Car tracking by quantised input LMS, QX-LMS algorithm in traffic scenes

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Abstract: The tracking algorithm is an important tool for motion analysis in computer vision. A new car tracking algorithm is proposed which is based on a new clipping technique in the field of adaptive filter algorithms. The uncertainty and occlusion of vehicles increase the noise in vehicle tracking in a traffic scene, so the new clipping technique can control noise in prediction of vehicle positions. The authors present a new quantised version of the LMS, namely the QX-LMS algorithm, which has a better tracking capability in comparison with the clipped LMS (CLMS) and the LMS and also involves less computation. The threshold parameter of the QX-LMS algorithm causes controllability and the increase of tracking and convergence properties, whereas the CLMS and LMS algorithms do not have these capabilities. The QX-LMS algorithm is used for estimation of a noisy chirp signal, for system identification and in car tracking applications. Simulation results for noisy chirp signal detection show that this algorithm yields a considerable error reduction in comparison to the LMS and CLMS algorithms. The proposed algorithm, in tracking some 77 vehicles in different traffic scenes, shows a reduction of the tracking error relative to the LMS and CLMS algorithms.

1 Introduction

A lot of research has been done on intelligent transportation systems, and from it has resulted the surveillance of road traffic based on machine vision techniques. Normal behaviour recognition and identification of offending drivers are examples of vision-based surveillance systems. Analysis of the behaviour of any vehicle is possible by using the obtained trajectory of vehicles and quantitative parameters such as velocity and acceleration. Vehicle trajectory is an important feature for behaviour recognition, so in many research works, vehicle tracking has been studied despite numerous difficulties such as full or partial occlusion [1–7].

Vehicle tracking includes two main stages: segmentation and prediction or estimation algorithms. Segmentation is the first step in tracking in which objects are extracted using spatial and temporal methods [8–10]. Edge, region, texture and colour features are used in spatial segmentation, and frame differencing and background subtraction are used in temporal segmentation. The optical flow method is a typical spatio-temporal segmentation. After detection of vehicles in the scene, a matching algorithm is used in the search area for finding similar vehicles in two consecutive frames.

Abstract: The tracking algorithm is an important tool for motion analysis in computer vision. A new car tracking algorithm is proposed which is based on a new clipping technique in the field of adaptive filter algorithms. The uncertainty and occlusion of vehicles increase the noise in vehicle tracking in a traffic scene, so the new clipping technique can control noise in prediction of vehicle positions. The authors present a new quantised version of the LMS, namely the QX-LMS algorithm, which has a better tracking capability in comparison with the clipped LMS (CLMS) and the LMS and also involves less computation. The threshold parameter of the QX-LMS algorithm causes controllability and the increase of tracking and convergence properties, whereas the CLMS and LMS algorithms do not have these capabilities. The QX-LMS algorithm is used for estimation of a noisy chirp signal, for system identification and in car tracking applications. Simulation results for noisy chirp signal detection show that this algorithm yields a considerable error reduction in comparison to the LMS and CLMS algorithms. The proposed algorithm, in tracking some 77 vehicles in different traffic scenes, shows a reduction of the tracking error relative to the LMS and CLMS algorithms.

The second step of the tracking process includes estimation or prediction of the vehicle state (e.g. position, velocity, colour) and for this case a suitable tool is an adaptive filter. Different factors stimulate the use of adaptive filters, such as noise in the frame acquisition, vehicle detection algorithm and uncertainty due to occlusion [10–14]. The statistics and condition of noise are related to the type of detection algorithm, optical system in frame acquisition, lighting conditions and traffic congestion. A variety of work has been reported; for example in [15] estimation of motion parameters is studied in SNRs of 5, 10 and 15 dB, and in [10] a segmentation algorithm is analysed in SNRs of −10 to 3 dB and in [16] the proposed tracking algorithm for vehicle collision is tested by adding noise to the vehicle trajectory but the SNR range is not discussed.

