dimensional characteristic if one has $a\lambda - b\beta = 0$, that is to say

$$\frac{1}{\bar{R}_n}\frac{d\ln v}{ds} - \frac{1}{T_g}\left(\frac{1}{R_g} - \frac{1}{\bar{R}_g}\right) = 0, \tag{7.241}$$

the other hand by eliminating p between the two relations (7.230) we find $\lambda = -\alpha$, and hence we have the additional condition

$$\frac{d\ln(uv)}{ds} = 0. \tag{7.242}$$

It may give $1/\rho$, $1/\tau$ and ϖ and u as a function of s; v will be determined at a nearly constant factor, we will then $1/\bar{\rho}$, $1/\bar{\tau}$ and $\bar{\varpi}$ and then we going to experience $1/\bar{R}_n$, $1/\bar{R}_g$ and $1/\bar{T}_g$; that is to say $\cos\bar{\varpi}/\bar{\rho}$, $\sin\bar{\varpi}/\bar{\rho}$, and $d\bar{\varpi}/d\bar{s}+1/\bar{\tau}$. The data in this case will involve not only four arbitrary functions of one variable.

47. *Special case. Minimal surface.*
If the surface S is minimal, it will be the same for $\bar{S}$. Relations (7.230) reduce to $\lambda+\alpha=0$, $\mu+\cdot\!\!:=0$, which results in the equation

$$\frac{du}{u}+\frac{dv}{v}=0. \tag{7.243}$$

We then see that the last two quadratic equations (7.244) are consequences of the previous two. The minimum surface $\bar{S}$ is given, the surface must satisfy the three system

$$\begin{cases} \bar{\omega}^1 = u\,\omega^1, \qquad \bar{\omega}_{13} = \dfrac{m}{u}\,\omega_{13}, \qquad \bar{\omega}^3 = 0, \\ \bar{\omega}^2 = u\,\omega^2, \qquad \bar{\omega}_{23} = \dfrac{m}{u}\,\omega_{23}, \\ \omega^1 \wedge \dfrac{du}{u} + \omega^2 \wedge \left(\omega_{12} - \bar{\omega}_{12}\right) = 0, \\ \omega^2 \wedge \dfrac{du}{u} - \omega^1 \wedge \left(\omega_{12} - \bar{\omega}_{12}\right) = 0. \end{cases} \tag{7.244}$$

this system that is in involution and its solution is provided by an arbitrary minimum surface. Two arbitrary minimal surfaces can be mapped in accordance with correspondence of asymptotic lines and lines of curvature: it is a classical result.

If the surface S is not minimal, the surface $\bar{S}$, when it exists, depends on more arbitrary constants.

Problem 13. Pairs of surfaces in point correspondence to the lines of curvature and the second fundamental form

48. By relating the two surfaces to their trihedral of Darboux, one is led to the closes following differential system:

$$\begin{cases} \omega^3 = 0, \quad \bar{\omega}^1 = u\,\omega^1, \quad \omega_{13} = a\,\omega^1, \quad \bar{\omega}_{13} = \dfrac{a}{u}\,\omega^1, \quad (u, v > 0), \\ \bar{\omega}^3 = 0, \quad \bar{\omega}^2 = v\,\omega^2, \quad \omega_{23} = c\,\omega^2, \quad \bar{\omega}_{23} = \dfrac{c}{v}\,\omega^2, \\ \omega^1 \wedge da + (a-c)\,\omega^2 \wedge \omega_{12} = 0, \\ \omega^2 \wedge dc + (a-c)\,\omega^1 \wedge \omega_{12} = 0, \\ \omega^1 \wedge du + \omega^2 \wedge \Big(u\,\omega_{12} - v\bar{\omega}_{12}\Big) = 0, \\ \omega^2 \wedge dv + \omega^1 \wedge \Big(u\,\bar{\omega}_{12} - v\omega_{12}\Big) = 0, \\ \omega^1 \wedge \Big(\dfrac{da}{a} - \dfrac{du}{u}\Big) + \omega^2 \wedge \Big(\omega_{12} - \dfrac{cu}{av}\bar{\omega}_{12}\Big) = 0, \\ \omega^2 \wedge \Big(\dfrac{dc}{c} - \dfrac{dv}{v}\Big) - \omega^1 \wedge \Big(\omega_{12} - \dfrac{av}{cu}\bar{\omega}_{12}\Big) = 0. \end{cases} \tag{7.245}$$

This system is not in involution, two simple linear combinations lead to quadratic equations

$$\begin{cases} \omega^1 \wedge \big(uv\overline{(a+c)}\,\omega_{12} - \overline{(av^2+cu^2)}\,\bar{\omega}_{12}\big), \\ \omega^2 \wedge \big(uv\overline{(a+c)}\,\omega_{12} - \overline{(av^2+cu^2)}\,\bar{\omega}_{12}\big), \end{cases} \tag{7.246}$$

where the new equation

$$(av^2+cu^2)\,\bar{\omega}_{12} = uv(a+c)\,\omega_{12}, \tag{7.247}$$

that removes the last two example equations (7.244).

The new system obtained is provided by the equations

$$\begin{cases} \omega^3 = 0, \quad \bar{\omega}^1 = u\,\omega^1, \quad \omega_{13} = a\,\omega^1, \quad \bar{\omega}_{13} = \dfrac{a}{u}\,\omega^1, \quad (u, v > 0), \\ \bar{\omega}^3 = 0, \quad \bar{\omega}^2 = v\,\omega^2, \quad \omega_{23} = c\,\omega^2, \quad \bar{\omega}_{23} = \dfrac{c}{v}\,\omega^2, \\ (av^2+cu^2)\,\bar{\omega}_{12} = uv(a+c)\,\omega_{12}, \\ \omega^1 \wedge da + (a-c)\,\omega^2 \wedge \omega_{12} = 0, \\ \omega^2 \wedge dc + (a-c)\,\omega^1 \wedge \omega_{12} = 0, \\ \omega^1 \wedge du + cu\dfrac{u^2-v^2}{av^2+cu^2}\,\omega^2 \wedge \omega_{12} = 0, \\ \omega^2 \wedge dv + av\dfrac{u^2-v^2}{av^2+cu^2}\,\omega^1 \wedge \omega_{12} = 0, \\ \big[uv(u^2-v^2)(a\,dc - c\,da) + (a+c)(av^2-cu^2)\,(u\,dv - v\,du)\big] \wedge \omega^2 \\ \qquad + c(av^2+cu^2)\Big(uv(a+c) - \dfrac{av^2+cu^2}{uv}\Big)\,\omega^1 \wedge \omega^2 = 0. \end{cases} \tag{7.248}$$

The generic integral element in two dimensions depends on five arbitrary parameters. *The system is in involution and its general solution depends on five arbitrary functions of one variable.*

The characteristics are given by the equation

$$2uv(u^2-v^2)\frac{av^2-cu^2}{av^2+cu^2}\,\omega_{12}.\big(a(\omega^1)^2-c(\omega^2)^2\big).\omega^1.\omega^2+\Big[uv(u^2-v^2)\times$$

$$\times(a\,dc-c\,da)+(a+c)(av^2-cu^2)\,(u\,dv-v\,du)\Big].(\omega^1)^2.(\omega^2)^2=0. \quad (7.249)$$

Some possibilities have been left coast. We will discard first the case of developable surfaces. The last linear equation (7.229) could be the identity ei had $a+c=0$, $av^2+cu^2=0$, the two surfaces would be minimal, with $u=v$; they would be consistent with conservation of matching lines asymptotic case studied in the previous problem.

If we had, $av^2+cu^2=0$ without $a+c=0$, then the form ω_{12} is zero, and $ac=0$, the case excluded.

46. *Singular solution.*
Equation (7.230) may be an identity, the form w,, is not identically zero, if one has

$$\begin{aligned}(u^2-v^2)(av^2-cu^2)&=0,\\ uv(u^2-v^2)(a\,dc-c\,da)+(a+c)(av^2-cu^2)\,(u\,dv-v\,du)&=0.\end{aligned} \quad (7.250)$$

If $v=u$, falls on the previous problem, but with an additional condition, however narrowly, the two surfaces to be exactly the same second fundamental form. If the surfaces are not minimal, the last linear equation (7.229) shows that$\bar{\omega}_{12}=\omega_{12}$, where $uv(1-uv)=0$ and hence $u=1$. Both are equal surfaces, trivial case.

If $a^2v^2=c^2u^2$, $av=\varepsilon cu$ and if the surfaces are not minimum, we have $\bar{\omega}_{12}=\omega_{12}$, where $u=\sqrt{a/c}$, $v=\varepsilon\sqrt{c/a}$. Then, The quadratic equations (7.229) show that the product has a constant value ac. Both are surfaces constant positive curvature, the same for both surfaces; the principal curvatures are the same in two corresponding points, but to the tangent of curvature principal principal has in the surface S is the tangent of principal principal curvature c in $\bar{S}$. The surface S being given, the surface is determined by a displacement.

We leave aside consideration of the Cauchy problem.[7]

Problem 14. Surfaces $\bar{S}$ in point correspondence with a given surface S of way that the lines of curvature of each surface correspond to the asymptotics of the other.

50. We naturally assume that the surfaces S and $\bar{S}$ with opposite curvatures and we shall relate to their Darboux trihedral. On both surfaces correspond to the tangents harmonics conjugate both with respect to the tangents to the tangents and main asymptotic tangents represented by the equations

$$a(\omega^1)^2-c(\omega^2)^2=0, \qquad \bar{a}(\bar{\omega}^1)^2-\bar{c}(\bar{\omega}^2)^2=0. \quad (7.251)$$

[7] On the determination of surfaces admitting a given second fundamental form, see an article by E. Cartan [9].

We get a relation of the form

$$\bar{a}(\bar{\omega}^1)^2 - \bar{c}(\bar{\omega}^2)^2 = \rho\Big(a(\omega^1)^2 - c(\omega^2)^2\Big), \tag{7.252}$$

On the other hand, the equation $\bar{\omega}^1 = 0$ that is one of the asymptotic tangents of S. The result for example, taking into account the above relation,

$$\begin{cases} \sqrt{\bar{a}}\bar{\omega}^1 = \lambda\left(\sqrt{a}\omega^1 - \sqrt{-c}\omega^2\right), \\ \sqrt{-\bar{c}}\bar{\omega}^2 = \lambda\left(\sqrt{a}\omega^1 + \sqrt{-c}\omega^2\right), \end{cases} \tag{7.253}$$

assuming $a > 0$, $c < 0$, $\bar{a} > 0$, $\bar{c} < 0$. Is deduced $\rho = 2\lambda^2$. As a checking, we have

$$\begin{cases} \bar{\omega}^1 \wedge d\bar{a} + (\bar{a} - \bar{c})\,\bar{\omega}^2 \wedge \omega_{12} = 0, \\ \bar{\omega}^2 \wedge d\bar{c} + (\bar{a} - \bar{c})\,\bar{\omega}^1 \wedge \omega_{12} = 0, \\ \sqrt{\bar{a}}\,\bar{\omega}^1 \wedge \left(\dfrac{d\lambda}{\lambda} + \dfrac{\bar{a}+\bar{c}}{\bar{a}-\bar{c}}\dfrac{d\bar{a}}{2\bar{a}}\right) - H\,\bar{\omega}^1 \wedge \bar{\omega}^2 = 0, \\ \sqrt{-\bar{c}}\,\bar{\omega}^2 \wedge \left(\dfrac{d\lambda}{\lambda} - \dfrac{\bar{a}+\bar{c}}{\bar{a}-\bar{c}}\dfrac{d\bar{c}}{2\bar{c}}\right) - K\,\bar{\omega}^1 \wedge \bar{\omega}^2 = 0. \end{cases} \tag{7.254}$$

formulas agree with the statement of the problem.

