

Continuous-time Markov chains

Books

- Performance Analysis of Communications Networks and Systems (Piet Van Mieghem), Chap. 10
- Introduction to Stochastic Processes (Erhan Cinlar), Chap. 8

Definition

$\{X(t), t \geq 0\}$ is a continuous-time Markov chain if

$$\begin{aligned}\Pr[X(t + \tau) = j | X(\tau) = i, X(u) = x(u), 0 \leq u < \tau] \\ = \Pr[X(t + \tau) = j | X(\tau) = i]\end{aligned}$$

Stationarity of the transition probabilities

$$P_{ij}(t) = \Pr[X(t + \tau) = j | X(\tau) = i] = \Pr[X(t) = j | X(0) = i]$$

The state vector $s(t)$ with components $s_k(t) = \Pr[X(t) = k]$ obeys

from which
$$s(t + \tau) = s(\tau)P(t)$$

$$s(t + u + \tau) = s(\tau)P(t + u)$$

$$\begin{aligned}s(t + u + \tau) &= s(\tau + u)P(t) = s(\tau)P(u)P(t) \\ &= s(\tau + t)P(u) = s(\tau)P(t)P(u)\end{aligned}$$

Thus, the transition probability matrix $P(t)$ satisfies the **Chapman-Kolmogorov equation**

$$P(t + u) = P(u)P(t) = P(t)P(u)$$

for all $t, u > 0$.

For any state i

$$\sum_{j=1}^N P_{ij}(t) = 1$$

Initial condition of the transition probability matrix

$$P(0) = \lim_{t \downarrow 0} P(t) = I$$

The infinitesimal generator Q

Lemma The transition probability matrix $P(t)$ is continuous for all $t \geq 0$.

Additional assumption: the existence of

$$\lim_{h \downarrow 0} \frac{P(h) - I}{h} = P'(0) = Q$$

matrix Q

- is called the **infinitesimal generator** of the continuous-time Markov process

- corresponds to $P-I$ in discrete-time

The sum of the rows in Q is zero, with $\sum_{j=1, j \neq i}^N q_{ij} = -q_{ii}$

$$q_{ij} = \lim_{h \downarrow 0} \frac{P_{ij}(h)}{h} \geq 0 \quad \text{and} \quad q_{ii} \leq 0$$

We call q_{ij} "rates"

- they are derivatives of probabilities and reflect a change in transition probability from state i towards state j

We define $q_i = -q_{ii} > 0$. Then, $\sum_{j=1}^N |q_{ij}| = 2q_i$

– Q is bounded if and only if the rates q_{ij} are bounded

- It can be shown that q_{ij} is always finite.

– For finite-state Markov processes, q_j are finite (since q_{ij} are finite), but, in general, q_j can be infinite.

If $q_j = \infty$, state j is called instantaneous (when the process enters j , it immediately leaves j)

We consider only CTMCs with all states non-instantaneous

For small h

$$P_{ij}(t) = \Pr [X(t + \tau) = j | X(\tau) = i] = \Pr [X(t) = j | X(0) = i]$$

indicates that

$$\Pr [X(t + h) = j | X(t) = i] = q_{ij}h + o(h) \quad (i \neq j)$$

$$\Pr [X(t + h) = i | X(t) = i] = 1 - q_i h + o(h)$$

which generalizes the Poisson process and motivates to call

q_i the rate corresponding to state i

Lemma Given the infinitesimal generator Q , the transition probability matrix $P(t)$ is differentiable for all $t \geq 0$, with

$$\begin{aligned} P'(t) &= P(t)Q && \text{the forward equation} \\ &= QP(t) && \text{the backward equation} \end{aligned}$$

The probability $s_k(t)$ that the Markov process is in state k at time t is completely determined by

$$s'_k(t) = -q_k s_k(t) + \sum_{j=1, j \neq k}^N q_{jk} s_j(t)$$

with initial condition $s_k(0)$

It holds that $s(t + \tau) = s(\tau)P(t) \implies s_k(t + h) = \sum_{j=1}^N s_j(t)P_{jk}(h)$
from which

$$\frac{s_k(t + h) - s_k(t)}{h} = s_k(t) \frac{P_{kk}(h) - 1}{h} + \sum_{j=1, j \neq k}^N s_j(t) \frac{P_{jk}(h)}{h}$$

For $h \downarrow 0$ $q_{jk} = \lim_{h \downarrow 0} \frac{P_{jk}(h)}{h}$ and $q_k = \lim_{h \downarrow 0} \frac{1 - P_{kk}(h)}{h}$

Algebraic properties of Q

The solution of $P'(t) = P(t)Q = QP(t)$ with initial condition $P(0) = I$ is

$$P(t) = e^{Qt}$$

If all eigenvalues λ_k of Q are distinct, then

$$P(t) = e^{Qt} = X \text{diag}(e^{\lambda_k t}) Y^T$$

X contains as columns the right- eigenvectors of Q x_k

Y contains as columns the left- eigenvectors of Q y_k

Then

$$P(t) = \sum_{k=1}^N e^{\lambda_k t} x_k y_k^T$$

where $y_k^T x_k = 1$ and $x_k y_k^T$ is a $N \times N$ matrix

Assuming (thus omitting pathological cases) that $P(t)$ is stochastic, irreducible matrix for any time t , we may write

$$P(t) = u\pi + \sum_{k=2}^N e^{-|\operatorname{Re} \lambda_k t| + \operatorname{Im} \lambda_k t} x_k y_k^T$$

spectral or eigen decomposition
of the transition probability matrix

where $P_\infty = u\pi$ is the $N \times N$ matrix with each row containing the steady-state vector π

$$P(t) = e^{Qt} = \sum_{k=0}^{\infty} \frac{(Qt)^k}{k!}$$

Taylor expansion

$$P(t) = e^{Qt} = \lim_{n \rightarrow \infty} \left(I + \frac{Qt}{n} \right)^n$$

matrix equivalent of $e^x = \lim_{n \rightarrow \infty} (1 + x/n)^n$

Exponential sojourn times

Theorem The sojourn times τ_j of a continuous-time Markov process in a state j are independent, exponential random variables with mean $\frac{1}{q_j}$.

Proof

- The independence of the sojourn times follows from the Markov property.
- The exponential distribution is proved by demonstrating that the sojourn times τ_j satisfy the memoryless property. The only continuous distribution that satisfies the memoryless property is the exponential distribution.

[Cinlar, Ch. 8]

Define by W_t the length of time the process Y remains in the state being occupied at the time instant t :

$$W_t(\omega) = \inf \{s > 0 : Y_{t+s}(\omega) \neq Y_t(\omega)\}$$

Theorem: For any $i \in E$ and $t \geq 0$ $P \{W_t > u | Y_t = i\} = e^{-\lambda(i)u}$, $u \geq 0$

Proof: The previous conditional probability is independent of t (Time homogeneity of Y). Also, $\{W_t > u + v\} \sim \{W_t > u, W_{t+u} > v\}$

$$\begin{aligned} f(u + v) &= P \{W_t > u + v | Y_t = i\} \\ &= P \{W_t > u, W_{t+u} > v | Y_t = i\} \\ &= P \{W_t > u | Y_t = i\} \cdot P \{W_{t+u} > v | Y_t = i, W_t > u\} \end{aligned}$$

If $Y_t = i$ and $W_t > u$ then $Y_{t+u} = i$. Therefore, $P \{W_{t+u} > v | Y_{t+u} = i\} = f(v)$

Hence, $f(u + v) = f(u) \cdot f(v) \implies f(u) = e^{-cu}$ for some $c \geq 0$

Steady state

For an irreducible, finite-state Markov chain (all states communicate and $P_{ij}(t) > 0$), the steady-state π exists

By definition, the steady-state does not change over time, or $\lim_{t \rightarrow \infty} P'(t) = 0$. Thus, $P'(t) = P(t)Q = QP(t)$ implies

$$QP_{\infty} = P_{\infty}Q = 0$$

where $\lim_{t \rightarrow \infty} P(t) = P_{\infty}$

All rows of P_{∞} are proportional to the eigenvector of Q belonging to $\lambda = 0$