Uncertainty in dynamic activities is discussed in [13–17], a non-matching model with observation in vision system [18], and varieties of motion models [2] have also been discussed. Motion models of objects inside the scene have not been developed and can be considered to be either fixed or variable. The need of the Kalman filter to a model is one of problems when using this filter, thus data driven-based adaptive filters such as RLS and LMS algorithms are used in vision applications [19–23]. In noisy environments where the SNR is low, the LMS filter has a better tracking ability than the RLS filter [24]. Also the simplicity of implementation of the LMS filter causes new developments for this algorithm that enhance the capability and performance of this filter [24–29].

The LMS adaptive filter is very popular owing to its simplicity but even simpler approaches are required for many real-time applications. Reduction of the complexity of the LMS filter has received attention in the area of adaptive filters [29–32]. The sign algorithm and the clipped data algorithm are in this category [26, 29, 31–33].

Much effort from the viewpoint of reduction of the computational complexity of the LMS algorithm is seen in the aforementioned references. The present work is
concerned with the presentation of a modified version of the clipped LMS (CLMS) algorithm which has much better tracking capability than the CLMS and LMS and has less computations as well.

The description of the proposed new algorithm, which is a modification of the aforementioned algorithm, includes the background of clipping techniques, behaviour of convergence and tracking of the CLMS algorithm. A car tracking algorithm is presented and results are discussed.

2 Proposed quantised LMS algorithm

Here we propose a new modification to the clipped LMS algorithm for further controllability of the LMS algorithm, after first briefly introducing the main existing variants of the LMS algorithm.

2.1 Variants of the LMS algorithm

In order to clarify the background of the new and the conventional algorithms, it is necessary to show how they are interrelated and how they have evolved. The LMS algorithm has been studied in [34, 35] as

\[ W_{n+1} = W_n + \mu e_n X_n \]  

where

\[ e_n = d_n - X_n^T W_n \]  

\[ W_n = [w_n(1), w_n(2), \ldots, w_n(N)]^T \] is the weight vector of the estimator, and \( X_n \) is the vector of the input data sequence, which is assumed to be a stationary random process. \( N \) is the number of filter taps, \( e_n \) is the estimation error, \( d_n \) is the desired response and \( \mu \) is the step size of the weight update.

A simple change can be made to the LMS algorithm to obtain the CLMS algorithm [2, 36, 37]

\[ W_{n+1} = W_n + \mu \tilde{e}_n \tilde{X}_n \]  

where \( \tilde{X} \) is the clipped input signal vector, whose \( i \)th component is \( \tilde{x}_n(i) = \text{sgn}[x_n(i)] \). Other variations of the LMS algorithm that have been studied are the ‘sign’ algorithm [38, 39]

\[ W_{n+1} = W_n + \mu \tilde{e}_n X_n \]  

where

\[ \tilde{e}_n = \text{sgn}[e_n] \]  

and ‘the zero-forcing’ algorithm [40, 41]

\[ W_{n+1} = W_n + \mu \tilde{e}_n \tilde{X}_n \]  

The CLMS algorithm involves clipping the input signal vector in the weight-update formula (3). This quantisation scheme can best be illustrated by Fig. 1.

2.2 New quantisation scheme

Here we propose a new modification to the CLMS algorithm in order to further simplify the implementation. Rather than representing the input signal, \( X_n \), by a two-level signal as shown earlier by (3), we quantise it into a three-level signal according to the quantisation scheme shown in Fig. 2. Thus, the adaptation equation can be written as

\[ W_{n+1} = W_n + \mu \tilde{e}_n \tilde{X}_n \]  

where \( \tilde{X}_n \) is the modified clipped input signal vector whose \( i \)th component is \( \tilde{x}_n(i) = \text{msgn}[x_n(i), \delta] \), where \( \text{msgn}[\cdot] \) is the modified sign function defined as

\[ \text{msgn}[x_n(i), \delta] = \begin{cases} +1, & \delta \leq x_n(i) \\ 0, & -\delta < x_n(i) < \delta \\ -1, & x_n(i) \leq -\delta \end{cases} \]