By adding to the equations (7.256) linear equations

$$\bar{\omega}^2 = 0, \qquad \bar{\omega}_{12} = \bar{a}\,\bar{\omega}^1, \qquad \bar{\omega}_{23} = \bar{c}\,\bar{\omega}^2, \tag{7.255}$$

and quadratic equations resulting from the outer exterior differentiation of (7.256) and (7.255), we obtain

$$\begin{cases} \sqrt{\bar{a}}\bar{\omega}^1 = \lambda\left(\sqrt{a}\omega^1 - \sqrt{-c}\omega^2\right), \\ \sqrt{-\bar{c}}\bar{\omega}^2 = \lambda\left(\sqrt{a}\omega^1 + \sqrt{-c}\omega^2\right), \end{cases} \tag{7.256}$$

The closed system of differential problem is constituted by the equations (7.256, 7.255, and 7.254). We find that

$$\begin{aligned} H\,\bar{\omega}^1 \wedge \bar{\omega}^2 &= \lambda d\left(\sqrt{a}\omega^1 - \sqrt{-c}\omega^2\right), \\ &= \lambda\frac{a+c}{2}\left(\frac{h}{\sqrt{a}} + \frac{k}{\sqrt{-c}}\right)\omega^1 \wedge \omega^2, \\ &= \frac{(a+c)\sqrt{-\bar{a}\bar{c}}}{4\lambda\sqrt{-ac}}\left(\frac{h}{\sqrt{a}} + \frac{k}{\sqrt{-c}}\right)\bar{\omega}^1 \wedge \bar{\omega}^2, \\ K\,\bar{\omega}^1 \wedge \bar{\omega}^2 &= \lambda d\left(\sqrt{a}\omega^1 + \sqrt{-c}\omega^2\right), \\ &= \lambda\frac{a+c}{2}\left(\frac{h}{\sqrt{a}} - \frac{k}{\sqrt{-c}}\right)\omega^1 \wedge \omega^2, \\ &= \frac{(a+c)\sqrt{-\bar{a}\bar{c}}}{4\lambda\sqrt{-ac}}\left(\frac{h}{\sqrt{a}} - \frac{k}{\sqrt{-c}}\right)\bar{\omega}^1 \wedge \bar{\omega}^2, \end{aligned} \tag{7.257}$$

where h and k denote the coefficients that enter into the form ω_{12}:

$$\omega_{12} = h\,\omega^1 + k\,\omega^2. \tag{7.258}$$

The elements integral to both dimensions of four dependent parameters and the number of quadratic equations (7.254) being equal to four, the system is in involution and its general solution depends on four arbitrary functions of one variable.

The characteristics are given by the following equation, obtained by annihilating the determinant of the polar matrix whose columns correspond to the forms $d\bar{a}$, $d\bar{c}$, $\bar{\omega}_{12}$, $d\lambda/\lambda$:

$$\begin{vmatrix} \bar{\omega}^1 & 0 & \bar{\omega}^2 & 0 \\ 0 & \bar{\omega}^2 & \bar{\omega}^1 & 0 \\ \dfrac{1}{2\sqrt{\bar{a}}}\dfrac{\bar{a}+\bar{c}}{\bar{a}-\bar{c}}\,\omega^1 & 0 & 0 & \sqrt{\bar{a}}\,\omega^1 \\ 0 & \dfrac{1}{2\sqrt{-\bar{c}}}\dfrac{\bar{a}+\bar{c}}{\bar{a}-\bar{c}}\,\omega^2 & 0 & \sqrt{-\bar{c}}\,\omega^2 \end{vmatrix} = 0. \tag{7.259}$$

A one factor that is neither zero nor infinite, this equation is

$$(\bar{a}+\bar{c})\,\bar{\omega}^1.\bar{\omega}^2.\big(a(\omega^1)^2 - c(\omega^2)^2\big) = 0. \tag{7.260}$$

The characteristics of this complete differential system are lines of curvature and asymptotic lines of integral surfaces.

51. *Singular solution.*
These are the integral surfaces minimal ($\bar{a}+\bar{c}=0$). The last two equations (7.254) results, by an easy calculation,

$$\frac{2d\lambda}{\lambda} = \frac{a+c}{\sqrt{-ac}}\big(h\,\omega^1 + k\,\omega^2\big). \tag{7.261}$$

The function λ exists only if the second member is an exact differential, if so λ is defined in an almost arbitrary constant factor. The surface is then $\bar{S}$ an arbitrary minimum area, and the point correspondence between S and $\bar{S}$ depends on the arbitrary constant that enters the λ expression. If the minimum surface is S, λ is an arbitrary constant non-zero.

We will not address the problem of determining the surfaces S for which the equation (7.261) is completely integrable, that is to say that satisfy the equation

$$\omega^1 \wedge d\Big(h\frac{a+c}{\sqrt{-ac}}\Big) + \omega^2 \wedge d\Big(h\frac{a+c}{\sqrt{-ac}}\Big) = 0. \tag{7.262}$$

51. *General solution. Cauchy problem.* Let correspond to a given line C of S which is neither a line nor a curve asymptotic line given a line and a *barC* oriented developable circumscribed, so that $1/\bar{\rho}$, $1/\bar{\tau}$ and angle $\bar{\boldsymbol{\varpi}}$ are functions of data $\bar{s}$, and also give us the point correspondence between two curves. Linear equations (7.256) and

(7.255) provide, by appointing the angles θ and $\bar{\theta}$ which the tangents to the positive curves C and $\bar{C}$ with the vectors $\mathbf{e}_1$ corresponding

$$\begin{cases} \sqrt{\bar{a}}\cos\bar{\theta}\,d\bar{s} = \lambda\left(\sqrt{a}\cos\bar{\theta} - \sqrt{-c}\sin\theta\right)ds, \\ \sqrt{-\bar{c}}\sin\bar{\theta}\,d\bar{s} = \lambda\left(\sqrt{a}\cos\bar{\theta} + \sqrt{-c}\sin\theta\right)ds, \\ \dfrac{\cos\bar{\varpi}}{\bar{\rho}} = \bar{a}\cos^2\bar{\theta} + \bar{c}\sin\bar{\theta}, \\ \dfrac{d\bar{\varpi}}{d\bar{s}} + \dfrac{1}{\bar{\tau}} = (\bar{c} - \bar{a})\sin\bar{\theta}\cos\bar{\theta}. \end{cases} \tag{7.263}$$

The report $d\bar{s}/ds$ being known as θ, we have

$$\sqrt{\bar{a}}\cos\bar{\theta} = m\lambda, \qquad \sqrt{-\bar{c}}\sin\bar{\theta} = n\lambda, \tag{7.264}$$

where m and n have well known by the curve C; We then

$$(m^2 - n^2)\lambda^2 = \frac{\cos\bar{\varpi}}{\bar{\rho}}, \tag{7.265}$$

equation which gives λ; we finally

$$m^2\lambda^2\tan\bar{\varpi} + n^2\lambda^2\cot\bar{\varpi} = -\frac{d\bar{\varpi}}{d\bar{s}} - \frac{1}{\bar{\tau}}, \tag{7.266}$$

where $\bar{\theta}$ is drawn; finally we deduce $\bar{a}$ and $\bar{c}$. The one dimensional solution of the system is thus completely determined. As verification, the data depend effectively four arbitrary functions of one variable.

Suppose now that the line C is an asymptotic line of S, for example

$$\sqrt{a}\cos\theta + \sqrt{-c}\sin\theta = 0. \tag{7.267}$$

Equations (7.263) then give $\bar{\theta} = 0$ and

$$\begin{cases} \sqrt{\bar{a}}\,d\bar{s} = 2\lambda\sqrt{a}\cos\theta\,ds, \\ \dfrac{\cos\bar{\varpi}}{\bar{\rho}} = \bar{a}, \\ \dfrac{d\bar{\varpi}}{d\bar{s}} + \dfrac{1}{\bar{\tau}} = 0. \end{cases} \tag{7.268}$$

There is an additional condition, the last equation (7.254) shows that we must have along $\bar{C}$

$$\frac{d\lambda}{\lambda} - \frac{\bar{a}+\bar{c}}{\bar{a}-\bar{c}} + \frac{(a+c)\sqrt{\bar{a}}}{4\lambda\sqrt{-ac}}\left(\frac{h}{\sqrt{a}} - \frac{k}{\sqrt{-c}}\right)d\bar{s} = 0. \tag{7.269}$$

Give us the line C, the third at equation (7.268) gives $\bar{\varpi}$ a constant, the second gives $\bar{a}$. If we now give the point transformation between C and $\bar{C}$, the first equation

(7.268) is λ, and equation (7.269) gives a differential equation in $\bar{c}$, allowing the function to have an arbitrary constant pressure. This time dependent data of three arbitrary functions of one variable. Assume that the line C is a line of curvature.

Assume that the line C is a line of curvature of S, for example $\theta = 0$. Equations (7.263) then becomes

$$\begin{cases} \sqrt{\bar{a}}\,d\bar{s} = \sqrt{-\bar{c}}\sin\bar{\theta}\,d\bar{s} = \lambda\sqrt{a}\,ds, \\ \dfrac{\cos\bar{\varpi}}{\bar{\rho}} = \bar{a}\cos^2\bar{\theta} + \bar{c}\sin^2\bar{\theta} = 0, \\ \dfrac{d\bar{\varpi}}{d\bar{s}} + \dfrac{1}{\bar{\tau}} = (\bar{c}-\bar{a})\sin\bar{\theta}\cos\bar{\theta}. \end{cases} \tag{7.270}$$

We must add an additional condition. A linear combination of the quadratic equations (7.254) leads to the equation

$$\left(\sqrt{-\bar{a}}\,\bar{\omega}^1 - \sqrt{-\bar{c}}\,\bar{\omega}^2\right) \wedge \left(\frac{d\lambda}{\lambda} + \frac{\bar{a}+\bar{c}}{2\sqrt{-1\bar{a}\bar{c}}}\,\bar{\omega}_{12}\right) = (H-K)\,\bar{\omega}^1 \wedge \bar{\omega}^2, \tag{7.271}$$

hence the required condition

$$\frac{d\lambda}{\lambda} + \frac{\bar{a}+\bar{c}}{2\sqrt{-1\bar{a}\bar{c}}}\,\bar{\omega}_{12} = \frac{\lambda(H-K)}{\sqrt{-1\bar{a}\bar{c}}}\,d\bar{s} = \frac{n\,d\bar{s}}{\sqrt{-1\bar{a}\bar{c}}}, \tag{7.272}$$

where n is a known function.

Let us then the curve $\bar{C}$, with $\bar{\varpi} = \pi/2$ and the point transformation between C and $\bar{C}$. This we will have

$$\sqrt{\bar{a}}\cos\bar{\theta} = \sqrt{-\bar{c}}\sin\bar{\theta} = m\lambda, \tag{7.273}$$

m is known, then

$$m^2\lambda^2 = -\frac{1}{\bar{\tau}}\sin\bar{\theta}\cos\bar{\theta}. \tag{7.274}$$

By bringing the value of λ^2 in (XIV, 9), where $\bar{\omega}_{12}$ is replaced by its value $-d\bar{\theta} + \sin\bar{\varpi}\,d\bar{s}/\bar{\rho}$, we obtain a differential equation in $\bar{\theta}$, which determines $\bar{\theta}$ up to constant, hence the values of λ, $\bar{a}$ and $\bar{c}$. The data are still dependent on three arbitrary functions of one variable.