The steady (row) vector π is a solution of $\pi Q = 0$

A single component of π obeys $\pi_i q_i = \sum_{j=1, j \neq i}^N \pi_j q_{ji}$

Steady state

The steady (row) vector π is a solution of $\pi Q = 0$

A single component of π obeys

balance equations

$$\pi_i q_i = \sum_{j=1, j \neq i}^N \pi_j q_{ji}$$

Long-run rate at which the process leaves state i

=

aggregate long-run rate towards state i (sum of the long-run rates of transitions towards state i from other states)

The balance equations follow also from

$$s'_k(t) = -q_k s_k(t) + \sum_{j=1, j \neq k}^N q_{jk} s_j(t)$$

since $\lim_{t \rightarrow \infty} s_k(t) = \pi_k$ and $\lim_{t \rightarrow \infty} s'_k(t) = 0$

The embedded Markov chain

The probability that, if a transition occurs, the process moves from state i to a different state $j \neq i$ is

$$\begin{aligned} V_{ij}(h) &= \Pr [X(h) = j | X(h) \neq i, X(0) = i] \\ &= \frac{\Pr [\{X(h) = j\} \cap \{X(h) \neq i\} | X(0) = i]}{\Pr [X(h) \neq i | X(0) = i]} = \frac{P_{ij}(h)}{1 - P_{ii}(h)} \end{aligned}$$

$$\text{For } h \downarrow 0 \quad V_{ij} = \lim_{h \downarrow 0} V_{ij}(h) = \lim_{h \downarrow 0} \frac{\frac{P_{ij}(h)}{h}}{\frac{1 - P_{ii}(h)}{h}} = \frac{q_{ij}}{q_i}$$

V_{ij} : transition probabilities of the **embedded Markov chain**

Given a transition, it is a transition to another state $j \neq i$ since

$$\sum_{j=1, j \neq i} V_{ij} = 1$$

The embedded Markov chain

The rate q_{ij} can be expressed in terms of the transition probabilities of the embedded Markov chain as

$$q_{ij} = q_i V_{ij}$$

rate (number of transitions per unit time) of the process in state i

probability that a transition from state i to state j occurs (By definition, $V_{ii}=0$)

$V_{ii}=0$: in the embedded Markov chain specified by V there are no self transitions

The steady-state vector π_j of the CTMC obeys

$$\left. \begin{array}{l} \pi_i q_i = \sum_{j=1, j \neq i}^N \pi_j q_{ji} \\ V_{ii} = 0 \quad q_{ij} = q_i V_{ij} \end{array} \right\} \pi_i q_i = \sum_{j=1}^N \pi_j q_j V_{ji}$$

The embedded Markov chain

The steady-state vector π_j of the CTMC obeys

$$\pi_i q_i = \sum_{j=1}^N \pi_j q_j V_{ji}$$

The steady-state vector v_j of the embedded MC obeys

$$v_i = \sum_{j=1}^N v_j V_{ji} \quad \text{and} \quad \|v\|_1 = 1$$

The relations between the steady-state vectors of the CTMC and of its corresponding embedded DTMC are

$$\pi_i = \frac{v_i / q_i}{\sum_{j=1}^N v_j / q_j} \quad v_i = \frac{\pi_i q_i}{\sum_{j=1}^N \pi_j q_j}$$

The classification in the discrete-time case into transient and recurrent can be transferred via the embedded MC to continuous MCs

Uniformization

The restriction that there are no self transitions from a state to itself can be removed

$P(t) = e^{Qt}$ can be rewritten as

$$P(t) = \exp \left[-\beta I t + \beta t \left(I + \frac{Q}{\beta} \right) \right] = e^{-\beta t} \exp \left[\beta t \left(I + \frac{Q}{\beta} \right) \right]$$

or
$$P_{ij}(t) = e^{-\beta t} \sum_{k=0}^{\infty} \frac{(\beta t)^k}{k!} T_{ij}^k(\beta)$$

where $T(\beta) = I + \frac{Q}{\beta}$ and $\beta \geq \max_i q_i$

Uniformization

$\beta T(\beta) = Q + \beta I$ can be regarded as a rate matrix with the property that

$$\sum_{j=1}^N \beta T_{ij}(\beta) = \sum_{j=1}^N Q_{ij} + \beta \sum_{j=1}^N \delta_{ij} = \beta$$

(for each state i the transition rate in any state i is precisely the same, equal to β)

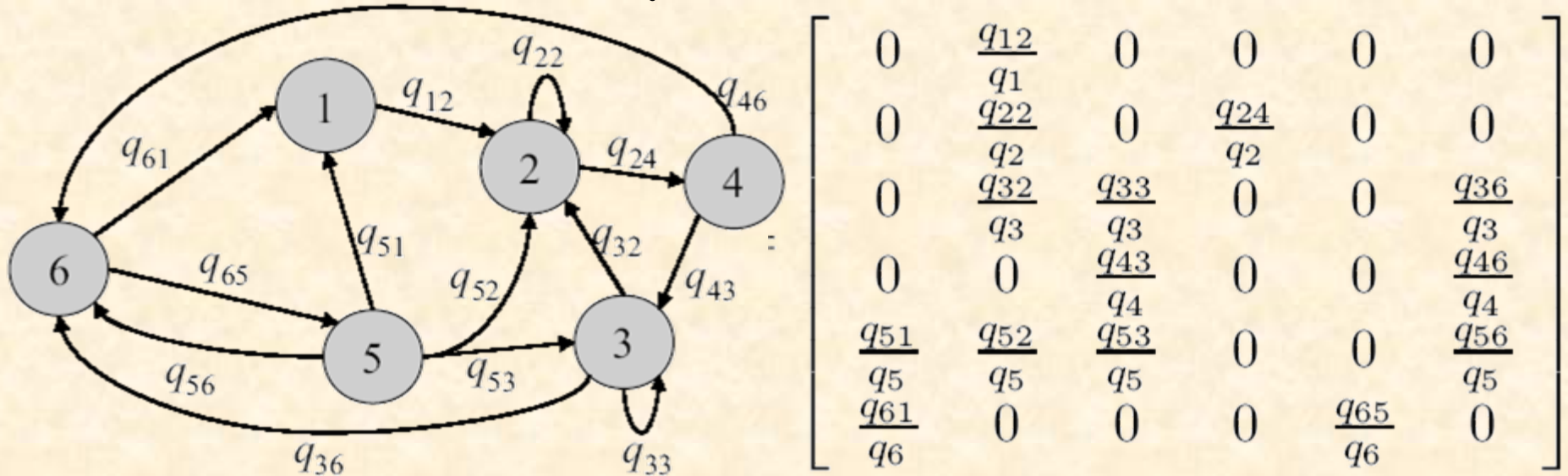
$$T_{ii}(\beta) = 1 - \frac{1}{\beta} \sum_{j=1; j \neq i}^N Q_{ij} = 1 - \frac{q_i}{\beta} \geq 0$$

$$T_{ij}(\beta) = \frac{q_{ij}}{\beta}$$

$T(\beta)$ can be interpreted as an embedded Markov chain that

- allows self transitions and
- the rate for each state is equal to β

Uniformization



The transition rates q_i follow from $\sum_{j=1}^6 V_{ij} = 1$ with $V_{ij} = \frac{q_{ij}}{q_i}$

The change in transition rates changes

- the steady-state vector (since the balance equations change)
- the number of transitions during some period of time

However, the Markov process $\{X(t), t \geq 0\}$ is not modified

(a self-transition does not change $X(t)$ nor the distribution of the time until the next transition to a different state)

Uniformization

When the transition rate q_j at each state j are the same, the embedded Markov chain $T(\beta)$ is called a uniformized chain.