It should be noted that the implementation of such an adaptive filter has potentially greater throughput because for those times when the tap input signal, \( x_n(i) \), is less than the specified threshold, \( \delta \), then \( \tilde{x}_n(i) \) will be equal to zero and no coefficient adaptation for the corresponding weight needs to be performed. This means that some of the time-consuming operations in the weight update formula (7) can be omitted, thereby leading to a reduction of the computational load on the processor. Whether this potential can be realised depends on the architecture used in the processor and also the application.

2.3 QX-LMS specifications

The convergence of the mean of the weight vector for QX-LMS and the tracking capability are proved in the Appendix. It is shown that the mean of the weight vector of the QX-LMS algorithm converges to the optimum weight vector of the Wiener filter, but the time constant for the exponential relaxation of the weight vector to its optimal value (see (36)) and the tracking capability (see (44)) of the QX-LMS algorithm depends on threshold \( \delta \). The following results are obtained from those equations: (a) the threshold, \( \delta \), can control the convergence speed and tracking performance; and (b) the tracking performance and convergence speed behaviour are related. It can be seen that the QX-LMS algorithm has better capabilities of controllability on tracking and convergence behaviour compared to LMS and CLMS algorithm. In the next sub-section, experimental results in identification and chirp tracking problems are explained.

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Fig. 1 Quantisation scheme for the CLMS algorithm

![Quantisation scheme for the CLMS algorithm](image1.png)

Fig. 2 Quantisation scheme for the QX-LMS algorithm

![Quantisation scheme for the QX-LMS algorithm](image2.png)
2.4 Applying the QX-LMS algorithm in identification and tracking problems

In this Section, the QX-LMS algorithm is used in an identification problem and for noise reduction in noisy chirp sinusoids and the results are compared to those of the LMS and CLMS algorithms.

2.4.1 Application of the QX-LMS algorithm in the identification problem: In order to demonstrate the convergence behaviour of the LMS, CLMS, and the new QX-LMS algorithm, some 100 runs of simulation experiments have been performed (with $\mu = 0.17$ and $\delta = 0.7$ for QX-LMS and $\mu = 0.105$ for LMS and $\mu = 0.13$ for CLMS, which were the best parameters for maximum speed of convergence). In all the system identification experiments, a stationary white noise sequence was used and the system was a seven-tap FIR transversal filter having parameters that were arbitrarily chosen.

The input data were normalised to have unit variance. The norm of the difference between the plant FIR weights and adaptive filter weights generated by each algorithm was averaged over 100 independent simulation runs and plotted as a function of time, as depicted in Fig. 3. The norm of the difference weight vector

$$\text{norm}(h, W) = \left( \sum_i (h_i - W_i)^2 \right)^{1/2}$$

(9)

It can be seen that the QX-LMS algorithm has a much better convergence than the CLMS algorithm and it is also almost as good as the LMS algorithm in terms of the convergence speed. Nevertheless, the convergence rate of the QX-LMS algorithm is decreased by increasing the threshold $\delta$, but tracking capability is increased.

For example, with a threshold of 0.7, the difference weight norm of the CLMS is improved by 12%, whereas, the CLMS in comparison to the LMS has both lower convergence speed and higher weight error. Fig. 4 shows the ratio of the tracking error norm for weights of the proposed QX-LMS algorithm to the optimum weights of the LMS and CLMS algorithms. The existence of the second parameter in the QX-LMS algorithm, i.e. $\delta$, in comparison with the LMS and CLMS algorithms gives rise to an increase in its efficiency.