Remark. The famous transformation of S. Lie changing the lines into spheres and vice versa passes a surface S to a surface $\bar{S}$ whose asymptotic lines correspond to lines of curvature of S and vice versa. But this transformation does not exist in the real domain; remains of the surfaces $\bar{S}$ that correspond to a surface S are independent as of arbitrary constants.

Problem 15. Pairs of convex surfaces such that the point transformation

of one asymptotic lines correspond to lines of the other minimum.

53. We say that a surface is convex if its total curvature is everywhere positive. Clearly we can restrict ourselves to surface portions that have this property.

The lines of curvature are on both surfaces, since the principal tangents are both harmonics conjugate with respect to the asymptotic tangents and the minimum tangents.

We will attach to different points of the two surfaces of the trihedral corresponding right-handed Darboux trihedral. The closed system attached to the differential problem is formed of linear equations

$$\begin{cases} \omega^3 = 0, \quad \bar{\omega}^1 = u\sqrt{a}\,\omega^1, \quad \omega_{13} = a\,\omega^1, \quad \bar{\omega}_{13} = \dfrac{v}{\sqrt{a}}\,\omega^1, \\ \bar{\omega}^3 = 0, \quad \bar{\omega}^2 = u\sqrt{c}\,\omega^2, \quad \omega_{23} = c\,\omega^2, \quad \bar{\omega}_{23} = \dfrac{v}{\sqrt{c}}\,\omega^2, \end{cases} \tag{7.275}$$

it was assumed the Darboux trihedral of the first area selected so that the principal curvatures are positive, we can also assume $u > 0$.

The differential system is completed by the quadratic equations

$$\begin{cases} \omega^1 \wedge \dfrac{du}{u} + \omega^2 \wedge \left(\dfrac{a+c}{2a}\omega_{12} - \sqrt{\dfrac{c}{a}}\bar{\omega}_{12}\right) = 0, \\ \omega^2 \wedge \dfrac{du}{u} - \omega^1 \wedge \left(\dfrac{a+c}{2c}\omega_{12} - \sqrt{\dfrac{a}{c}}\bar{\omega}_{12}\right) = 0, \\ \omega^1 \wedge \dfrac{dv}{v} + \omega^2 \wedge \left(\dfrac{3a-c}{2a}\omega_{12} - \sqrt{\dfrac{a}{c}}\bar{\omega}_{12}\right) = 0, \\ \omega^2 \wedge \dfrac{dv}{v} - \omega^1 \wedge \left(\dfrac{3a-c}{2c}\omega_{12} - \sqrt{\dfrac{c}{a}}\bar{\omega}_{12}\right) = 0. \end{cases} \tag{7.276}$$

The generic integral element in two dimensions depends on six arbitrary parameters, the system is in involution as independent quadratic equations (7.276) are six in number, and *the general solution depends on six arbitrary functions of one variable.*

The determinant of the matrix polar, whose columns correspond to the forms $da/(a-c)$, $dc/(a-c)$, du/u, dv/v, ω_{12}, $\bar{\omega}_{12}$ is, after division by $a-c$ of the first two lines,

$$\begin{vmatrix} \omega^1 & 0 & 0 & 0 & \omega^2 & 0 \\ 0 & \omega^2 & 0 & 0 & \omega^1 & 0 \\ 0 & 0 & \omega^1 & 0 & \dfrac{a+c}{2a}\omega^2 & -\sqrt{\dfrac{c}{a}}\,\omega^2 \\ 0 & 0 & \omega^2 & 0 & -\dfrac{a+c}{2a}\omega^1 & \sqrt{\dfrac{a}{c}}\,\omega^1 \\ 0 & 0 & \omega^1 & 0 & \dfrac{3a-c}{2c}\omega^2 & -\sqrt{\dfrac{a}{c}}\,\omega^2 \\ 0 & 0 & \omega^2 & 0 & -\dfrac{3c-a}{2c}\omega^1 & \sqrt{\dfrac{c}{a}}\,\omega^1 \end{vmatrix} \tag{7.277}$$

its value is

$$\frac{(a-c)^2}{2ac\sqrt{ac}}\omega^1.\omega^2.\big((\omega^1)^2+(\omega^2)^2\big).\big(a(\omega^1)^2+c(\omega^2)^2\big). \tag{7.278}$$

The only actual characteristics correspond to the lines of curvature of the two surfaces.

There is no singular solution; of the manner in same after we posed the problem analytically, we exclude surface portions containing umbilici points. But it is obvious that any pair of spheres is a solution of the problem.

54. *Cauchy problem.* Do we give two curves C and $\bar{C}$, and the law of correspondence between these two curves, we finally give the angles ϖ and $\bar{\varpi}$ that is to say, the developable circumscribed Σ and $\bar{\Sigma}$. Equations (7.275) containing four unknown functions a, c, u, v, and the angles θ and $\bar{\theta}$ that are the tangents to the curves C and $\bar{C}$ with the vectors $\mathbf{e}_1$ corresponding to write as

$$\begin{aligned}
&\cos\bar{\theta}\,d\bar{s} = u\sqrt{a}\cos\bar{\theta}\,ds, && \sin\bar{\theta}\,d\bar{s} = u\sqrt{c}\sin\bar{\theta}\,ds,\\
&\frac{\cos\varpi}{\rho} = a\cos^2\bar{\theta}+c\sin^2\theta, && \frac{d\varpi}{ds}+\frac{1}{\tau} = (c-a)\sin\theta\cos\theta,\\
&\frac{\cos\bar{\varpi}}{\bar{\rho}} = \frac{v}{uc}\cos^2\bar{\theta}+\frac{v}{uc}\sin^2\bar{\theta}, && \frac{d\bar{\varpi}}{d\bar{s}}+\frac{1}{\bar{\tau}} = \frac{v}{u}\frac{a-c}{ac}\sin\bar{\theta}\cos\bar{\theta}.
\end{aligned} \tag{7.279}$$

It was therefore finished six equations with six unknowns $\bar{\theta}$, θ, a, c, u, v.

If the angle θ is zero (C be the line of curvature of S), the angle $\bar{\theta}$ will also be zero, and equations will be reduced to

$$\begin{cases}
d\bar{s} = u\sqrt{a}\,ds, & \dfrac{d\varpi}{ds}+\dfrac{1}{\tau} = 0, \qquad a = \dfrac{\cos\varpi}{\rho},\\
\dfrac{v}{au} = \dfrac{\cos\bar{\varpi}}{\bar{\rho}}, & \dfrac{d\bar{\varpi}}{d\bar{s}}+\dfrac{1}{\bar{\tau}} = 0.
\end{cases} \tag{7.280}$$

But we must add an additional condition which is obtained by searching a linear combination of equations (7.276) not containing. Thus we find the new condition

$$a\frac{dv}{v}-c\frac{du}{u}+\frac{c-a}{2c}dc = 0. \tag{7.281}$$

Give us the two curves C, $\bar{C}$ smells and the point transformation between them. The third and fifth equations (7.280) give ϖ and $\bar{\varpi}$, gives the second was the first to give u, and the fourth gives the equation (7.281) gives c using a differential equation. The data depend on five arbitrary functions of one variable. We shall use the corresponding portions of the curves C and $\bar{C}$ for which the functions a and c are positive.

55. Note. The problem that has been treated and the previous one have a close analogy, but there is however an essential difference between them. Given a surface S has opposite curvatures, there is always an infinity of surfaces $\bar{S}$ that can be matched with point S so as to match the asymptotic surfaces of any of the lines of curvature of the other. Instead, given a convex surface S, it is usually impossible to find a surface $\bar{S}$ can be mapped point with S so has to match the asymptotic lines of surfaces of any of the lines minima of the other surface.

General note. The problems we have reviewed have been treated in Euclidean geometry, but they can arise in non-Euclidean geometry without the analytical apparatus and put essentially different results without undergoing significant change. The only difference comes from the structure equations. Instead of the equations

$$d\omega_{23} = \omega_{12} \wedge \omega_{31}, \qquad d\omega_{31} = \omega_{23} \wedge \omega_{12}, \qquad d\omega_{12} = \omega_{31} \wedge \omega_{23}, \tag{7.282}$$

we have

$$\begin{aligned} d\omega_{23} &= \omega_{12} \wedge \omega_{31} - \mathbf{C}\,\omega^2 \wedge \omega^3, \\ d\omega_{31} &= \omega_{23} \wedge \omega_{12} - \mathbf{C}\,\omega^3 \wedge \omega^1, \\ d\omega_{12} &= \omega_{31} \wedge \omega_{23} - \mathbf{C}\,\omega^1 \wedge \omega^2, \end{aligned} \tag{7.283}$$

In these formulas $\mathbf{C}$ denotes the constant curvature of space.

In applications we have made of structural formulas, the form ω^3 was always zero, the only change is in the expression of $d\omega_{12}$.

The notions of fundamental forms of a surface remains unaltered; the formulas

$$\begin{cases} \omega_{13}\cos\theta + \omega_{23}\sin\theta = \dfrac{\cos\varpi}{\rho}\,ds, \\ \omega_{23}\cos\theta - \omega_{13}\sin\theta = \dfrac{\cos\varpi}{\rho}\,ds, \\ d\theta + \omega_{12} = \dfrac{\sin\varpi}{\rho}\,ds, \end{cases} \tag{7.284}$$

remain unchanged.

Part of the problems treated also keeps a sense of Riemannian geometry in three dimensions. It could also deal with problems of affine differential geometry, projective line, etc., using in each case the structure equations of the group of the corresponding geometry (See E. Cartan [7]).

Chapter 8
Geometric problems with more than two independent variables

8.1 Orthogonal triple systems

57. The search for orthogonal triple systems in ordinary space of three dimensions is reduced to a very simple closed differential system where we take the rectangular trihedral unknowns attached to different points A of space whose unit vectors base $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are respectively normal to the three surfaces of the system which pass through A.

First recall the Darboux equations (structure equations) of Chapter 7:

$$\begin{cases} d\omega^1 = -\omega^2 \wedge \omega_{12} + \omega^3 \wedge \omega_{31}, \\ d\omega^2 = -\omega^3 \wedge \omega_{23} + \omega^1 \wedge \omega_{12}, \\ d\omega^3 = -\omega^1 \wedge \omega_{31} + \omega^2 \wedge \omega_{23}, \end{cases} \tag{8.1}$$

where ω^1, ω^2, ω^3 are the projections on the axes of the trihedral with origin A of a vector $\overrightarrow{AA'}$ joining A to the point infinitely close A', and ω_{23}, ω_{31}, ω_{12}, are the components of the vector which infinitesimal represents the rotation which brings the trihedral attached to point A to be equipollent to the trihedral attached to point A'.

It was to express each of the equations that $\omega^1 = 0$, $\omega^2 = 0$, $\omega^3 = 0$ is completely integrable, which gives

$$\omega^1 \wedge d\omega^1 = 0, \qquad \omega^2 \wedge d\omega^2 = 0, \qquad \omega^3 \wedge d\omega^3 = 0. \tag{8.2}$$

It follows immediately, taking account of (8.1), the three equations

$$\omega^2 \wedge \omega^3 \wedge \omega_{23} = 0, \qquad \omega^3 \wedge \omega^1 \wedge \omega_{31} = 0, \qquad \omega^1 \wedge \omega^2 \wedge \omega_{12} = 0. \tag{8.3}$$

There is no need to exterior differentiate these equations since this would lead to equations of the fourth degree, identically satisfied for any three-dimensional element.