In a uniformized chain, the steady-state vector $t(\beta)$ of $T(\beta)$ is the same as the steady-state vector π .

$$T(\beta) \stackrel{\text{def}}{=} I + \frac{Q}{\beta}$$
$$t_j(\beta) = \sum_{k=1}^N T_{kj}(\beta) t_k(\beta) = \sum_{k=1}^N \left(\delta_{kj} + \frac{q_{kj}}{\beta} \right) t_k(\beta)$$
$$= t_j(\beta) + \frac{1}{\beta} \sum_{k=1}^N t_k(\beta) q_{kj}$$

$$t_k(\beta) q_k = \sum_{k=1; k \neq j}^N t_k(\beta) q_{kj} \quad , \quad \text{where } t_k(\beta) = \pi_k \text{ (independent of } \beta)$$

since it satisfies the balance equation and the steady-state of a positive recurrent chain is unique

Uniformization

$\{X_k(\beta)\}$: uniformized (discrete) process

$N(t)$: total number of transitions in $[0, t]$ in $\{X_k(\beta)\}$

the rates $q_i = \beta$ are all the same $\implies N(t)$ is Poisson rate β

(for any continuous-time Markov chain, the inter-transition or sojourn times are i.i.d. exponential random variables)

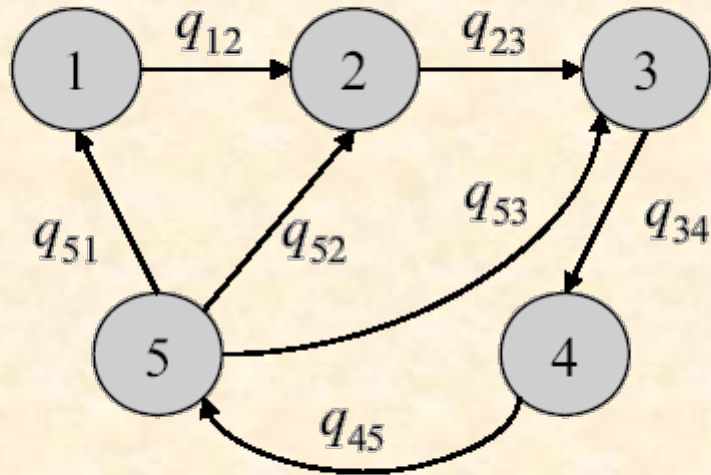
$\Pr [N(t) = k] = e^{-\beta t} \frac{(\beta t)^k}{k!}$ Prob. that the number of transitions in $[0, t]$ in $\{X_k(\beta)\}$ is k

$T_{ij}^k(\beta) = \Pr [X_k(\beta) = j | X_0(\beta) = i]$ k -step transition probability of $\{X_k(\beta)\}$

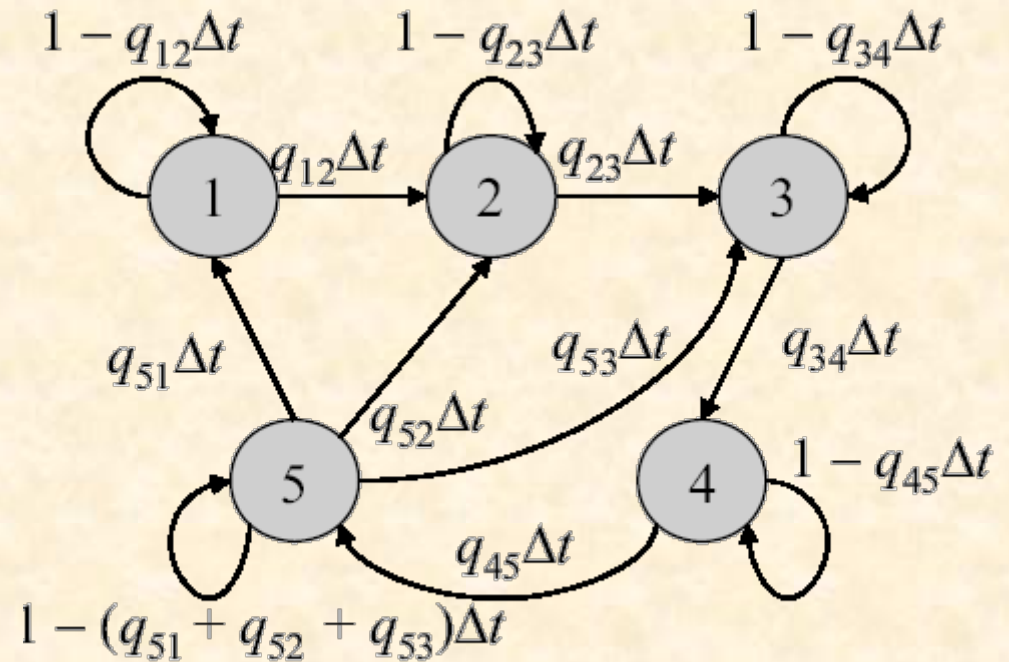
$P_{ij}(t) = e^{-\beta t} \sum_{k=0}^{\infty} \frac{(\beta t)^k}{k!} T_{ij}^k(\beta)$ can be interpreted as

$$P_{ij}(t) = \sum_{k=0}^{\infty} \Pr [X_k(\beta) = j | X_0(\beta) = i, N(t) = k] \Pr [N(t) = k]$$

Sampled time Markov chain



Continuous-time Markov process



Sampled-time Markov chain

Approximates the continuous time Markov process

transition probabilities of the sampled time Markov chain

$$P_{ij} = q_{ij}\Delta t \quad (i \neq j)$$

$$P_{ii} = 1 - q_i\Delta t$$

(are obtained by expanding $P_{ij}(t)$ to first order with fixed step $h = \Delta t$)

Sampled time Markov chain

The steady-state vector of the sampled-time Markov chain \mathcal{x} with $\|\mathcal{x}\|_1 = 1$ satisfies for each component j

$$\mathcal{x}_j = \sum_{k=1}^N P_{kj} \mathcal{x}_k = \Delta t \sum_{k=1; k \neq j}^N q_{kj} \mathcal{x}_k + (1 - q_j \Delta t) \mathcal{x}_j$$

or

$$q_j \mathcal{x}_j = \sum_{k=1; k \neq j}^N q_{kj} \mathcal{x}_k$$

is exactly equal to that of the CTMC for any sampling step by sampling every Δt we miss the smaller-scale dynamics, however the steady-state behavior is exactly captured

The transitions in a CTMC

Based on the embedded Markov chain all properties of the continuous Markov chain may be deduced.

Theorem Let V_{ij} denote the transition probabilities of the embedded Markov chain and q_{ij} the rates of the infinitesimal generator. The transition probabilities of the corresponding continuous-time Markov chain are found as

$$P_{ij}(t) = \delta_{ij}e^{-q_i t} + q_i \sum_{k \neq i} V_{ik} \int_0^t e^{-q_i u} P_{kj}(t-u) du$$

By a change of variable $s = t - u$

$$P_{ij}(t) = \delta_{ij}e^{-q_it} + q_i \sum_{k \neq i} V_{ik} \int_0^t e^{-q_i u} P_{kj}(t - u) du$$

we have

$$P_{ij}(t) = \delta_{ij}e^{-q_it} + q_i \sum_{k \neq i} V_{ik} e^{-q_it} \int_0^t e^{q_i s} P_{kj}(s) ds$$

After differentiation wrt t

$$\begin{aligned} P'_{ij}(t) &= -q_i \delta_{ij} e^{-q_it} - q_i \sum_{k \neq i} q_k V_{ik} e^{-q_it} \int_0^t e^{q_i s} P_{kj}(s) ds + q_i \sum_{k \neq i} V_{ik} P_{kj}(t) \\ &= -q_i \delta_{ij} e^{-q_it} - q_i (P_{ij}(t) - \delta_{ij} e^{-q_it}) + q_i \sum_{k \neq i} V_{ik} P_{kj}(t) \\ &= -q_i P_{ij}(t) + q_i \sum_{k \neq i} V_{ik} P_{kj}(t) \end{aligned}$$

Evaluating at $t=0$, recalling that $P'(0)=Q$ and $P(0)=I$,

$$P'_{ij}(0) = -q_i P_{ij}(0) + q_i \sum_{k \neq i} V_{ik} P_{kj}(0)$$

$$q_{ij} = -q_i \delta_{ij} + q_i \sum_{k \neq i} V_{ik} \delta_{kj} = -q_i \delta_{ij} + q_i V_{ij}$$

which is precisely $q_{ij} = q_i V_{ij}$

with $q_i = -q_{ii}$
and $q_{ij} = q_i V_{ij}$ we arrive at $P'_{ij}(t) = \sum_{k=1}^N q_{ik} P_{kj}(t)$ backward equation

$$\text{Hence, } P_{ij}(t) = \delta_{ij} e^{-q_i t} + q_i \sum_{k \neq i} V_{ik} \int_0^t e^{-q_i u} P_{kj}(t-u) du$$

can be interpreted as an integrated form of the backward equation and thus of the entire CTMC

The two-state CTMC

is defined by the infinitesimal generator $Q = \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix}$

the forward equation is $P'(t) = P(t)Q$, or

$$\begin{bmatrix} P'_{11}(t) & P'_{12}(t) \\ P'_{21}(t) & P'_{22}(t) \end{bmatrix} = \begin{bmatrix} P_{11}(t) & P_{12}(t) \\ P_{21}(t) & P_{22}(t) \end{bmatrix} \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix}$$

Due to $P_{12}(t) = 1 - P_{11}(t)$ and $P_{21}(t) = 1 - P_{22}(t)$

the forward equation simplifies to
$$\begin{aligned} P'_{11}(t) &= -(\lambda + \mu)P_{11}(t) + \mu \\ P'_{22}(t) &= -(\lambda + \mu)P_{22}(t) + \lambda \end{aligned}$$

Due to symmetry $\begin{matrix} \lambda \rightarrow \mu \\ \mu \rightarrow \lambda \end{matrix}$ it is sufficient to solve the one eq.