2.4.2 Noise reduction for a noisy chirp sinusoid signal: Adaptive recovery of a chirp sinusoid buried in noise is a standard method because the chirp sinusoid represents a well defined form of nonstationarity. In this experiment, we consider the tracking of a chirped sinusoid. The chirped input signal is given by

$$S(k) = \sqrt{P_s} \exp(j(2\pi f_c + \varphi k) + \varphi))$$

(10)

where $\sqrt{P_s}$ denotes the signal amplitude, $f_c$ is the centre frequency, $\psi$ is the chirp rate and $\varphi$ is an arbitrary phase shift. The signal $S(k)$ is deterministic but nonstationary because of the chirping. $S(k)$ is added with noise $n(k)$, then tracking of the noisy chirp is a benchmark for testing the QX-LMS and LMS algorithms. The signal-to-noise ratio $\text{SNR}$ is denoted by

$$\text{SNR} = 10 \log \left( \frac{\sqrt{P_s}}{A_n} \right)$$

(11)

where $A_n$ is the amplitude of the noise. In our experiment, a noisy chirp with a centre frequency of 50 Hz and a sampling rate of 1 kHz is utilised.

Two criteria are tested for showing the capability of the QX-LMS relative to CLMS and LMS algorithms in tracking: mean square error and error variance.
Mean square error shows the difference of prediction of the signal with the actual value and error variance in tracking mode on noisy chirp which can show the degree of smoothing the estimation or prediction.

It is expected that increasing $d$ in the QX-LMS algorithm for low SNRs causes more clipping noise and increases performance or decreases the mean square error. Fig. 5 shows a comparison of the MSE of three algorithms for the best value of $m$ and $d = 0.05$ for the QX-LMS algorithm. For a low SNR, we increase $d$ to 0.8 and the result is shown in Fig. 6. Also, different thresholds show that increasing the threshold in the low SNR region gives better tracking results.

Now, we examine the error variance in the tracking phase. Fig. 7 shows the error variances of all compared algorithms (in QX-LMS, $d = 0.87$ is selected). At low SNR, it can be seen that the QX-LMS algorithm gives better performance relative to LMS and CLMS algorithms. So the proposed approach gives smoother estimation for noisy chirp tracking.

### 2.5 Reduction of computational complexity of the QX-LMS relative to the CLMS algorithm

The proposed algorithm has less computational complexity relative to the CLMS algorithm. If we assume that the input signal has a Gaussian distribution with zero mean and standard deviation $\sigma_s$, then the probability that the signal falls in the interval $[-\delta \sigma_s, \delta \sigma_s]$ is

$$P(-\delta \sigma_s < x < \delta \sigma_s) = \int_{-\delta \sigma_s}^{\delta \sigma_s} N(\mu_x, \sigma_x) \, dx$$  \hspace{1cm} (12)

where $N(\mu_x, \sigma_x)$ is the input probability density function and $P(-\delta \sigma_s < x < \delta \sigma_s)$ which, in addition to being the probability of the occurrence of the signal in the interval $[-\delta \sigma_s, \delta \sigma_s]$, is also the computational reduction of the QX-LMS relative to the CLMS. The reason is that the signal is falling between the two thresholds with a probability of $P(-\delta \sigma_s < x < \delta \sigma_s)$, and within this interval, the proposed algorithm has no weight update, since according to (8), $\text{msgn}[x_s(t), \delta \sigma_s]$ is equal to zero.

It is interesting to note that regarding (12) and Fig. 3 for $\delta = 0.7$, the computational complexity of the weight update formula can be reduced about 52% without any noticeable change in the convergence behaviour.

### 3 Car tracking using the QX-LMS algorithm

The tracking algorithm has become a significant tool in traffic scene analysis. Incident monitoring, traffic parameter extraction and behaviour recognition of moving objects in a scene requires reliable tracking of the entities in the scene. In this Section, we are interested in noise reduction of trajectory using the new quantisation scheme in adaptive filters.