It is clear that the system (8.3) imposes no condition on a linear element has a dimension or two dimensions to, be complete, we therefore have, for reduced characters s_0 and s_1,

$$s_0 = 0, \qquad s_1 = 0. \tag{8.4}$$

Is now a two-dimensional element, which, among ω^1, ω^2, ω^3 a linear relationship[1]

$$u_1\,\omega^1 + u_2\,\omega^2 + u_3\,\omega^3 = 0, \tag{8.5}$$

the polar system reduces this integral element is reduced obviously

$$u_1\,\omega_{23} + u_2\,\omega_{31} + u_3\,\omega_{12} = 0. \tag{8.6}$$

As its rank is 3, we have $s_2 = 3$, and hence $s_3 = 0$.

On the other hand, the integral three-dimensional elements of six parameters depend arbitrary, since the equations (8.3) give

$$\begin{cases} \omega_{23} = \alpha_1 \wedge \omega^2 - \beta_1 \wedge \omega^3, \\ \omega_{31} = \alpha_2 \wedge \omega^3 - \beta_2 \wedge \omega^1, \\ \omega_{12} = \alpha_3 \wedge \omega^1 - \beta_3 \wedge \omega^2. \end{cases} \tag{8.7}$$

Since the sum $s_1 + 2s_2 + 3s_3 = 2s_2$, is equal to 6, it follows that the system (8.3) is in involution and *its general solution depends on three arbitrary functions of two variables.*

58. *Cauchy problem.* Solutions that do not two-dimensional characteristic of the system (8.3) uniquely determines an orthogonal triple system. Such a solution will be obtained by giving an arbitrary analytic surface Σ, and attaching to each point a arbitrary rectangular trihedral, c is to say by giving each point A in three rectangular unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$. There will, in a sufficiently small neighbourhood of Σ, an orthogonal triple system such that at point A of Σ the surfaces of the three families that go through A are respectively normal to $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$. If we take in. In particular a specific plan, we thus obtain the orthogonal triple systems the most general, each once and only once, and the data depend effectively of three arbitrary functions of points of Σ.

Case where the data are characteristics. The data are characteristic of the equations (8.6), if one of the coefficients u_1, u_2, u_3, the equation $u_1\,\omega^1 + u_2\,\omega^2 + u_3\,\omega^3 = 0$ the tangent plane at a generic point of Σ is zero, or if at any point A of the trihedral Σ attached at this point has one of its axes, the first example, tangent to Σ. In this case it is easy to see that the problem is usually impossible. This follows from equations (8.7), if one moves on Σ in the direction $\mathbf{e}_1$ was $\omega^2 = \omega^3 = 0$, and

[1] The coefficients u_1, u_2, u_3 are none other than the Plückerian coordinates u^{23}, u^{31}, u^{13} with respect to the trihedral with origin A and bi-vector formed by the two-dimensional integral element considered.

consequently the form ω_{23} must be zero. Therefore, for the problem is possible that moving on Sigma along a trajectory of the vector $\mathbf{e}_1$, we have the relation

$$\mathbf{e}_2 \cdot d\mathbf{e}_3 = 0. \tag{8.8}$$

This relation expresses that moving along a path $\mathbf{e}_1$ (that is to say along the intersection of the surfaces of the last two families of triple system unknown) vector $\mathbf{e}_3$ leads me developable surface: it the same is true for the vector $\mathbf{e}_2$: this is a consequence of the classical theorem of Dupin.

If two of coefficients of the equation of the tangent plane at a generic point in Σ were zero, is that one of the axes of the attached to that point, the third example, would be normal to Σ. It is then necessary for the possibility of the problem that moving on Σ in the direction $\mathbf{e}_1$ form ω_{23} is zero and that moving in the direction $\mathbf{e}_2$ form ω_{31}, is zero. A necessary condition of possibility is that two axes of the Σ are attached to the main tangents at this point of the surface Σ. This condition is of sufficient rest, as the family of surfaces parallel to Σ and the two families of normal developable of Σ is an orthogonal triple system corresponding to the data. *This does not mean they are the only solution of the problem.*

Finally, note that given a triple orthogonal surfaces characteristics of this system are formed by the surface areas of intersection curves of two families of triple system.

Remark. We used only the structure equations (8.1) space. As a result, all results are valid in a space of constant curvature, and even in any Riemannian space.

8.2 Triple systems with constant angles

59. We can generalize the problem of orthogonal triple systems in three families searching one-parameter surfaces intersecting two to two angles given constant. If we attach to each point A of space with an orthogonal unit vectors are tangent to the three basic curves of intersection of pairs of three surfaces of the system which pass through this point, the faces of the trihedral angles are constant which we denote its cosines by α, β, γ.

We then relations

$$\begin{cases} dA = \omega^i \mathbf{e}_i, \\ d\mathbf{e}_i = \omega_i^j \mathbf{e}_j, \end{cases} \tag{8.9}$$

with 9 coefficients ω_i^j that satisfy 6 linear relations with constant coefficients. They are obtained by differentiating the relations

$$\mathbf{e}_i^2 = 1, \qquad \mathbf{e}_2 \cdot \mathbf{e}_3 = \alpha, \qquad \mathbf{e}_3 \cdot \mathbf{e}_1 = \beta, \qquad \mathbf{e}_1 \cdot \mathbf{e}_2 = \gamma, \tag{8.10}$$

which gives

$$
\begin{cases}
\omega_1^1 + \gamma\omega_1^2 + \beta\,\omega_1^3 = 0, \\
\omega_2^2 + \alpha\,\omega_2^3 + \gamma\omega_2^1 = 0, \\
\omega_3^3 + \beta\,\omega_3^1 + \alpha\,\omega_3^2 = 0, \\
\omega_2^3 + \omega_3^2 + \alpha\left(\omega_2^2 + \omega_3^3\right) + \beta\,\omega_2^1 + \gamma\omega_3^1 = 0, \\
\omega_3^1 + \omega_1^3 + \alpha\left(\omega_3^3 + \omega_1^1\right) + \gamma\omega_3^2 + \alpha\,\omega_1^2 = 0, \\
\omega_1^2 + \omega_2^1 + \alpha\left(\omega_1^1 + \omega_2^2\right) + \alpha\,\omega_1^3 + \beta\,\omega_2^3 = 0.
\end{cases}
\tag{8.11}
$$

Forms ω_i^j are three in number independent, which agrees with the fact that the orientation of a mobile trihedral remains equal to itself depends on three parameters.

The structure equations that result from the exterior differentiation of the first equation (8.9) (we will not need the other) are

$$
\begin{cases}
d\omega^1 = \omega^1 \wedge \omega_1^1 + \omega^2 \wedge \omega_2^1 + \omega^3 \wedge \omega_3^1 = \omega^i \wedge \omega_i^1, \\
d\omega^2 = \omega^1 \wedge \omega_1^2 + \omega^2 \wedge \omega_2^2 + \omega^3 \wedge \omega_3^2 = \omega^i \wedge \omega_i^2, \\
d\omega^3 = \omega^1 \wedge \omega_1^3 + \omega^2 \wedge \omega_2^3 + \omega^3 \wedge \omega_3^3 = \omega^i \wedge \omega_i^3.
\end{cases}
\tag{8.12}
$$

Setting equations of the problem. Is expressed, as in the case of orthogonal triple systems, each of the equations that $\omega^1 = 0$, $\omega^2 = 0$, $\omega^3 = 0$ is completely integrable, which gives, according to (8.12),

$$
\begin{cases}
\omega^1 \wedge \omega^2 \wedge \omega_2^1 - \omega^3 \wedge \omega^1 \wedge \omega_3^1 = 0, \\
\omega^2 \wedge \omega^3 \wedge \omega_3^2 - \omega^1 \wedge \omega^2 \wedge \omega_1^2 = 0, \\
\omega^3 \wedge \omega^1 \wedge \omega_1^3 - \omega^2 \wedge \omega^3 \wedge \omega_2^3 = 0,
\end{cases}
\tag{8.13}
$$

A new exterior differentiation is unnecessary, the equations (8.13) form the closed system of differential problem.

All the elements in one dimension or two dimensions are integral; the polar system of the integral two-dimensional element $u_1\,\omega^1 + u_2\,\omega^2 + u_3\,\omega^3 = 0$ is

$$
\begin{cases}
u_3\,\omega_2^1 - u_2\,\omega_3^1 = 0, \\
u_1\,\omega_3^2 - u_3\,\omega_1^2 = 0, \\
u_2\,\omega_1^3 - u_1\,\omega_2^3 = 0.
\end{cases}
\tag{8.14}
$$

Its rank s_2 is equal to 3; result, we

$$
s_1 = 0, \qquad s_2 = 3, \qquad s_3 = 0. \tag{8.15}
$$

The generic integral element in three dimensions depends on the other hand $s_1 + 2s_2 + 3s_3 = 6$ arbitrary parameters[2], the system is in involution and its general solution depends on three functions of two arbitrary variables.

[2] There is indeed omega 9 forms, and their expression in terms of 27 coefficients ω^1, ω^2, ω^3 introduced; equations (8.11) each provide three relations between these coefficients, making 18 in all; each equation (8.14) provides a new relation, making a total of $18 + 3 = 21$ relations more. It

60. *Cauchy problem.* Any two-dimensional solution uncharacteristic of the system (8.13) uniquely determines a solution of the problem. We have such a solution by giving us an arbitrary surface Σ and at each point of this surface equal to a trihedral trihedral considered. If we take a fixed plane of Σ, the data depend effectively of three arbitrary functions of two variables.

The data will be characteristic if the equations (8.14), where u_1, u_2, u_3 are the parameters of the tangent plane to the reported trihedral attempts to this point, are among this least three independent, or if the rank of the system formed by the nine equations (8.11) and (8.14) is less than to 9, or finally if the determinant of the coefficients of four in the nine equations is zero.