The two-state CTMC

The solution of $y'(x) + p(x)y(x) = r(x)$ is of the form $y = e^{-a(x)} \left(\int r(x)e^{a(x)} dx + \kappa \right)$ where κ is the constant of integration, and $a(x) = \int p(x)dx$.

Thus, the solution of $P'_{11}(t) = -(\lambda + \mu)P_{11}(t) + \mu$ is of the form

$$\frac{\mu}{(\lambda + \mu)} + c e^{-(\lambda + \mu)t}$$

The constant c follows from the initial condition $P(0) = I$
 $\implies P_{11}(0) = 1$

Thus,

$$P_{11}(t) = \frac{\mu}{\lambda + \mu} + \frac{\lambda}{\lambda + \mu} e^{-(\lambda + \mu)t} \quad \text{and} \quad \pi = \begin{bmatrix} \frac{\mu}{\lambda + \mu} & \frac{\lambda}{\lambda + \mu} \end{bmatrix}$$
$$P_{22}(t) = \frac{\lambda}{\lambda + \mu} + \frac{\mu}{\lambda + \mu} e^{-(\lambda + \mu)t}$$

Time reversibility

Ergodic MCs with non-zero steady-state distribution

Suppose the process is in the steady-state and consider the time-reversed process defined by the sequence X_n, X_{n-1}, \dots

Theorem The time-reversed Markov process is a MC.

A MC is said to be *time reversible* if for all i and j $P_{ij} = R_{ij}$

Condition for time reversibility $\pi_i P_{ij} = \pi_j P_{ji}$ for all i, j

rate from $i \rightarrow j$

rate from $j \rightarrow i$

For time reversible MCs any vector x satisfying $\|x\|_1 = 1$ and $x_i P_{ij} = x_j P_{ji}$ is a steady state vector

Time reversibility

Let V_{ij} be the transition probabilities of the discrete-time embedded MC

Let U_{ij} be the transition probabilities of the time-reversed embedded MC and r_{ij} the rates of the corresponding CTMC

$$R_{ij} = \frac{\pi_j P_{ji}}{\pi_i} \implies r_{ij} = r_i U_{ij}$$

The sojourn time in state i of the time-reversed process is exponentially distributed with precisely the same rate $r_i = q_i$ as the forward time process.

For the continuous time rates it holds $\pi_i r_{ij} = \pi_j q_{ji}$

$$\pi_i r_{ij} = \pi_j q_{ji}$$

rate at which the time-reversed process moves from state i to j

=

rate at which the forward process moves from state j to i

Applications of Markov chains

Books

- Performance Analysis of Communications Networks and Systems
(Piet Van Mieghem), Chap. 11

Examples of DTMCs

$\{Y_n\}_{n \geq 1}$: set of positive integer, independent random variables that are identically distributed with $\Pr[Y = k] = a_k$

Examples of DTMCs

$$X_n = Y_n \quad X_n = \max[Y_1, Y_2, Y_3, \dots, Y_n] \quad X_n = \sum_{k=1}^n Y_k$$

$$X_n = Y_n \quad P = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & \cdots \\ a_1 & a_2 & a_3 & a_4 & \cdots \\ a_1 & a_2 & a_3 & a_4 & \cdots \\ a_1 & a_2 & a_3 & a_4 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

All rows are identical and $\Pr[X_{n+1} = j | X_n = i] = a_j$ shows that the states X_{n+1} and X_n are independent from each other.

Examples of DTMCs

$\{Y_n\}_{n \geq 1}$: set of positive integer, independent random variables that are identically distributed with $\Pr [Y = k] = a_k$

$$X_n = \max [Y_1, Y_2, Y_3, \dots, Y_n]$$

$X_{n+1} = \max [X_n, Y_{n+1}]$ reflects the Markov property

For $j < i$ $\Pr [X_{n+1} = j | X_n = i] = 0$

For $j > i$ $\Pr [X_{n+1} = j | X_n = i] = \Pr [Y_{n+1} = j] = a_j$

For $j = i$

$$\Pr [X_{n+1} = j | X_n = i] = \Pr [Y_{n+1} \leq j] = \sum_{k=1}^j \Pr [Y_{n+1} = k] = \sum_{k=1}^j a_k = A_j$$

$$P = \begin{bmatrix} A_1 & a_2 & a_3 & a_4 & \cdots \\ 0 & A_2 & a_3 & a_4 & \cdots \\ 0 & 0 & A_3 & a_4 & \cdots \\ 0 & 0 & 0 & A_4 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

Examples of DTMCs

$\{Y_n\}_{n \geq 1}$: set of positive integer, independent random variables that are identically distributed with $\Pr [Y = k] = a_k$

$$X_n = \sum_{k=1}^n Y_k$$

$X_{n+1} = X_n + Y_{n+1}$ reflects the Markov property

For $j \leq i$. $\Pr [X_{n+1} = j | X_n = i] = 0$

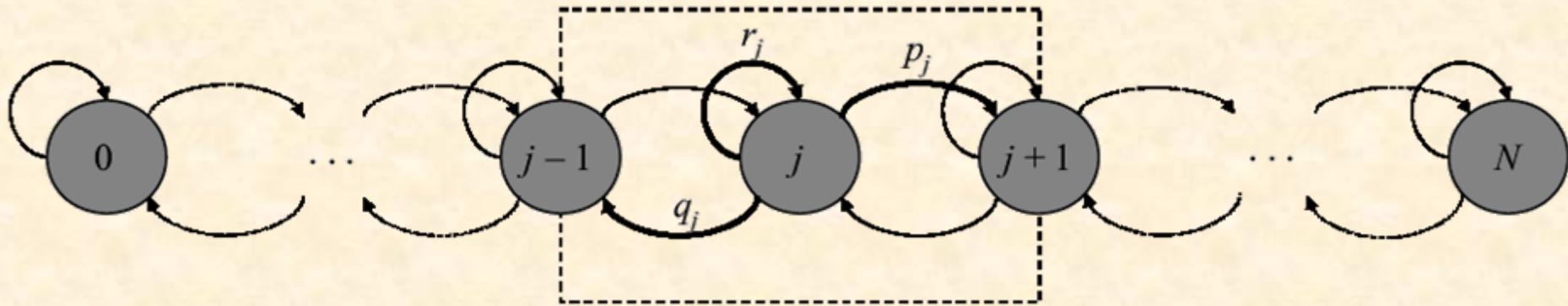
For $j > i$. $\Pr [X_{n+1} = j | X_n = i] = \Pr [X_n + Y_{n+1} = j | X_n = i]$
 $= \Pr [Y_{n+1} = j - i] = a_{j-i}$

$$P = \begin{bmatrix} 0 & a_1 & a_2 & a_3 & \cdots \\ 0 & 0 & a_1 & a_2 & \cdots \\ 0 & 0 & 0 & a_1 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

The general random walk

Describes a “three-possibility motion” of an item

In general, the transition probabilities depend on the position



If the process is in state j it has three possibilities:

- remain in state j with probability $r_j = \Pr [X_{k+1} = j | X_k = j]$
- move to the next state $j + 1$ with prob $p_j = \Pr [X_{k+1} = j + 1 | X_k = j]$
- jump back to state $j - 1$ with prob $q_j = \Pr [X_{k+1} = j - 1 | X_k = j]$