A variety of methods exist for tracking objects in outdoor scenes. Many types of object tracking methods have been presented, the standard types which include feature-based [5, 42], region-based [6, 43] and model-based [7] tracking algorithms.

Vision-based tracking algorithms include vehicle segmentation and state estimation which are explained as follows.

#### 3.1 Segmentation method

A classic technique in segmentation is the background subtraction method. Static targets are obtained using background subtraction according to

$$O_k = I_k - B_k$$  \hspace{1cm} (13)

where $I_k$ is $k$th input frame and $B_k$ is the background at $k$th frames. $O_k$ is a static target. $B_k$ is obtained by combination

### Fig. 8 Segmentation method using adaptive background subtraction

$a$ Blobs and mask detection in frame 171

$b$ In frame 173, the black car is lost by threshold 55
of non-moving blobs according to (14). A non-moving blob is part of the image which has had changing grey value smaller than threshold (e.g. 15) between two frames.

\[ B_k(i,j) = \alpha B_k(i,j) + (1 - \alpha) \text{NMB}(i,j) \quad (14) \]

where NMB is a blob of the image which is chosen as a non-moving blob and \(i, j\) refer to the pixels of this area of the picture. Then, each pixel in the background image is combined with the corresponding pixels of the received non-moving image blob. \(\alpha\) is the coefficient of the effect of the last background and \(1 - \alpha\) is the coefficient of the received non-moving blob. \(\alpha\) is a number between 0 to 1 which is chosen here to be equal to 0.9. Fig. 8a shows the result of above segmentation method including the main picture, the result of background subtraction and after using threshold (in this scene the threshold is 55).

### 3.2 Car localisation

After performing the background subtraction and finding the existing objects in the scene, vehicle localisation is used for determining the object boundaries. Localisation helps to solve the partial occlusion problem and is carried out by finding feature points and grouping them. We use corners as features and extract those according to [44].

Detection of different parts of the vehicle because of partial occlusion as in Fig. 8b; and

\[ Q_{EG} = \left\{ \frac{\sqrt{\lambda_1 \lambda_2}}{(\lambda_1 + \lambda_2)/2} \right\}^2 = \frac{4(s_{xx} s_{yy} - s_{xy}^2)}{(s_{xx} + s_{yy})^2} \quad (16) \]

where \(\lambda_1\) and \(\lambda_2\) are eigenvalues of \(H_1\). \(Q_{EG}\) shows the corner in a small window if it is near to 1. Fig. 9 shows a sample of detected corners on segmented image for localisation with \(Q_{EG}\) bigger than 0.95. After corner detection, a grouping technique based on distance criteria that appear in [5] is used for finding the vehicle boundary. Fig. 10 shows grouping technique for car localisation, despite partial occlusion.

### 3.3 Car searching in consecutive frames

After finding the vehicle boundary, a colour histogram is used for the similarity measurement in consecutive frames. For each detected boundary, an HSV colour histogram is achieved and quantised to \(18 \times 6 \times 6\) bins, such that one vector of length 648 is obtained. Matching is obtained when the Euclidean distance between each pair of vectors is smaller than 0.05. Minimum distance is given in search rectangular windows with length of 50 pixels. Then centre of gravity is applied to an estimator or predictor in order that after the convergence of the predictor for each vehicle attribution of similar blobs to an object and generating a smoothed trajectory is assisted.

### 3.4. Prediction of position using the QX-LMS algorithm

The different types of noise observed in car tracking provide an incentive for using adaptive filtering for noise reduction. Some sources of noise are as follows:

- (a) attributing the whole or part of the vehicle to other vehicles because of shadow and partial occlusion as in Fig. 8b;
- (b) Detection of different parts of the vehicle because of type of segmentation method, e.g. Fig. 11; and

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<thead>
<tr>
<th>Frame</th>
<th>Image</th>
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<tbody>
<tr>
<td>450</td>
<td><img src="image" alt="Frame 450" /></td>
</tr>
<tr>
<td>476</td>
<td><img src="image" alt="Frame 476" /></td>
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Fig. 9: Detected corners with \(Q_{EG}\) more than 0.95