To form the equation which expresses that this determinant is zero, we can solve first the equations (8.14) by putting

$$\begin{aligned} \omega_2^1 &= \lambda^1 u_2, & \omega_3^1 &= \lambda^1 u_3, \\ \omega_3^2 &= \lambda^2 u_3, & \omega_1^2 &= \lambda^2 u_1, \\ \omega_1^3 &= \lambda^3 u_1, & \omega_2^3 &= \lambda^3 u_2, \end{aligned} \tag{8.16}$$

where λ^1, λ^2, λ^3 designate three auxiliary unknowns. The first three equations (8.11) then give

$$\begin{aligned} \omega_1^1 &= -(\beta\lambda^3 + \gamma\lambda^2)\, u_1, \\ \omega_2^2 &= -(\gamma\lambda^1 + \alpha\lambda^3)\, u_2, \\ \omega_3^3 &= -(\alpha\lambda^2 + \beta\lambda^1)\, u_3, \end{aligned} \tag{8.17}$$

by bringing in the last three equations (8.11), we obtain three equations in λ^1, λ^2, λ^3:

$$\begin{aligned} \big((\beta-\alpha\gamma)\, u_2 + (\gamma-\alpha\beta)\, u_3\big)\, \lambda^1 + (1-\alpha^2)u_3\, \lambda^2 + (1-\alpha^2)u_2\, \lambda^3 &= 0, \\ (1-\beta^2)u_3\, \lambda^1 + \big((\gamma-\alpha\beta)\, u_3 + (\alpha-\beta\gamma)\, u_1\big)\, \lambda^2 + (1-\beta^2)u_1\, \lambda^3 &= 0, \\ (1-\gamma^2)u_2\, \lambda^1 + (1-\gamma^2)u_1\, \lambda^2 + \big((\alpha-\beta\gamma)\, u_1 + (\beta-\gamma\alpha)\, u_2\big)\, \lambda^3 &= 0. \end{aligned} \tag{8.18}$$

The elimination of λ^1, λ^2, λ^3 leads to the required equation

$$\begin{vmatrix} (\beta-\alpha\gamma)\, u_2 + (\gamma-\alpha\beta)\, u_3 & (1-\alpha^2)u_3 & (1-\alpha^2)u_2 \\ (1-\beta^2)u_3 & (\gamma-\alpha\beta)\, u_3 + (\alpha-\beta\gamma)\, u_1 & (1-\beta^2)u_1 \\ (1-\gamma^2)u_2 & (1-\gamma^2)u_1 & (\alpha-\beta\gamma)\, u_1 + (\beta-\gamma\alpha)\, u_2 \end{vmatrix} = 0. \tag{8.19}$$

This is the equation of a cone of the third class (characteristic cone). The plan elements are singular integrals at each point A the tangent planes at characteristic

therefore remains at least $27-21=6$ arbitrary parameters. But on the other hand we know that the number $s_1+2s_2+3s_2$ can not be exceeded cone attached to that point. Being given a triple systems considered, the surfaces corresponding characteristics are those whose tangent plane at any point is tangent to the cone (characteristic) attached to that point.

The characteristic cone is divided into three axes of the in the case of orthogonal triple systems. It decomposes when two faces of the trihedral are right angles; have such $\mathbf{e}_3$ is the vector that is perpendicular to both vectors $\mathbf{e}_1$, $\mathbf{e}_2$, the cone is divided into right and bringing $\mathbf{e}_3$ in two straight plan $A\mathbf{e}_1 \wedge \mathbf{e}_2$ which form an angle $\pi/4$ with the bisector of the angle formed by the two vectors $\mathbf{e}_1$, $\mathbf{e}_2$. There is no other case of decomposition characteristic of the cone.

8.3 p-tuples systems orthogonal to the $p-$dimensional space

61. It is found in the p-dimensional Euclidean space, p families of hypersurfaces to $p-1$-dimensional intersecting orthogonally. We will use the same method as in space has three dimensions by attaching to each point A of space a rectangular $p-$hedral formed by p unit vectors $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_p$, and each rectangular each of which is orthogonal to a of the system of p hypersurfaces passing through A.

By moving from point A to point infinitely close, there will be formulas

$$\begin{cases} dA = \omega^i \mathbf{e}_i, \\ d\mathbf{e}_i = \omega_i^j \mathbf{e}_j, \qquad (i = 1, 2, \cdots, p), \end{cases} \tag{8.20}$$

with forms $\omega_{ij} = -\omega_{ji}$, the last relations that just expressing vectors $\mathbf{e}_i$ are of constant length and intersect orthogonally.

The structure equations of the space are obtained by differentiation of exterior (8.20) and give

$$\begin{cases} d\omega^i = \omega^k \wedge \omega_{ki}, \\ d\omega_{ij} = \omega_{ik} \wedge \omega_{kj}. \end{cases} \tag{8.21}$$

62. This granted, the problem of differential equations simply express that each of the equations $\omega^i = 0$ is completely integrable. Now the equation $\omega^1 \wedge \omega^1 = 0$, for example, describes, taking account of (8.21),

$$\omega^1 \wedge \omega^k \wedge \omega_{k1} = 0. \tag{8.22}$$

The system of equations analogous p is not in involution. But the calculations for $p = 3$, it follows that if we $\omega^2 = \omega^3 = \cdots = \omega^p = 0$, has the form ω_{12} does not depend on ω^3, so it does not depend of $\omega^4, \cdots, \omega^p$. As a result, any solution of the problem considered, we must have

$$\omega^i \wedge \omega^j \wedge \omega_{ij} = 0, \qquad (i, j = 1, 2, \cdots, p) \qquad \text{(not add up)}. \tag{8.23}$$

Reciprocally, the equations (8.23) result in the complete integrability of all equations $\omega^i = 0$.

Complete The equation of the problem demands the addition to equations (8.23) those which are deduced by exterior differentiation, i.e.

$$\omega^j \wedge \omega^k \wedge \omega_{ki} \wedge \omega_{ij} - \omega^i \wedge \omega^k \wedge \omega_{kj} \wedge \omega_{ij} + \omega^i \wedge \omega^j \wedge \omega_{ik} \wedge \omega_{kj} = 0, \qquad (i, j = 1, 2, \cdots, p), \tag{8.24}$$

equations in which we must sum only over the index k.

We will show that the system (8.24) is in involution.

First of all $p-$dimensional integral element is defined by (8.23), by relations of the form

$$\omega_{ij} = \alpha_{ij}\,\omega^j - \alpha_{ji}\,\omega^i, \qquad (i, j = 1, 2, \cdots, p), \tag{8.25}$$

and it is easy to see that, whatever the numerical values given to $p(p-1)$ coefficients α_{ij} $(i \neq j)$, equations (8.24) are consequences of equations (8.25). Indeed, each term of the first member of one of equations (8.24), is of the form $\omega^i \wedge \omega^j \wedge \omega_{ik} \wedge \omega_{kj}$, i, j, k denote three indices still taken in the sequence $1, 2, \cdots, p$, from (8.25), the monomial $\omega^i \wedge \omega^j \wedge \omega_{ik}$ is a multiple of $\omega^i \wedge \omega^j \wedge \omega_{ik}$ and $\omega^i \wedge \omega^j \wedge \omega^k$ its exterior product by ω_{jk} is zero, since ω_{jk} is linear in ω^j and ω^k.

The $p-$dimensional integral elements therefore depend on $p(p-1)$ arbitrary parameters.

Now looking characters reduced of the system (8.23 and 8.24). All two-dimensional element is integral. The polar element of an integral two-dimensional Plückerian components u^{ij} is defined by the equations reduced

$$u^{ij}\,\omega_{ij} = 0, \qquad (i, j = 1, 2, \cdots, p), \tag{8.26}$$

the rank of this system is

$$s_2 = \frac{p(p-1)}{2}. \tag{8.27}$$

Since $p(p-1)/2$ is the number of forms ω_{ij} distinct forms of ω^i, we have

$$s_1 = o, \qquad s_2 = \frac{p(p-1)}{2}, \qquad s-3 = \cdots = s_p = 0, \tag{8.28}$$

and hence

$$s_1 + s_2 + 3s_3 + \cdots + ps_p = p(p-1), \tag{8.29}$$

number of arbitrary parameters upon which the generic $p-$dimensional integral element.

The system (8.23 and 8.24) is in involution and *its general solution depends on $p(p-1)/2$ arbitrary functions of two variables.*

63. *Cauchy problem.* Any two-dimensional non-characteristic solution of the system (8.23 and 8.24) provides one and only one solution of the problem. Now any element integral to both dimensions, we have the most general solution to two dimensions by giving a surface (analytical) Σ arbitrary two-dimensional, and, at each point A of this surface, a $p-$hedral in a rectangular analytical law arbitrary. If data are not typical, there will be close to Σ a $p-$tuple orthogonal and one such that at each point A of Σ the p hypersurfaces of this system are respectively the normal basis vectors of the corresponding $p-$hedral. Surface Σ is fixed, the data involve $p(p-1)/2$ functions of the two curvilinear coordinates of the surface Σ.

The data will be characteristic if the rank of the polar system of the integral element (u^{ij}) tangent to Σ is less than $p(p-1)/2$ that is to say, if at least one component is zero u^{ij}. It is then necessary additional conditions to the problem is possible.

For example, suppose there is only one component zero or u^{12}, which means that each point A of Σ, one of the tangents to the Σ is perpendicular to the plane determined by A and two vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and can not the rest will be more than one, because then all the components and u^{1i} and u^{2i} would be zero, what we exclude. This put it on Sigma exist a family of curves possessing the property that each of their points the tangent is perpendicular to the plane determined by the point A and the vectors $\mathbf{e}_1$, $\mathbf{e}_2$. By moving along one of these curves, it would $\omega^1 = \omega^2 = 0$ and consequently, by (8.25) it would $\omega_{12} = 0$, that is to say $\mathbf{e}_1 \cdot \mathbf{e}_2 = 0$. This is a necessary condition of possibility of the problem. It can be expressed by saying that when moving along any one of two curves orthogonal plans $A\mathbf{e}_1 \wedge \mathbf{e}_2$, the trace of any point $M = A + x^1\mathbf{e}_1 + x^2\mathbf{e}_2$ coordinate x^1, x^2 fixed is orthogonal to the plane $A\mathbf{e}_1 \wedge \mathbf{e}_2$ that contains: we in fact

$$\mathbf{e}_1 d\mathbf{M} = \omega^1 + x^2\omega_{21} = 0, \qquad \mathbf{e}_2 d\mathbf{M} = \omega^2 + x^1\omega_{12} = 0. \tag{8.30}$$

64. There may be more complicated cases. Let us just consider what happens for $p = 4$. We easily see that toue possible cases are reduced to the following six cases:

1° a component u^{ij} zero, since we can assume u^{12};
2° two components are zero, we can assume u^{12}, u^{34};
3° and 4° three components u^{ij} are zero we may assume either u^{23}, u^{31}, u^{12}, or u^{14}, u^{24}, u^{34};
5° four components u^{ij} are zero, we can assume u^{23}, u^{31}, u^{12}, u^{34} ;
6° five components are zero, we can assume u^{12}, u^{13}, u^{14}, u^{23}, u^{24}.

We will see that in each case, there are as many additional conditions of possibility that there components u^{ij} are zero, except for the case (6), which includes six conditions.

The first two cases have already been considered potentially.

3° Suppose $u^{23} = u^{31} = u^{12} = 0$. – The equations of the planar tangent to a point Σ are

$$\frac{\omega^1}{u^{14}} = \frac{\omega^2}{u^{24}} = \frac{\omega^3}{u^{34}}, \tag{8.31}$$

as a result there is at each point A of a tangent determined undermined Σ along which we have $\omega^1 = \omega^2 = \omega^3$ bone, the three forms ω_{23}, ω_{31}, ω_{12}, becoming zero. There is thus a one parameter family of lines (C) of Σ along which the quantities $\mathbf{e}_2 d\mathbf{e}_3$, $\mathbf{e}_3 d\mathbf{e}_1$, $\mathbf{e}_1 d\mathbf{e}_2$ must be zero for the problem is possible. This means that even if the three-dimensional space $A\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$ attach each point A to Σ we consider a point M of coordinates x^1, x^2, x^3 fixed, the location of the point M, when A point describes a curve (C), is an orthogonal trajectory corresponding spaces $A\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$.

4° Now suppose $u^{14} = u^{24} = u^{34} = 0$. – The equations of the element tangent plane at a point of Σ are

$$u^{23}\omega^1 + u^{31}\omega^2 + u^{12}\omega^3 = 0, \qquad \omega^4 = 0. \tag{8.32}$$

there in the surface Σ has three families of curves one parameter: the first consists of lines (C_1) which was the along $\omega^1 = 0$, the second lines (C_2) along which was $\omega^2 = 0$, the third lines (C_3) along which is a $\omega^3 = 0$. The form ω_{i4} is zero if one moves on the line (C_i) $(i = 1,2,3)$. It thus gives three conditions of possibility, each with a geometric interpretation similar to that which was given in case one of the components u^{ij} is zero. It came out of the conditions necessary for it to pass through the surface Σ hypersurface belonging orthogonal to a quadruple system.