The gambler's ruin problem

A state j reflects the capital of a gambler

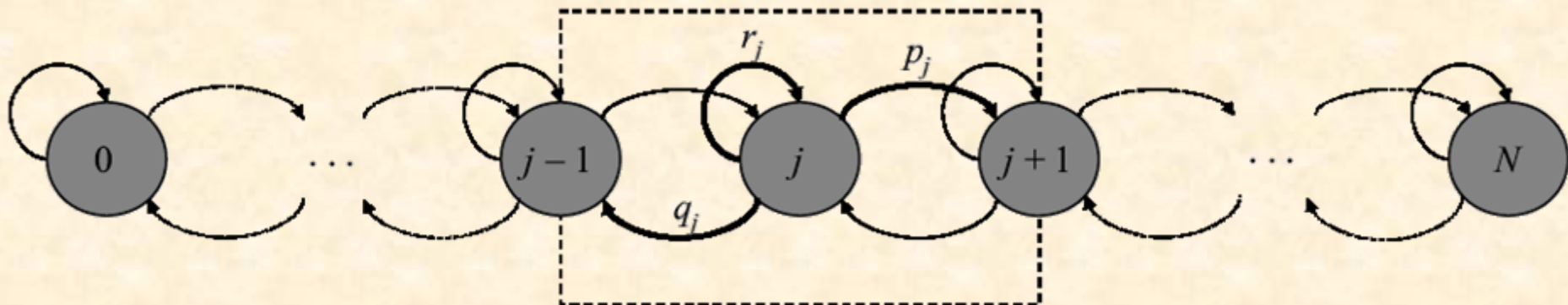
- p_j is the chance that the gambler wins
- q_j is the probability that he loses

The gambler

- achieves his target when he reaches state N
- is ruined at state 0

States 0 and N are absorbing states with $r_0 = r_N = 1$

In most games $p_j = p$, $q_j = q$ and $r_j = 1 - p - q$



The probability of gambler's ruin

$T = \min_k \{X_k = 0\}$ is the hitting time to state 0

The probability of gambler's ruin is defined as

$$u_j = \Pr [X_T = 0 | X_0 = j] \text{ or equivalently, } u_j = \Pr [T_0 < \infty | X_0 = j]$$

By definition, $u_0 = 1$ and $u_N = 0$

The law of total probability gives

$$\Pr [X_T = 0 | X_0 = j] = \sum_{k=0}^N \Pr [X_T = 0 | X_0 = j, X_1 = k] \Pr [X_1 = k | X_0 = j]$$

$$= q_j \Pr [X_T = 0 | X_1 = j - 1]$$

$$+ r_j \Pr [X_T = 0 | X_1 = j]$$

$$+ p_j \Pr [X_T = 0 | X_1 = j + 1]$$

The probability of gambler's ruin

$$\Pr [X_T = 0 | X_0 = j] = q_j \Pr [X_T = 0 | X_1 = j - 1] + r_j \Pr [X_T = 0 | X_1 = j] + p_j \Pr [X_T = 0 | X_1 = j + 1]$$

After the first transition, the probability $\Pr [X_T = 0 | X_1 = j] = u_j$ remains the same as the initial $\Pr [X_T = 0 | X_0 = j]$ because T is a random variable depending on the state and not on the discrete-time. Hence,

$$u_0 = 1$$

$$u_j = q_j u_{j-1} + r_j u_j + p_j u_{j+1} \quad (1 \leq j < N)$$

Substituting $r_j = 1 - p_j - q_j$ gives $u_{j+1} = -\frac{q_j}{p_j} u_{j-1} + \left(1 + \frac{q_j}{p_j}\right) u_j$

Hence,

$$u_j = -\sum_{k=1}^{j-1} \prod_{m=1}^k \frac{q_m}{p_m} + \left(1 + \sum_{k=1}^{j-1} \prod_{m=1}^k \frac{q_m}{p_m}\right) u_1$$

u_1 is determined by $u_N = 0$

Hence,

$$\Pr [X_T = 0 | X_0 = j] = \frac{\sum_{k=j}^{N-1} \prod_{m=1}^k \frac{q_m}{p_m}}{1 + \sum_{k=1}^{N-1} \prod_{m=1}^k \frac{q_m}{p_m}}$$

The probability of gambler's ruin

The mean hitting time $\eta_j = E[T|X_0 = j]$ follows by

$$\eta_j = 1 + q_j\eta_{j-1} + r_j\eta_j + p_j\eta_{j+1}$$

with $\eta_0 = \eta_N = 0$ and $r_j = 1 - p_j - q_j$

Hence,
$$\eta_{j+1} = -\frac{1}{p_j} - \frac{q_j}{p_j}\eta_{j-1} + \left(1 + \frac{q_j}{p_j}\right)\eta_j$$

and
$$\eta_j = -\sum_{k=1}^{j-1} \frac{1}{p_k} \left(1 + \sum_{n=1}^{k-1} \prod_{m=1}^n \frac{q_{k-m+1}}{p_{k-m}}\right) + \left(1 + \sum_{k=1}^{j-1} \prod_{m=1}^k \frac{q_m}{p_m}\right) \eta_1$$

Eliminating η_1 from $\eta_N = 0$, finally leads to mean hitting time to ruin or the mean duration of the game

$$\eta_j = -\sum_{k=1}^{j-1} \frac{1}{p_k} \left(1 + \sum_{n=1}^{k-1} \prod_{m=1}^n \frac{q_{k-m+1}}{p_{k-m}}\right) + \left(1 + \sum_{k=1}^{j-1} \prod_{m=1}^k \frac{q_m}{p_m}\right) \frac{\sum_{k=1}^{N-1} \frac{1}{p_k} \left(1 + \sum_{n=1}^{k-1} \prod_{m=1}^n \frac{q_{k-m+1}}{p_{k-m}}\right)}{1 + \sum_{k=1}^{N-1} \prod_{m=1}^k \frac{q_m}{p_m}}$$

The probability of gambler's ruin

In the special case where $q_k = q$ and $p_k = p$

- The probability of gambler's ruin becomes

$$\Pr [X_T = 0 | X_0 = j] = \frac{\sum_{k=j}^{N-1} \left(\frac{q}{p}\right)^k}{\sum_{k=0}^{N-1} \left(\frac{q}{p}\right)^k} = \frac{\left(\frac{q}{p}\right)^j - \left(\frac{q}{p}\right)^N}{1 - \left(\frac{q}{p}\right)^N}$$

- The mean duration of the game becomes

$$\eta_j = - \sum_{k=1}^{j-1} \left(\frac{1 - \left(\frac{q}{p}\right)^k}{p - q} \right) + \left(\frac{1 - \left(\frac{q}{p}\right)^j}{1 - \left(\frac{q}{p}\right)^N} \right) \sum_{k=1}^{N-1} \frac{1 - \left(\frac{q}{p}\right)^k}{p - q}$$

or

$$E [T | X_0 = j] = \frac{1}{p - q} \left[N \left(\frac{1 - \left(\frac{q}{p}\right)^j}{1 - \left(\frac{q}{p}\right)^N} \right) - j \right]$$

The probability of gambler's ruin

The steady-state equation for the vector component π_j becomes, for $1 \leq j < N$,

$$\pi_j = p_{j-1}\pi_{j-1} + r_j\pi_j + q_{j+1}\pi_{j+1}$$

and, for $j = 0$ and $j = N$,

$$\pi_0 = r_0\pi_0 + q_1\pi_1$$

$$\pi_N = p_{N-1}\pi_{N-1} + r_N\pi_N$$

These equations, in combination with $\|\pi\|_1 = 1$, yield to

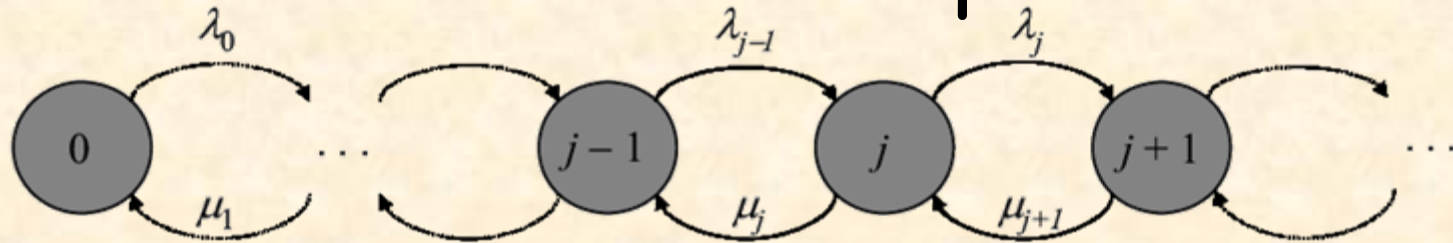
$$\pi_j = \frac{\prod_{m=0}^{j-1} \frac{p_m}{q_{m+1}}}{1 + \sum_{j=1}^N \prod_{m=0}^{j-1} \frac{p_m}{q_{m+1}}}$$

These relations remain valid even when the number of states N tends to infinity provided the infinite sum converges.