Fig. 10: Car localisation using feature grouping

Fig. 11: Detection of different parts of vehicle

Fig. 12: Noise in position because of partial occlusion
range of SNR (QX-LMS with threshold 17 pixel) can be controlled by the threshold and we modify it to QX-LMS whose tracking performance is better than the unmodified algorithms. This method showed the tracking property of QX-LMS can be controlled by the threshold $d$. We tested the proposed algorithm on 77 vehicles and compared it to LMS and CLMS algorithms.

Fig. 13 shows mean square error, MSE, of 77 cars’ trajectories in different scenes. Trajectories were captured under different congestion conditions and then they are tracked by LMS, CLMS and QX-LMS algorithms. According to the results illustrated in Fig. 13, the QX-LMS algorithm has smaller MSE relative to CLMS and LMS. We compared the algorithms in the acceptable range of SNR that has been reported in [10, 15]. Fig. 14 shows the MSE of the LMS, CLMS and QX-LMS algorithms for 77 obtained vehicle trajectories. For completion of the experiment results are presented for different thresholds for QX-LMS (threshold $d$ at pixel) in Fig. 15. We conclude that the tracking property of QX-LMS can be controlled by the threshold $d$, confirming previous findings in Section 2.3.

4 Conclusions

Multi-object tracking algorithms are based on prediction techniques. One of the most commonly used algorithms in prediction is the LMS algorithm. This algorithm, owing to its good tracking capabilities in noisy environments, has many applications. In this paper we have proposed a new variant of the LMS, namely, the QX-LMS algorithm. This algorithm uses a three-level quantisation (+1, 0, −1) scheme which involves the threshold clipping of the input signal in the filter weight-update formula. Mathematical analysis shows the convergence of the filter weights to the optimum Wiener filter weights. Also, it can be proved that the proposed QX-LMS algorithm has a better tracking capability than the LMS algorithm. In addition, this algorithm has a reduced computational complexity relative to the unmodified algorithms. This method showed the tracking performance of the proposed algorithm in multi-object tracking of vehicles in traffic scenes.

5 References


Fig. 13 MSE of 77 vehicles trajectories by the LMS, CLMS and QX-LMS algorithms in different scenes

Fig. 14 MSE of LMS, CLMS and QX-LMS algorithms in wide range of SNR (QX-LMS with threshold 17 pixel)

Fig. 15 MSE of the LMS, CLMS and QX-LMS algorithms with different thresholds for QX-LMS (threshold $d$ at pixel)
6 Appendix

6.1 Expectation of quantised variables theorem

Theorem: If two random variables \( u \) and \( v \) both have a Gaussian distribution \( N(0, \sigma_u) \) and \( N(0, \sigma_v) \) respectively and \( E[|v|] = \rho \sigma_u \sigma_v \), \( \bar{v} = \text{msgn}(v, \delta) \) then

\[
E[|v|] = \frac{\sigma'}{\sigma_v} = E[|v|]
\]

(17)

where \( \sigma' = \sqrt{2/\pi} \exp(-\delta^2/2\sigma_v^2) \).

Proof: We define the random variable

\[
z = \frac{u}{\sigma_u} - \frac{\rho v}{\sigma_v}
\]

(18)

Now we have

\[
E[|v|] = E\left[ \left( \frac{u}{\sigma_u} - \frac{\rho}{\sigma_v} v \right) \right] = \frac{E[u]}{\sigma_u} - E\left[ \frac{\rho v^2}{\sigma_v} \right]
\]

(19)

With regard to the assumption of the theorem

\[
E[|v|] = \frac{\rho \sigma_u \sigma_v}{\sigma_u} - \frac{\sigma_v}{\sigma_v} = 0
\]

(20)