5° Suppose $u^{23} = u^{31} = u^{12} = u^{34} = 0$. – The equations of the planar member Σ are tangent to

$$u^{14}\omega^2 - u^{24}\omega^1 = 0, \qquad \omega^3 = 0. \tag{8.33}$$

In Σ there a family of curves (C_1) along which it $\omega^1 = \omega^2 = 0$ and a family of curves (C_2) along which was $\omega^4 = 0$. Along the curves (C_1) forms ω_{12}, ω_{13}, ω_{23}, are zero along curves (C_2) the form ω_{34} is zero. The consideration of lines (C_1) provides three conditions of possibility, that of curves (C_2) a fourth condition.

6° Suppose finally $u^{12} = u^{13} = u^{23} = u^{14} = u^{24} = 0$. – The equations of the element tangent plane to Σ are

$$\omega^1 = \omega^2 = 0. \tag{8.34}$$

There are in Σ a family of curves (C_1) along which we have $\omega^3 = 0$, and a family of curves (C_2) along which we have $\omega^4 = 0$. Along curves (C_1) forms ω_{12}, ω_{13}, ω_{23} are zero along the curves (C_2) formsω_{12}, ω_{14}, ω_{24} are zero. This gives six conditions of possibility. These are six conditions for the surface Σ can be regarded as the intersection of two hypersurfuces a quadruple orthogonal system (Here a hypersurface of the first family and a hypersurface of the second family of the system).

65. *Varieties characteristics.* Given a $p-$tuple orthogonal system, variety characteristics of this system are those whose all two-dimensional elements are tangent to singular, that is to say that at least one components u^{ij} are zero. Assume for simplicity $p=4$, and denote by ξ^1, ξ^2, ξ^3, ξ^4 parameters of hypersurfaces of each of the four families of the system. The six cases considered in the previous issue are six classes of characteristic varieties.

1° $u^{12}=0$. – We have varieties satisfying an equation $f(\xi^1, xi^2)=0$;
2° $u^{12}=u^{34}=0$. – We have varieties satisfying an equation

$$f(\xi^1, xi^2)=0, \qquad \varphi(\xi^3, xi^4)=0. \tag{8.35}$$

3° $u^{23}=u^{31}=u^{12}=0$. – We have varieties satisfying an independent equation

$$f(\xi^1, xi^2, \xi^3)=0, \qquad \varphi(\xi^1, xi^2, \xi^3)=0. \tag{8.36}$$

4° $u^{14}=u^{24}=u^{34}=0$. – We have the varieties $\xi^4=$ cte., that is to say, hypersurfaces of one of four families of the given system and all the varieties in two dimensions in such a hypersurface.
5° $u^{23}=u^{31}=u^{12}=u^{34}=0$. – We have varieties satisfying an independent equation

$$\xi^3=\text{cte.}, \qquad \varphi(\xi^1, xi^2)=0. \tag{8.37}$$

6° $u^{12}=u^{13}=u^{23}=u^{14}=u^{24}=0$. – We have the intersections of two hypersurfaces given orthogonal quadruple system.

66. *Remark.* There is nothing to change in the solution of systems of p-tuples orthogonal if $p-$dimensional space has constant curvature. But it is more even if one is in an arbitrary Riemannian space[3], the reason is that equations (8.24) involve the exterior differential forms ω_{ij}, and in general, every element that satisfies the three-dimensional equations (8.23) does not satisfy equations (8.24). This drawback does not present itself in the problem of orthogonal triple systems. Besides the existence of a $p-$tuple orthogonal in a p-dimensional Riemannian space leads to the possibility of representing the ds^2 of this space as a quadratic form

$$g_1\,(d\xi^1)^2+g_1\,(d\xi^2)^2+\cdots+g_1\,(d\xi^p)^2, \tag{8.38}$$

or ds^2 the most general $p-$dimensions contains $p(p-1)/2$ arbitrary functions of p variables, it is true that one can always make a change of variables to reduce the p coefficients to have fixed numerical values, but will not generally cancel $p(p-1)/2$ of these coefficients, since $p(p-1)/2$ is greater than that of p, and p is greater than 3.

[3] below the note No. 67

8.4 Realization of a three-dimensional Riemannian space with a manifold of Euclidean space

67. We have already seen (No. 20 and 21) the problem of application of surfaces, which can be seen as the problem of finding a surface having a given ds^2. A similar problem arises if we are given a quadratic differential form defined three variables can be found in one. Euclidean space a three-dimensional variety whose ds^2 is precisely this particular form? We will show that the problem is always possible in a six-dimensional Euclidean space, but it is generally impossible in Euclidean space of five or four dimensions.

In Riemannian geometry (See E. Cartan [10]), can be attached to each point A of a three-dimensional space as a rectangular trihedral in Euclidean geometry, the infinitesimal displacement of the trihedral may, by a suitable convention, also be defined by six forms ϖ^1, ϖ^2, ϖ^3, $\varpi_{23} = -\varpi_{32}$, $\varpi_{31} = -\varpi_{13}$, $\varpi_{12} = -\varpi_{21}$. ds^2 the space is equal to the sum of squares $(\varpi^1)^2 + (\varpi^2)^2 + (\varpi^3)^2$. It was further the formulas

$$\begin{cases} d\mathbf{A} = \varpi^1 \mathbf{e}_1 + \varpi^2 \mathbf{e}_2 + \varpi^3 \mathbf{e}_3, \\ \mathrm{D}\mathbf{e}_1 = \varpi_{12} \mathbf{e}_2 - \varpi_{31} \mathbf{e}_3, \\ \mathrm{D}\mathbf{e}_2 = -\varpi_{12} \mathbf{e}_1 + \varpi_{23} \mathbf{e}_3, \\ \mathrm{D}\mathbf{e}_3 = \varpi_{31} \mathbf{e}_1 - \varpi_{23} \mathbf{e}_2, \end{cases} \tag{8.39}$$

Symbol $\mathrm{D}\mathbf{e}_i$ is a symbol of *covariant differentiation*. The equations of structure is also generalized, but in part; they are written

$$\begin{cases} d\varpi^1 = -\varpi^2 \wedge \varpi_{12} + \varpi^2 \wedge \varpi_{31}, \\ d\varpi^2 = \varpi^1 \wedge \varpi_{12} - \varpi^3 \wedge \varpi_{23}, \\ d\varpi^3 = -\varpi^1 \wedge \varpi_{31} + \varpi^2 \wedge \varpi_{23}, \\ d\varpi_{23} = \varpi_{12} \wedge \varpi_{31} - K_{11}\,\varpi^2 \wedge \varpi^3 - K_{12}\,\varpi^3 \wedge \varpi^1 - K_{13}\,\varpi^1 \wedge \varpi^2, \\ d\varpi_{31} = \varpi_{23} \wedge \varpi_{12} - K_{21}\,\varpi^2 \wedge \varpi^3 - K_{22}\,\varpi^3 \wedge \varpi^1 - K_{23}\,\varpi^1 \wedge \varpi^2, \\ d\varpi_{12} = \varpi_{31} \wedge \varpi_{23} - K_{31}\,\varpi^2 \wedge \varpi^3 - K_{32}\,\varpi^3 \wedge \varpi^1 - K_{33}\,\varpi^1 \wedge \varpi^2. \end{cases} \tag{8.40}$$

The coefficients define the $K_{ij} = K_{ji}$ Riemannian curvature of space.

If one attaches to each point in space a fixed rectangular trihedral, or forms, ϖ^i, ϖ_{ij} are linear combinations of differentials of the coordinates u^1, u^2, u^3 of a point in space, coordinates defined following by any act. If one attaches to each point in the opposite rectangular trihedral as general as possible with this point as origin, ϖ_{ij} forms depend linearly on the differential parameters of three new, distinct from the original point coordinates, and used to set the direction of the trihedral. The formulas (8.39) and (8.40) is valid in both cases.

68. Before addressing the problem of realizing a given Riemannian space with a variety of three-dimensional Euclidean space of six dimensions, recall the basic formulas of the method of rectangular hexahedron mobile formulas that are a special case of formulas (8.39) and (8.40) No. 61. By attaching to each point A of the space defined by a rectangular hexahedron six rectangular unit vectors e. , We have the relations

$$\begin{cases} dA = \omega^i \mathbf{e}_i, \\ d\mathbf{e}_i = \omega_{ij}\mathbf{e}_j, \qquad (\omega_{ij} = -\omega_{ji}), \qquad (i = 1, 2, \cdots, 6), \end{cases} \tag{8.41}$$

with the structure equations

$$\begin{cases} d\omega^i \ = \omega^k \wedge \omega_{ki}, \qquad (i = 1, 2, \cdots, 6), \\ d\omega_{ij} = \omega_{ik} \wedge \omega_{kj}, \qquad (i, j = 1, 2, \cdots, 6). \end{cases} \tag{8.42}$$

69. This being established, let us given a Riemannian space $\mathscr{E}$ and its three-dimensional equations of structure (8.40) for a choice of rectangular trihedral attached to its different points. The problem that we will ask involves matching each of these trihedral a rectangular hexahedron in Euclidean space six-dimensional E_6, whose origin point will describe a three-dimensional variety V such that the three vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ of the hexahedron are tangential to V, the other three $\mathbf{e}_4$, $\mathbf{e}_5$, $\mathbf{e}_6$ it being normal, so that finally, in the infinitesimal displacement of the hexahedron which corresponds to an infinitesimal displacement of the trihedral of the Riemannian space, we have

$$(\omega^1)^2 + (\omega^2)^2 + (\omega^3)^2 = (\varpi^1)^2 + (\varpi^2)^2 + (\varpi^3)^2. \tag{8.43}$$

This relation shows that we can pass to ϖ^i to ω^i by orthogonal substitution, which means we can say, in the three-dimensional space tangent to V at any point M of V, to subject the vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ from this point of rotation M together around so as to obtain $\varpi^i = \omega^i$.

Denote by Latin letters $i, j, \cdots$ the indices $1, 2, 3$ and by Greek letters α, β, γ indices $4, 5, 6$. The problem will come back to integrate the system.

$$\omega^i = \varpi^i, \quad (i = 1, 2, 3), \qquad\qquad \omega^\alpha = 0, \quad (\alpha = 4, 5, 6), \tag{8.44}$$

The exterior differentiation of the first three equations gives, from (8.40) and (8.42),

$$\begin{aligned} &\omega^2 \wedge (\omega_{12} - \varpi_{12}) - \varpi^3 \wedge (\omega_{31} - \varpi_{31}) = 0, \\ &\omega^3 \wedge (\omega_{23} - \varpi_{23}) - \varpi^1 \wedge (\omega_{12} - \varpi_{12}) = 0, \\ &\omega^1 \wedge (\omega_{31} - \varpi_{31}) - \varpi^2 \wedge (\omega_{23} - \varpi_{23}) = 0, \end{aligned} \tag{8.45}$$

Equations from which we deduce

$$\omega_{23} = \varpi_{23}, \qquad \omega_{31} = \varpi_{31}, \qquad \omega_{12} = \varpi_{12}. \tag{8.46}$$

As for the last three equations (IV, 5), they shall, by exterior differentiation,

$$\varpi^k \wedge \omega_{k\alpha} = 0, \qquad (\alpha = 1,2,3), \tag{8.47}$$

where the summation index k takes the values $1,2,3$.