In the simple case where $p_k = p$ and $q_k = q$, we obtain with $\rho = \frac{p}{q}$,

$$\pi_j = \frac{(1 - \rho) \rho^j}{1 - \rho^{N+1}}$$

Birth and death process



$$Q = \begin{bmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \mu_1 & -(\lambda_1 + \mu_1) & \lambda_1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & \mu_2 & -(\lambda_2 + \mu_2) & \lambda_2 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & \mu_3 & -(\lambda_3 + \mu_3) & \lambda_3 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & \mu_4 & -(\lambda_4 + \mu_4) & \lambda_4 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

The embedded Markov chain of the birth and death process is a random walk with transition probabilities

$$V_{i,i-1} = \frac{\mu_i}{\lambda_i + \mu_i}, \quad V_{i,i+1} = \frac{\lambda_i}{\lambda_i + \mu_i} \quad \text{and} \quad V_{ik} = 0 \quad \text{for} \quad k \neq i-1 \neq i+1$$

$s_k(t) = \Pr [X(t) = k]$ Differential equations that describe the BD process

$$s'_0(t) = -\lambda_0 s_0(t) + \mu_1 s_1(t)$$

$$s'_k(t) = -(\lambda_k + \mu_k) s_k(t) + \lambda_{k-1} s_{k-1}(t) + \mu_{k+1} s_{k+1}(t)$$

with initial condition $s_k(0) = \Pr [X(0) = k]$

The steady-state

The steady-state follows from $\pi Q = 0$

$$Q = \begin{bmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \mu_1 & -(\lambda_1 + \mu_1) & \lambda_1 & 0 & 0 & 0 & 0 & \dots \\ 0 & \mu_2 & -(\lambda_2 + \mu_2) & \lambda_2 & 0 & 0 & 0 & \dots \\ 0 & 0 & \mu_3 & -(\lambda_3 + \mu_3) & \lambda_3 & 0 & 0 & \dots \\ 0 & 0 & 0 & \mu_4 & -(\lambda_4 + \mu_4) & \lambda_4 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

Hence,
$$-\lambda_0\pi_0 + \mu_1\pi_1 = 0$$

$$\lambda_{j-1}\pi_{j-1} - (\lambda_j + \mu_j)\pi_j + \mu_{j+1}\pi_{j+1} = 0$$

Hence, the steady-state of the birth and death process is

$$\pi_0 = \frac{1}{1 + \sum_{j=1}^{\infty} \prod_{m=0}^{j-1} \frac{\lambda_m}{\mu_{m+1}}}$$

$$\pi_j = \frac{\prod_{m=0}^{j-1} \frac{\lambda_m}{\mu_{m+1}}}{1 + \sum_{j=1}^{\infty} \prod_{m=0}^{j-1} \frac{\lambda_m}{\mu_{m+1}}} \quad j \geq 1$$

with initial condition $s_k(0) = \Pr[X(0) = k]$

The steady-state

The process is transient if and only if the embedded MC is transient.

For a recurrent chain, $r_{ij} = \Pr [T_j < \infty | X_0 = i]$ equals 1

(every state j is certainly visited starting from initial state i)

For the embedded MC (gambler's ruin), it holds that

$$\Pr [T_0 < \infty | X_0 = j] = 1 - \frac{\sum_{k=0}^{j-1} \prod_{m=1}^k \frac{q_m}{p_m}}{\sum_{k=0}^{N-1} \prod_{m=1}^k \frac{q_m}{p_m}}$$

Can be equal to one for $N \rightarrow \infty$ only if $\lim_{N \rightarrow \infty} \sum_{k=0}^{N-1} \prod_{m=1}^k \frac{q_m}{p_m} = \infty$

Transformed to the birth and death rates $\Sigma_2 = \sum_{j=1}^{\infty} \prod_{m=0}^{j-1} \frac{\mu_m}{\lambda_m} = \infty$

Furthermore, the infinite series $\Sigma_1 = \sum_{j=1}^{\infty} \prod_{m=0}^{j-1} \frac{\lambda_m}{\mu_{m+1}}$ must converge to have a steady-state distribution

- If $\Sigma_1 < \infty$ and $\Sigma_2 = \infty$ the BD process is positive recurrent
- If $\Sigma_1 = \infty$ and $\Sigma_2 = \infty$, it is null recurrent
- If $\Sigma_2 < \infty$, it is transient

A pure birth process

$\{X(t), t \geq 0\}$ is a pure birth process if for any state i it holds that $\mu_i = 0$

A pure birth process can only jump to higher states

In the simplest case all birth rates are equal $\lambda_i = \lambda$ and

$$P_{jj}(t) = \Pr[\tau_j > t | X(0) = j] = e^{-\lambda t}$$

The transition probabilities of a pure birth process have a Poisson distribution

$$P_{i,i+k}(t) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

and are only function of the difference in states $k = j - i$

Moreover, for $0 \leq u \leq t$, the increment $X(t) - X(u)$ has a Poisson distribution,

$$\Pr[X(t) - X(u) = k] = \frac{(\lambda(t-u))^k}{k!} e^{-\lambda(t-u)}$$

If all birth rates are equal the birth process is a Poisson process

The general birth process

In case the birth rates λ_k depend on the actual state k , the pure birth process can be regarded as the simplest generalization of the Poisson. The Laplace transform difference equations

$$(\lambda_0 + z) S_0(z) = s_0(0) + \mu_1 S_1(z)$$

$$(\lambda_k + \mu_k + z) S_k(z) = s_k(0) + \lambda_{k-1} S_{k-1}(z) + \mu_{k+1} S_{k+1}(z)$$

reduce to the set $S_0(z) = \frac{s_0(0)}{\lambda_0 + z}$ $S_k(z) = \frac{s_k(0)}{\lambda_k + z} + \frac{\lambda_{k-1}}{\lambda_k + z} S_{k-1}(z)$

which has the solution $S_k(z) = \sum_{j=0}^k \frac{s_j(0) \prod_{m=j}^{k-1} \lambda_m}{\prod_{m=j}^k (\lambda_m + z)}$

The form of $S_k(z)$ is a ratio that can always be transformed back to the time-domain provided that λ_k is known. If all $\lambda_k > 0$ are distinct

$$s_k(t) = - \sum_{j=0}^k s_j(0) \sum_{n=j}^k \frac{e^{-\lambda_n t} \prod_{m=j}^{k-1} \lambda_m}{\prod_{m=j; m \neq n}^k (\lambda_m - \lambda_n)}$$

The Yule process

Yule process $\lambda_k = k\lambda$

$$s_k(t) = - \sum_{j=0}^k s_j(0) \sum_{n=j}^k \frac{e^{-\lambda_n t} \prod_{m=j}^{k-1} \lambda_m}{\prod_{m=j; m \neq n}^k (\lambda_m - \lambda_n)}$$

Is simplified to

$$s_k(t) = \sum_{j=0}^k s_j(0) \binom{k-1}{j-1} e^{-j\lambda t} (1 - e^{-\lambda t})^{k-j}$$

In practice, $s_j(0) = \delta_{jn}$ if the process starts from state n (implying $s_k(t) = 0$ for $k < n$ because the process moves to the right for $t \geq 0$) and the general form simplifies to

$$s_k(t) = \binom{k-1}{n-1} e^{-n\lambda t} (1 - e^{-\lambda t})^{k-n}$$

The Yule process

The Yule process has been used as a simple model for the evolution of a population in which

- each individual gives birth at exponential rate λ and
- there are no deaths

$X(t)$ denotes the number of individuals in the population at time t

(At state k there are k individuals and the total birth rate is $\lambda_k = k \lambda$)

If the population starts at $t = 0$ with one individual $n = 1$, the evolution over time has the distribution $s_k(t) = e^{-\lambda t} (1 - e^{-\lambda t})^{k-1}$ which is a geometric distribution with mean $e^{\lambda t}$