Therefore \( z \) and \( v \) are uncorrelated. Also, therefore, since \( z \) and \( \bar{v} \) are uncorrelated, we have

\[
E[|v|] = E[|z|]E[|\bar{v}|] = E[|z|] \times 0 = 0
\]

(25)

(21)

On the other hand

\[
\bar{v} = v \times \text{msgn}(v, \delta) = \begin{cases} |v| & |v| > \delta \\ 0 & |v| \leq \delta \end{cases}
\]

(23)

The density function of \( \bar{v} \) is also Gaussian with distribution \( N(0, \sigma_v) \), hence

\[
E[|v|] = \int_{-\infty}^{\infty} |v| \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{v^2}{2\sigma_v^2}\right) dv
\]

(24)

\[
= 2 \int_{\delta}^{\infty} |v| \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{v^2}{2\sigma_v^2}\right) dv
\]

(24)

\[
= \sqrt{\frac{2}{\sigma_v}} \exp\left(-\frac{\delta^2}{2\sigma_v^2}\right)
\]

(25)
Now regarding (21) and (24) we have
\[
E[\hat{w}_n] = \frac{\rho \sigma_n}{\sigma_x} \sqrt{\frac{2}{\pi}} \sigma_c \exp \left( -\frac{\delta^2}{2\sigma_x^2} \right)
\]
\[
= \frac{1}{\sigma_x} \sqrt{\frac{2}{\pi}} \exp \left( -\frac{\delta^2}{2\sigma_x^2} \right) \rho \sigma_n \sigma_c
\]
Finally, with regard to \( E[\hat{w}_n] = \rho \sigma_n \sigma_c \) in (26) we have proved the theorem.

### 6.2 Convergence and tracking performance of the QX-LMS algorithm

#### 6.2.1 Derivation of the convergence of the QX-LMS algorithm

**Algorithm**: Now, we want to prove that the mean of the weight vector converges in the limit to the optimum Wiener weight vector. Taking expectations on both sides of (7) yields
\[
E[W_{n+1}] = E[W_n] + \mu E[e_n \hat{x}_n]
\]
Substituting (2) in (27) gives
\[
E[W_{n+1}] = E[W_n] + \mu E[d_n \hat{x}_n - \hat{x}_n X_n^T W_n]
\]
Assuming lack of correlation between the weights and \( \hat{x}_n X_n^T \) as in [16], (28) gives
\[
E[W_{n+1}] = E[W_n] + \mu (E[d_n \hat{x}_n] - E[\hat{x}_n X_n^T]) E(W_n)
\]
Now, with regard to (17) in Section 6.1, we have
\[
E[W_{n+1}] = E[W_n] + \mu \left( \frac{\alpha'}{\sigma_x} P - \frac{\alpha'}{\sigma_x} R E[W_n] \right)
\]
\[
= \left( I - \mu \frac{\alpha'}{\sigma_x} R \right) E[W_n] + \mu \frac{\alpha'}{\sigma_x} P
\]
where
\[
\alpha' = \sqrt{\frac{2}{\pi}} \exp \left( -\frac{\delta^2}{2\sigma_x^2} \right)
\]
and \( \sigma_x \) is the standard deviation of the input signal. We know that the optimum Wiener weight vector is \( W^* = R^{-1} P \). Substituting in (31) yields
\[
E[W_{n+1}] = \left( I - \mu \frac{\alpha'}{\sigma_x} A \right) E[W_n]
\]
If \( V_n = W_n - W^* \) and according to [34], the principal axes are rotated according to \( V = QV' \), where the rows of \( Q \) are eigenvectors of \( R = Q \Lambda Q^{-1} \) and \( \Lambda \) is a diagonal matrix whose elements are eigenvalues of \( R \). Thus, we have the following relation after simplification
\[
E[V_{n+1}] = \left( I - \mu \frac{\alpha'}{\sigma_x} A \right) E[V_n]
\]
If \( I - \mu (\alpha' / \sigma_x) A \) is in the limit converges to zero, then
\[
\lim_{n \to \infty} E[V_{n+1}] = 0
\]
in this case, \( \lim_{n \to \infty} E[V_{n+1}] = 0 \) and consequently \( \lim_{n \to \infty} E[W_{n+1}] = W^* \) i.e. the QX-LMS algorithm will converge. In order that \( \lim_{n \to \infty} (I - \mu (\alpha' / \sigma_x) A)^n = 0 \), it is necessary to find a condition for \( \mu \) in terms of the eigenvalues. Therefore, the convergence condition is that for the largest eigenvalue \( \lambda_{\text{max}} \), \( \mu \) satisfies the following relation that is the convergence condition for the QX-LMS
\[
0 < \mu < \left( \frac{1}{\delta} \right) \lambda_{\text{max}} \quad 1 \leq i \leq N
\]
If \( \mu \) satisfies this relation for the largest eigenvalue \( \lambda_{\text{max}} \), then (34) is also satisfied for all other eigenvalues. Thus, the convergence condition for QX-LMS is as follows
\[
0 < \mu < \left( \frac{1}{\delta} \right) \frac{1}{\lambda_{\text{max}}}
\]
Also, the time constant for the exponential relaxation of the weight vector to its optimal value is
\[
\tau_{\text{QX-LMS}} = \frac{1}{(\alpha' / \sigma_x) \mu \lambda_{\text{max}}}
\]