Finally equations (8.46) gives differentiated

$$\omega_{ik} \wedge \omega_{jk} + \omega_{i\lambda} \wedge \omega_{j\lambda} = \varpi_{\ell 1} \wedge \varpi_{jk} + K_{\ell 1}\, \varpi^2 \wedge \varpi^3 + K_{\ell 2}\, \varpi^3 \wedge \varpi^1 + K_{\ell 3}\, \varpi^1 \wedge \varpi^2, \tag{8.48}$$

denoting by ℓ the Latin index, with both indices i, j, determines a permutation (i, j, ℓ) of the three indices $1,2,3$.

Finally we obtain the closed differential system

$$\begin{cases} \omega^i = \varpi^i, \\ \omega^\alpha = 0, \\ \omega_{ij} = \varpi_{ij}, \\ \varpi^i \wedge \omega_{k\alpha} = 0, \\ \omega_{i\lambda} \wedge \omega_{j\lambda} = K_{\ell 1}\, \varpi^2 \wedge \varpi^3 + K_{\ell 2}\, \varpi^3 \wedge \varpi^1 + K_{\ell 3}\, \varpi^1 \wedge \varpi^2. \end{cases} \tag{8.49}$$

Number of these equations is 15, 9 of which are linear and 6 quadratic, in three independent variables of the coordinates of a point in given Riemannian space; forms ϖ^i and ϖ_{ij} are known linear combinations of differentials of these coordinates, which the first three are independent. There are 21 unknown functions, namely the parameters needed for the more general rectangular hexahedron in $6-$dimensional Euclidean space; forms ω^1, ω^α, ω_{ij}, $\omega_{i\alpha}$ are 18 independent linear forms constructed with these 21 parameters and their differentials. But in reality, the number of unknown functions is not equal to 18 parasites, because the equations expressed as the origin point of the hexahedron moving remains fixed and the vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, and these equations are completely integrable system whose general solution is formed by all the figures made a point and three rectangular unit vectors from this point, figures which in fact depend of 18 parameters (6 for the coordinates of point 5 for the components of $\mathbf{e}_1$, 4 for those $\mathbf{e}_2$ and 3 for those $\mathbf{e}_3$).

The system (8.49) does so in fact involves only the points of V, the three-dimensional space tangent at each point in this space and the tangent rectangular trihedral defined by the vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$. The positions of the vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ provide unknown parasites. There is in reality only 18 unknown functions, whose differential included in the 18 linear forms of independent, ω^1, ω^α, ω_{ij}, $\omega_{i\alpha}$.

70. *Integral three-dimensional elements.* They are obtained by solving 6 quadratic equations (8.49) from 9 forms $\omega_{i\alpha}$. The first three quadratic equations gives

$$\begin{cases} \omega_{14} = a_{11}\,\varpi^1 + a_{12}\,\varpi^2 + a_{13}\,\varpi^3, \\ \omega_{24} = a_{21}\,\varpi^1 + a_{22}\,\varpi^2 + a_{23}\,\varpi^3, \\ \omega_{34} = a_{31}\,\varpi^1 + a_{32}\,\varpi^2 + a_{33}\,\varpi^3, \\ \omega_{15} = b_{11}\,\varpi^1 + b_{12}\,\varpi^2 + b_{13}\,\varpi^3, \\ \omega_{25} = b_{21}\,\varpi^1 + b_{22}\,\varpi^2 + b_{23}\,\varpi^3, \\ \omega_{35} = b_{31}\,\varpi^1 + b_{32}\,\varpi^2 + b_{33}\,\varpi^3, \\ \omega_{16} = c_{11}\,\varpi^1 + c_{12}\,\varpi^2 + c_{13}\,\varpi^3, \\ \omega_{26} = c_{21}\,\varpi^1 + c_{22}\,\varpi^2 + c_{23}\,\varpi^3, \\ \omega_{36} = c_{31}\,\varpi^1 + c_{32}\,\varpi^2 + c_{33}\,\varpi^3, \end{cases} \tag{8.50}$$

where $a_{ij} = a_{ji}$, $b_{ij} = b_{ji}$, $c_{ij} = c_{ji}$. The ω_{i4} are the semi-partial derivatives with respect to ϖ^i of the quadratic form

$$\Phi_4 = a_{ij}\,\varpi^i.\varpi^j, \tag{8.51}$$

the ω_{ij} and $\omega_{i\alpha}$ produced therefore and also two other quadratic forms

$$\Phi_5 = b_{ij}\,\varpi^i.\varpi^j, \qquad \Phi_6 = c_{ij}\,\varpi^i.\varpi^j. \tag{8.52}$$

The geometrical meaning of these forms is as follows. If we consider a curve in the manifold V and if we denote by $\overrightarrow{1/R_n}$ the projection of the vector curvature of the normal three-dimensional space, we have

$$\overrightarrow{\frac{1}{R_n}}\,ds^2 = \Phi_4\,\mathbf{e}_4 + \Phi_5\,\mathbf{e}_5 + \Phi_6\,\mathbf{e}_6. \tag{8.53}$$

Other quadratic equations (8.49) lay between the coefficients of Φ_4, Φ_5, Φ_6 relations

$$A_{ij} + B_{ij} + C_{ij} = K_{ij}, \tag{8.54}$$

where A_{ij}, B_{ij}, C_{ij} designate minors relating to elements a_{ij}, b_{ii}, c_{ij} in the determinants formed with the coefficients of the forms Φ_4, Φ_5, Φ_6.

Hence it follows that the integral element generic three-dimensional depends $6 \times 3 = 18$ parameters related by 6 relations (independent), making 12 independent parameters.

71. *Calculation of reduced characters.* Consider a linear integral element that we can always assume satisfy the relations $\omega^2 = \omega^3 = 0$, because of the arbitrariness of trihedral that we can attach to the Riemannian space. The polar system reduces of this integral element is

$$\begin{cases} \omega_{14} = \omega_{15} = \omega_{16} = 0, \\ a_{11}\,\omega_{24} + b_{11}\,\omega_{25} + c_{11}\,\omega_{26} = 0, \\ a_{11}\,\omega_{34} + b_{11}\,\omega_{35} + c_{11}\,\omega_{36} = 0, \\ a_{13}\,\omega_{34} + b_{12}\,\omega_{35} + c_{12}\,\omega_{36} - \left(a_{13}\,\omega_{24} + b_{13}\,\omega_{25} + c_{13}\,\omega_{26}\right) = 0. \end{cases} \tag{8.55}$$

The rank of this system is $s_1 = 6$.

If now we take a two-dimensional integral element satisfies $\omega^3 = 0$, the reduced equations are of the polar element

$$\begin{cases} \omega_{14} = \omega_{15} = \omega_{16} = 0, \\ \omega_{24} = \omega_{25} = \omega_{26} = 0, \\ a_{11}\,\omega_{34} + b_{11}\,\omega_{35} + c_{11}\,\omega_{36} = 0, \\ a_{12}\,\omega_{34} + b_{12}\,\omega_{35} + c_{12}\,\omega_{36} = 0, \\ a_{22}\,\omega_{34} + b_{22}\,\omega_{35} + c_{22}\,\omega_{36} = 0. \end{cases} \tag{8.56}$$

We have, $s_1 + s_2 = 9$, where $s_2 = 3$, and consequently $s_3 = 0$.

As the sum $s_1 + 2s_2 + 3s_3 = 12$ is equal to the number of arbitrary parameters upon which the generic integral element in three dimensions, the system is inuolution and its general solution depends on three arbitrary functions of two variables.

72. *Cauchy problem.* It arises when one gives in Euclidean space has six dimensions Σ realize a surface given a variety of two-dimensional Riemannian space S. We may assume that at any point of S we have attached a rectangular trihedral with a third vector is normal to S. There will therefore $\omega^3 = 0$ for the two-dimensional solution of the differential system considered (8.49). At each point of Sigma will be attached a rectangular hexahedral whose vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ are determined and are tangent to Σ in order to satisfy equations

$$\omega^1 = \varpi^1, \qquad \omega^2 = \varpi^2. \tag{8.57}$$

However one chooses the other unit vectors of the hexahedral, we will $\omega^3 = \omega^4 = \omega^5 = \omega^6 = 0$. The two equations (8.57) lead to

$$\varpi^1 \wedge \left(\omega_{12} - \varpi_{12}\right) = 0, \qquad \varpi^2 \wedge \left(\omega_{12} - \varpi_{12}\right) = 0, \tag{8.58}$$

thus

$$\omega_{12} = \varpi_{12}. \tag{8.59}$$

For the linear equations (8.49) are all verified, also demands that we have

$$\omega_{13} = \varpi_{13}, \qquad \omega_{23} = \varpi_{23}. \tag{8.60}$$

Now the equation $\omega^3 = 0$ resulting according to the third equation (8.40) that

$$\varpi^1 \wedge \varpi_{13} + \varpi^2 \wedge \varpi_{23} = 0, \tag{8.61}$$

therefore

$$\varpi_{13} = a\,\varpi^1 + b\,\varpi^2, \qquad \varpi_{23} = b\,\varpi^1 + c\,\varpi^2. \tag{8.62}$$

To determine the complete two-dimensional solution, we must choose the vector $\mathbf{e}_3$ normal to Σ in order to have

$$\omega_{13} = a\,\varpi^1 + b\,\varpi^2, \qquad \omega_{23} = b\,\varpi^1 + c\,\varpi^2. \tag{8.63}$$

To make this election, suppose first that we have taken, following some law, the vectors $\mathbf{e}_3$, $\mathbf{e}_4$, $\mathbf{e}_5$, $\mathbf{e}_6$ normal to Σ; denote them by ε_3, ε_4, ε_5, ε_6, and asking

$$\begin{aligned} d\mathbf{e}_1 &= \widehat{\omega}_{13}\,\varepsilon_3 + \widehat{\omega}_{14}\,\varepsilon_4 + \widehat{\omega}_{15}\,\varepsilon_5 + \widehat{\omega}_{16}\,\varepsilon_6, \\ d\mathbf{e}_2 &= \widehat{\omega}_{23}\,\varepsilon_3 + \widehat{\omega}_{24}\,\varepsilon_4 + \widehat{\omega}_{25}\,\varepsilon_5 + \widehat{\omega}_{26}\,\varepsilon_6, \end{aligned} \tag{8.64}$$

with

$$\widehat{\omega}_{1\alpha} = h_\alpha\,\varpi^1 + k_\alpha\,\varpi^2, \qquad \widehat{\omega}_{2\alpha} = k_\alpha\,\varpi^1 + \ell_\alpha\,\varpi^2, \qquad (\alpha = 3,4,5,6). \tag{8.65}$$

The desired vector $\mathbf{e}_3$ will be of the form

$$\mathbf{e}_3 = x^3\,\varepsilon_3 + x^4\,\varepsilon_4 + x^5\,\varepsilon_5 + x^6\,\varepsilon_6, \tag{8.66}$$

with

$$\begin{cases} (x^3)^2 + (x^4)^2 + (x^5)^2 + (x^6)^2 = 1, \\ h_3 x^3 + h_4 x^4 + h_5 x^5 + h_6 x^6 = a, \\ k_3 x^3 + k_4 x^4 + k_5 x^5 + k_6 x^6 = b, \\ l_3 x^3 + l_4 x^4 + l_5 x^5 + l_6 x^6 = c. \end{cases} \tag{8.67}$$

These four equations with four unknowns can be reduced to linear equations by calculating the first determinant

$$\delta = \begin{vmatrix} x^3 & x^4 & x^5 & x^6 \\ h_3 & h_4 & h_5 & h_6 \\ k_3 & k_4 & k_5 & k_6 \\ l_3 & l_4 & l_5 & l_6 \end{vmatrix}, \tag{8.68}$$

whose square is equal to

$$\varDelta = \begin{vmatrix} 1 & a & b & c \\ a & \mathbf{h}^2 & \mathbf{h}\cdot\mathbf{k} & \mathbf{h}\cdot\mathbf{l} \\ b & \mathbf{h}\cdot\mathbf{k} & \mathbf{k}^2 & \mathbf{k}\cdot\mathbf{l} \\ c & \mathbf{l}\cdot\mathbf{h} & \mathbf{l}\cdot\mathbf{k} & \mathbf{l}^2 \end{vmatrix}, \tag{8.69}$$

where **h**, **k**, **l** designate in the normal four-dimensional space to Sigma, three vectors having components respectively for the quantities h_i, k_i, l_i; $(i = 3,4,5,6)$.