Since the sojourn times of a Markov process are i.i.d. exponential random variables, the average time T_k to reach k individuals from one ancestor equals

$$[T_k] = \sum_{j=1}^k \frac{1}{\lambda_j} = \frac{1}{\lambda} \sum_{j=1}^k \frac{1}{j} \approx \frac{\log(k+1) + \gamma}{\lambda},$$

where $\gamma = 0.577215$ is Euler's constant

Constant rate birth and death process

In a constant rate birth and death process, both the birth rate $\lambda_k = \lambda$ and death rate $\mu_k = \mu$ are constant for any state k

The steady-state for all states j with $\rho = \frac{\lambda}{\mu} < 1$ is

$$\pi_j = (1 - \rho) \rho^j$$

only depends on the ratio of birth over death rate

If $s_k(0) = \delta_{kj}$ (the constant rate BD process starts in state j) it can be proved that

$$s_k(t) = e^{-(\lambda+\mu)t} \left[\rho^{(k-j)/2} I_{k-j}(at) + \rho^{(k-j-1)/2} I_{k+j+1}(at) \right] \\ + e^{-(\lambda+\mu)t} (1 - \rho) \rho^k \sum_{m=k+j+2}^{\infty} \rho^{-m/2} I_m(at)$$

where $\rho = \frac{\lambda}{\mu}$, $a = 2\mu\sqrt{\rho}$ and $I_s(z)$ denotes the modified Bessel function

Constant rate birth and death process

Using the asymptotic formulas for the modified Bessel function, the behavior of $s_k(t)$ for large t can be derived

$$s_k(t) = (1 - \rho)\rho^k + \frac{\rho^{(k-j)/2} e^{-(1-\sqrt{\rho})^2 \mu t}}{2\sqrt{\pi} (\sqrt{\rho}\mu t)^{3/2}} \left[\left(k - \frac{\sqrt{\rho}}{1-\sqrt{\rho}} \right) \left(j - \frac{\sqrt{\rho}}{1-\sqrt{\rho}} \right) + O(t^{-1}) \right]$$
$$= \frac{1}{\sqrt{\pi\mu t}} [1 + O(t^{-1})] \quad \text{only if } \rho = 1$$

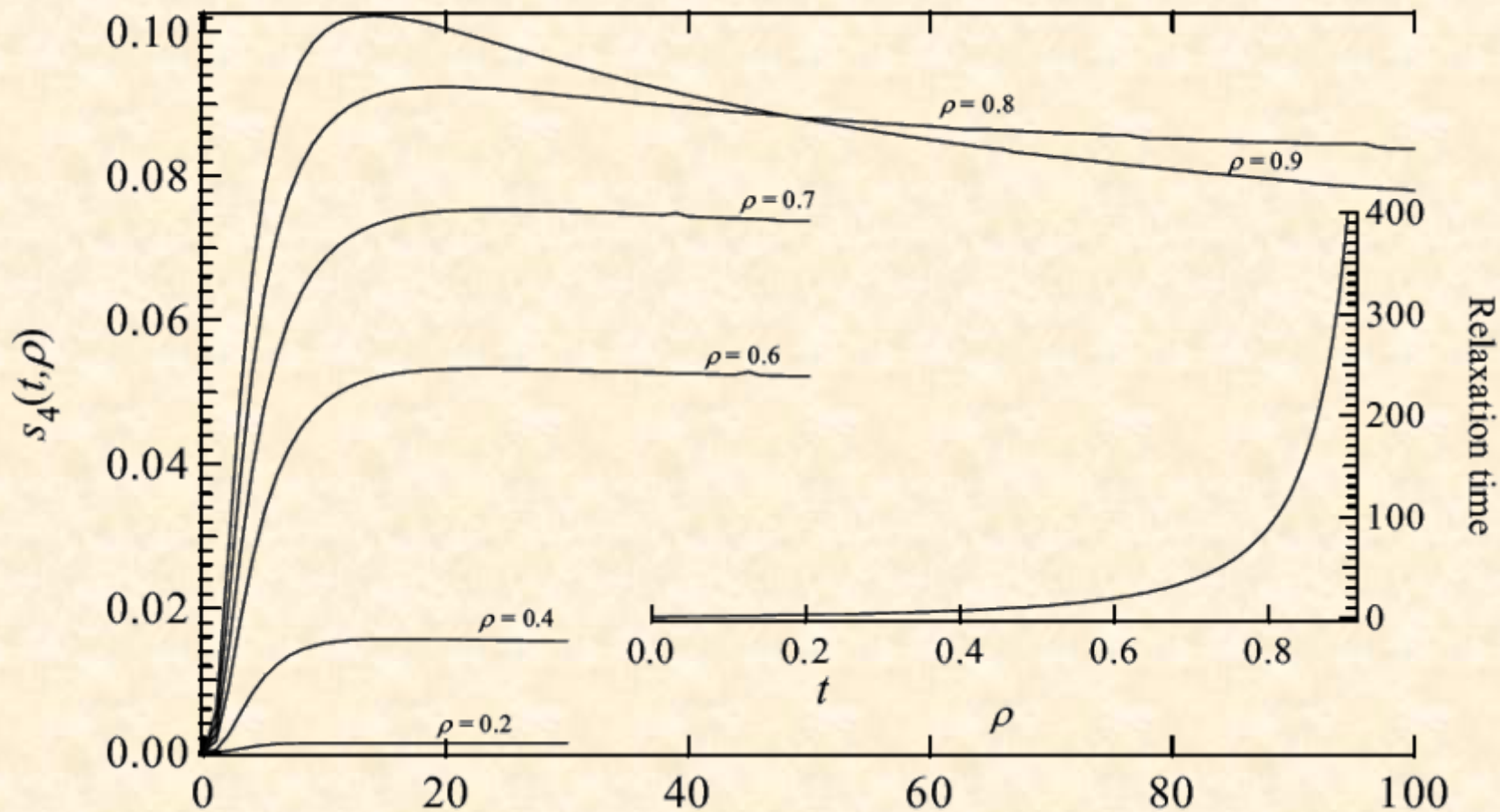
The constant rate birth death process converges to the steady-state $(1 - \rho)\rho^k$ with a relaxation rate $(1 - \sqrt{\rho})^2 \mu$

The higher ρ , the lower the relaxation rate and the slower the process tends to equilibrium

Intuitively, two effects play a role

- Since the probability that states with large k are visited increases with increasing ρ , the built-up time for this occupation will be larger
- In addition, the variability of the number of visited states increases with increasing ρ , which suggests that larger oscillations of the sample paths around the steady-state are likely to occur, enlarging the convergence time

Constant rate birth and death process



A random walk on a graph

$G(N, L)$: graph with N nodes and L links

Suppose that the link weight $w_{ij} = w_{ji}$ is proportional to the transition probability P_{ij} that a packet at node i decides to move to node j

Clearly, $w_{ii} = 0$

Specifically, with $\sum_{j=1}^N P_{ij} = 1$ $P_{ij} = \frac{w_{ij}}{\sum_{k=1}^N w_{ik}}$

This constraint destroys the symmetry in link weight structure ($w_{ij} = w_{ji}$) because, in general, $P_{ij} \neq P_{ji}$ since $\sum_{k=1}^N w_{ik} \neq \sum_{k=1}^N w_{jk}$

The sequence of nodes (or links) visited by that packet resembles a random walk on the graph $G(N, L)$ and constitutes a Markov chain

This Markov process can model an active packet that monitors the network by collecting state information (number of packets, number of lost or retransmitted packets, etc.) in each route

A random walk on a graph

The steady-state of this Markov process is readily obtained by observing that the chain is time reversible

The condition for time reversibility becomes
$$\frac{\pi_i w_{ij}}{\sum_{k=1}^N w_{ik}} = \frac{\pi_j w_{ji}}{\sum_{k=1}^N w_{jk}}$$
 or, since $w_{ij} = w_{ji}$,
$$\frac{\pi_i}{\sum_{k=1}^N w_{ik}} = \frac{\pi_j}{\sum_{k=1}^N w_{jk}}$$

This implies that $\pi_i = \alpha \sum_{k=1}^N w_{ik}$ and using the normalization $\|\pi\|_1 = 1$, we obtain the steady-state probabilities for all nodes i ,

$$\pi_i = \frac{\sum_{k=1}^N w_{ik}}{\sum_{i=1}^N \sum_{k=1}^N w_{ik}} = \frac{\sum_{k=1}^N w_{ik}}{2 \sum_{i=1}^N \sum_{k=i+1}^N w_{ik}}$$