#### 6.2.2 Evaluating the tracking performance of the QX-LMS algorithm

**Algorithm**: Tracking is a steady-state phenomenon that is different from the convergence, which is a transient phenomenon. In general, convergence and tracking are two different properties. That is, if an algorithm has a good convergence, its tracking ability is not necessarily fast and vice versa. In the tracking phase, a reasonable assumption is that the optimum weights vary necessarily fast and vice versa. In the tracking phase, a reasonable assumption is that the optimum weights vary.

**Misadjustment criterion in the QX-LMS algorithm**: According to [24], the algorithm misadjustment is usable as a criterion in tracking
\[
M_{\text{QX-LMS}} = \frac{E[|\omega_n X_n^T|^2]}{E[|v_n|^2]}
\]
The above relation shows that the weight misadjustment is related to the process noise power and \( \hat{x}_n \). Now, we calculate the misadjustment (39) in which the numerator can be written as
\[
E[|\omega_n X_n|^2] = E[|\omega_n X_n|^2]
\]
With the assumption of independence of \( \hat{x}_n \) and \( \omega_n \) and using relation (17) in the Section 6.1
\[
E[\omega_n X_n^T \omega_n] = \text{tr}[E[\omega_n X_n^T \omega_n] E[\hat{x}_n X_n^T \omega_n]]
\]
\[
= \left( \frac{\alpha'}{\sigma_x} \right)^2 \text{tr}(R\Phi)
\]
Also, the denominator of the fraction (39) is

\[ E[|v_n|^2] = \sigma_n^2 \]  

(42)

Hence, the QX-LMS algorithm misadjustment can be written as

\[ M_{QX-LMS} = \frac{1}{\sigma_n^2} \left( \frac{\alpha'}{\sigma_x} \right)^2 \text{tr}[R \Phi] \]  

(43)

Comparing this misadjustment value with that of the LMS algorithm, \( M_{LMS} = (1/\sigma_x^2) \text{tr}[R \Phi] \), the following relation can be obtained

\[ M_{QX-LMS} = \left( \frac{\alpha'}{\sigma_x} \right)^2 M_{LMS} \]  

(44)

The above relation shows that increasing the threshold \( \delta \) such that \( \alpha' \) is less than \( \sigma_x \) gives rise to a decrease in the misadjustment error relative to the LMS algorithm in tracking, but with regard to (26), it causes the QX-LMS algorithm to be slower in convergence.