Several cases are possible.

First assume the vectors **h**, **k**, **l** linearly independent. If Δ is negative, it is clear that equations (8.67) do not admit any real solution. If Δ is positive, they admit two distinct real solutions and we will see the next number that the data are not characteristic and that consequently it passes through the surface Σ two varieties producing three-dimensional in the neighbourhood Σ Riemannian space given $\mathscr{E}$. Finally if Δ is zero, equations (8.67) admit a unique solution which corresponds to the differential system (8.49) a two-dimensional solution, but that, as we will see the following number is characteristic.

If the vectors **h**, **k**, **l** are linearly dependent, and if the system (8.67) is compatible, at each of its solutions corresponds a two-dimensional solution of system (8.49), but, as we shall see, it is characteristic.

73. *Solutions two-dimensional characteristics of the system (8.49).*
The integral element in two dimensions $\omega^3 = 0$ is singular if the rank of the system (8.56) is less than nine, in other words if the determinant

$$\delta' = \begin{vmatrix} a_{11} & a_{11} & a_{11} \\ b_{12} & b_{12} & b_{12} \\ c_{22} & c_{22} & c_{22} \end{vmatrix}, \tag{8.70}$$

is zero. Start from a solution of the system in two dimensions (8.49) determined as it was exposed to the previous issue. If we take for vectors ε_3, ε_4, ε_5, ε_6, the vectors $\mathbf{e}_3$, $\mathbf{e}_4$, $\mathbf{e}_5$, $\mathbf{e}_6$, its the unknown x^3, x^4, x^5, x^6 of equations (8.67) have the values $1, 0, 0, 0$, the vectors **h**, **k**, **l** will for components

$$\begin{matrix} a & a_{11} & a_{11} & a_{11} \\ b & b_{12} & b_{12} & b_{12} \\ c & c_{22} & c_{22} & c_{22} \end{matrix} \tag{8.71}$$

That said, the determinant δ' is equal to the determinant δ (8.69), whose square is Δ. For the two-dimensional variety is considered characteristic, it is necessary and sufficient that Δ is zero, it can happen in two ways, either that the vectors **h**, **k**, **l** are linearly independent (the first case discussed in No. 72) either they are linearly dependent (second case discussed in No. 72).

If the two-dimensional solution obtained for the system (8.49) is characteristic, we will show that this solution must satisfy a (additional conditions to the problem is possible. These conditions we will be provided by the consideration of equations (8.72) of No. 70. We can assume that, for $\varpi^3 = 0$, the quadratic form Φ_6 is identically zero, that is to say that coefficients c_{11}, c_{12}, c_{22} are zero. the three equations

$$\begin{aligned} A_{11} + B_{11} + C_{11} &= K_{11}, \\ A_{12} + B_{12} + C_{12} &= K_{12}, \\ A_{22} + B_{22} + C_{22} &= K_{22}, \end{aligned} \tag{8.72}$$

are linear in a_{33} and b_{33}, and they are written in effect

$$\begin{cases} a_{11}a_{33}+b_{11}b_{33} = a_{23}^2+b_{23}^2+c_{13}^2+K_{11}, \\ a_{12}a_{33}+b_{12}b_{33} = a_{13}a_{23}+b_{13}b_{23}+c_{13}c_{23}-K_{12}, \\ a_{11}a_{33}+b_{11}b_{33} = a_{13}^2+b_{13}^2+c_{13}^2+K_{22}. \end{cases} \tag{8.73}$$

However, the manifold being given two-dimensional, in each point is known all the coefficients of linear forms Φ_4, Φ_5, Φ_6, with the exception of a_{33}, b_{33}, c_{33} (the coefficient a_{13} example is known because of the relation $d\mathbf{e}_3 = a_{31}\mathbf{e}_1 + a_{32}\mathbf{e}_2$ which takes place on two-dimensional manifold). By eliminating a_{33} and b_{33} between the three equations (8.73), we obtain the necessary condition to be met by any solution for two-dimensional characteristic that passes through this variety has a variety realizing the three-dimensional space $\mathscr{E}^4$, this requirement results in the equation

$$\begin{vmatrix} a_{22} & b_{22} & a_{23}^2+b_{23}^2+c_{13}^2+K_{11} \\ a_{33} & b_{12} & a_{13}a_{23}+b_{13}b_{23}+c_{13}c_{23}-K_{12} \\ a_{11} & b_{11} & a_{13}^2+b_{13}^2+c_{13}^2+K_{22} \end{vmatrix} = 0. \tag{8.74}$$

74. *The variety of characteristics a variety of three-dimensional realizing a given ds^2.* If the variety V three-dimensional Euclidean space to 6 dimensions realizes a given ds^2 ($\mathscr{E}$ given a Riemannian space), we obtain the two-dimensional characteristics varieties expressing the element tangent plane in three dimensions $u_1\,\omega^1 + u_2\omega^2 + u_3\omega^3 = 0$ has the property that moving along this plane element, forms Φ_4, Φ_5, Φ_6, are linearly dependent, in other words that there are three coefficients λ^4, λ^5, λ^6 not all zero such that the quadratic form $\lambda^4\Phi_4 + \lambda^5\Phi_5 + \lambda^6\Phi_6$ divisible by $u_1\,\varpi^1 + u_2\varpi^2 + u_3\varpi^3$. This results in relation

$$\begin{vmatrix} u_1 & 0 & 0 & 0 & u_3 & u_2 \\ 0 & u_1 & 0 & u_3 & 0 & u_1 \\ 0 & 0 & u_3 & u_2 & u_1 & 0 \\ a_{22} & a_{22} & a_{23} & 2a_{23} & 2a_{13} & 2a_{11} \\ b_{22} & b_{22} & b_{23} & 2b_{23} & 2b_{13} & 2b_{11} \\ c_{22} & c_{22} & c_{23} & 2c_{23} & 2c_{13} & 2c_{11} \end{vmatrix} = 0. \tag{8.75}$$

If this relation is not identically satisfied, that is to say if the manifold V is not a singular solution of system (8.49) we obtain a cubic equation in u_1, u_2, u_3 which defines V at each point of a cone with third class. The characteristic varieties are searched solutions of partial differential equation of first order defined by equation (8.75), the tangent plane in each point to be tangent to the cone of the third class defined by this equation.

75. *Singular solutions.* These are those for which the equation (8.75) in u_1, u_2, u_3 is an identity. This condition allows the characterization of singular solutions by a purely projective property.

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[4] This condition generalizes the well-known condition which has to satisfy a curve (C) of ordinary space so that it passes through (C) a surface which gave ds^2 is an asymptotic line (C): the square of its torsion must equal to the Riemannian curvature changed sign of ds^2 given.

Consider the integral manifold V a curve (C) whose tangent in each point A has the director parameters ϖ^1, ϖ^2, ϖ^3. The flat variety containing both the triplane (three dimensional flat variety) tangent to V in A and the biplane osculating at A of (C) contains the vector

$$\Phi_4\,\mathbf{e}_4 + \Phi_5\,\mathbf{e}_5 + \Phi_6\,\mathbf{e}_6, \tag{8.76}$$

as shown immediately the calculation of $d\mathbf{A}$ and $d^2\mathbf{A}$. This raises, if the biplane Π tangent to V at point A is a singular integral element, this means that instead of osculating planes to a tangent curves at point A is located in a five-dimensional hyperplane. The variety V is a singular solution if this takes place regardless of the plane element Π tangent to V. This is the projective property of singular solutions sought.

Two cases are possible, depending on whether the five-dimensional hyperplane corresponding to the biplane is independent of the biplane or not; in the first case this hyperplane is called the *osculating hyperplane* to the variety V at point A.

First case. – First case. - In this case can assume that the osculating hyperplane's normal vector at each point in $\mathbf{e}_6$, i.e. the form Φ_6 is identically zero. The equations

$$\omega_{16} = 0, \qquad \omega_{26} = 0, \qquad \omega_{36} = 0, \tag{8.77}$$

to cause a by exterior differentiation,

$$\begin{aligned} \omega_{14} \wedge \omega_{46} + \omega_{15} \wedge \omega_{56} &= 0, \\ \omega_{24} \wedge \omega_{46} + \omega_{25} \wedge \omega_{56} &= 0, \\ \omega_{34} \wedge \omega_{46} + \omega_{35} \wedge \omega_{56} &= 0. \end{aligned} \tag{8.78}$$

The result, in general, $\omega_{46} = \omega_{56} = 0$, unless the two conical $\Phi_4 = 0$, $\Phi_5 = 0$ are bi-tangents or have them contact of the second order. If we allow these two cases side, we see that $d\mathbf{e}_6$ and hence that the osculating hyperplane is fixed. The corresponding Riemannian spaces are those that are likely to be achieved by a variety of three-dimensional Euclidean space with five dimensions. It can be shown that in general their realization in the five-dimensional space is only possible in one way, has a displacement or up to a symmetry.

It could happen that two forms Φ_5 and Φ_6, are zero: then the Riemannian space would be likely to realization in a four-dimensional space. If all three forms Φ_4, Φ_5, Φ_6 were zero, the Riemannian space $\mathscr{E}$; Riemannian curvature would be zero and it would, at least locally, realizable by the three-dimensional Euclidean space.

Second case. – We will say little about the singular solutions of the second case. We prove easily that forms Φ_4, Φ_5, Φ_6 all decompose into two factors of the first degree, one of these factors being the same for all three forms, the other variable with the three forms. These singular solutions exist only for six spaces which the quadratic form of the Riemann coefficients K_{ij} has its discriminant zero. The integral manifolds are a very simple geometrical definition: each is generated by an arbitrary one-parameter family of planar two-dimensional varieties. They therefore depend on 11 arbitrary functions of one variable, which emphasizes the very exceptional character of Riemannian spaces for which the differential system (8.49) admits singular solutions falling in the second case.

As seen, there are still a number of points to clarify in the theory of singular solutions of differential system that gives three-dimensional varieties of six-dimensional Euclidean space capable of realizing a three-dimensional Riemannian space given.

As for the $n-$dimensional Riemannian spaces, known to be realizable by Varietal immersed in Euclidean space $n(n+1)/2$ dimensions of a given solution $n-1$ uncharacteristic dimensions of the problem determines differential system, as for $n=3$, a finite number of $n-$dimensional solutions (See on this general problem, M. Janet [18], and E. Cartan [6].)

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