For the collection of these data, the active packet should in steady-state visit all nodes about equally frequently or $\pi_i = 1/N$, implying that the Markov transition matrix P must be doubly stochastic

Slotted Aloha

N nodes that communicate via a shared channel using slotted Aloha

Time is slotted, packets are of the same size

A node transmits a newly arrived packet in the next timeslot

If two nodes transmit at the same timeslot (collision) packets must be retransmitted

Backlogged nodes (nodes with packets to be retransmitted) wait for some random number of timeslots before retransmitting

Packet arrivals at a node form a Poisson process with mean rate λ/N , where λ is the overall arrival rate at the network of N nodes

We ignore queuing of packets at a node (newly arrived packets are discarded if there is a packet to be retransmitted)

We assume, for simplicity, that p_r is the probability that a node retransmits in the next time slot

Slotted Aloha

Slotted Aloha constitutes a DTMC with $X_k \in \{0, 1, 2, \dots\}$, where

- state j counts the number of backlogged nodes
- k refers to the k -th timeslot

Each of the j backlogged nodes retransmits a packet in the next time slot with probability p_r

Each of the $N-j$ unbacklogged nodes transmits a packet in the next time slot iff a packet arrives in the current timeslot which occurs with probability $p_a = \Pr[A > 0] = 1 - \Pr[A = 0]$

For Poisson arrival process $p_a = 1 - \exp\left(-\frac{\lambda}{N}\right)$

The probability that n backlogged nodes in state j retransmit in the next time slot is binomially distributed

$$b_n(j) = \binom{j}{n} p_r^n (1 - p_r)^{j-n}$$

Slotted Aloha

Similarly, the probability that n unbacklogged nodes in state j transmit in the next time slot is

$$u_n(j) = \binom{N-j}{n} p_a^n (1-p_a)^{N-j-n}$$

A packet is transmitted successfully iff

(a) one new arrival and no backlogged packet or

(b) no new arrival and one backlogged packet is transmitted

The probability of successful transmission in state j and per slot is

$$p_s(j) = u_1(j)b_0(j) + u_0(j)b_1(j)$$

The transition probability $P_{j,j+m}$ equals

$$P_{j,j+m} = \begin{cases} u_m(j) & 2 \leq m \leq N-j \\ u_1(j)(1-b_0(j)) & m=1 \\ u_1(j)b_0(j) + u_0(j)(1-b_1(j)) & m=0 \\ u_0(j)b_1(j) & m=-1 \end{cases}$$

Slotted Aloha

The transition probability $P_{j,j+m}$ equals

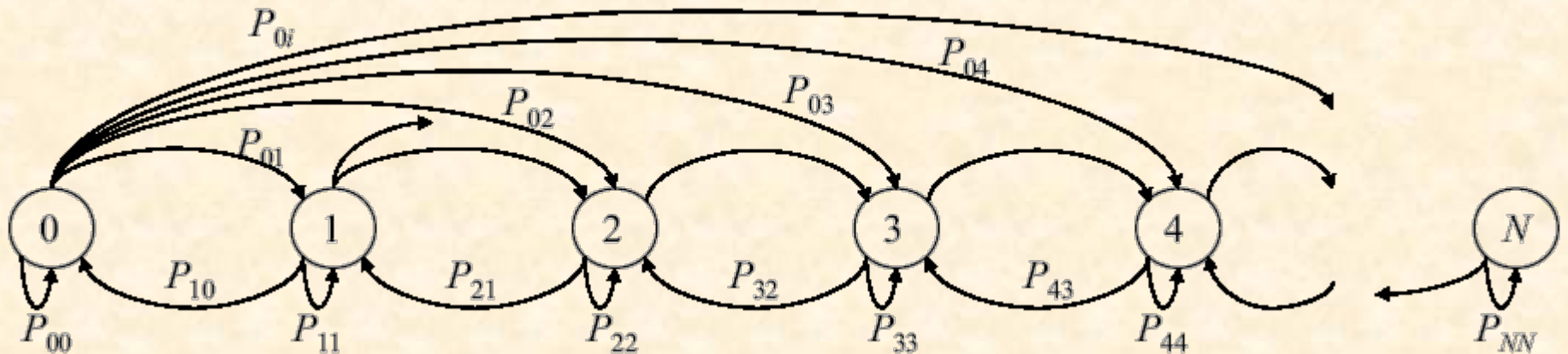
$$P_{j,j+m} = \begin{cases} u_m(j) & 2 \leq m \leq N - j \\ u_1(j) (1 - b_0(j)) & m = 1 \\ u_1(j) b_0(j) + u_0(j) (1 - b_1(j)) & m = 0 \\ u_0(j) b_1(j) & m = -1 \end{cases}$$

$$P = \begin{bmatrix} P_{00} & P_{01} & P_{02} & \cdots & \cdots & P_{0N} \\ P_{10} & P_{11} & P_{12} & \cdots & \cdots & P_{1N} \\ 0 & P_{21} & P_{22} & \cdots & \cdots & P_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & P_{N-1;N-2} & P_{N-1;N-1} & P_{N-1;N} \\ 0 & 0 & \cdots & 0 & P_{N;N-1} & P_{NN} \end{bmatrix}$$

Slotted Aloha

State j (j backlogged nodes) jumps to

- state $j-1$ if there are no new packets and one retransmission
- state j if
 - there is 1 new arrival and there are no retransmissions or
 - there are no new arrivals and none or more than 1 retransmissions
- state $j+1$ if there is 1 new arrival from a non-backlogged node and at least 1 retransmission (then there are surely collisions)
- state $j + m$ if m new packets arrive from m different non-backlogged nodes, which always causes collisions for $m > 1$



Slotted Aloha

For small N the steady-state equations can be solved

When N grows, slotted Aloha turns out to be instable

When $N \rightarrow \infty$, the steady-state vector π does not exist ($\lim_{N \rightarrow \infty} \pi = 0$)

The expected change in backlog per time slot is

$$E[X_{k+1} - X_k | X_k = j] = (N - j) p_a - p_s(j)$$

drift

=

expected number
of new arrivals

-

expected number
of successful transmissions

Since $p_s(j) \leq 1$ and $p_a = 1 - \exp(-\frac{\lambda}{N})$ $\lim_{N \rightarrow \infty} E[X_{k+1} - X_k | X_k = j] = \infty$

The drift tends to infinity, which means that, on average, the number of backlogged nodes increases unboundedly and suggests (but does not prove) that the Markov chain is transient for $N \rightarrow \infty$

Efficiency of slotted Aloha

For small arrival probability p_a and small retransmission probability p_r , the probability of successful transmission and of no transmission in state j is well approximated by

$$p_s(j) \simeq t(j) e^{-t(j)}$$
$$p_{no}(j) \simeq e^{-t(j)}$$

where $t(j) = (N - j)p_a + jp_r$ is the expected number of arrivals and retransmissions in state j (=total rate of transmission attempts in state j) and is also called the offered traffic G

for small p_a and p_r , the analysis shows that $p_s(j)$ and $p_{no}(j)$ are closely approximated in terms of a Poisson random variable with rate $t(j)$

p_s can be interpreted as the throughput $S_{SAloha} = G e^{-G}$, maximized if $G = 1$

The efficiency η_{SAloha} of slotted Aloha with $N \gg 1$ is defined as the maximum fraction of time during which packets are transmitted successfully which is $e^{-1} = 36\%$

Efficiency of pure Aloha

Pure Aloha: the nodes can start transmitting at arbitrary times

Performs half as efficiently as slotted Aloha with $\eta_{\text{PAloha}} = 18\%$

A transmitted packet at t is successful if no other is sent in $(t-1, t+1)$ which is equal to two timeslots and thus $\eta_{\text{PAloha}} = 1/2 \eta_{\text{SAloha}}$

In pure Aloha, $p_{no}(j) \simeq e^{-2t(j)}$ because in $(t-1, t+1)$ the expected number of arrivals and retransmissions is twice that in slotted Aloha

The throughput S roughly equals the total rate of transmission attempts G (which is the same as in slotted Aloha) multiplied by

$$p_{no}(j) \simeq e^{-2t(j)}, \text{ hence, } S_{\text{PAloha}} = Ge^{-2G}$$

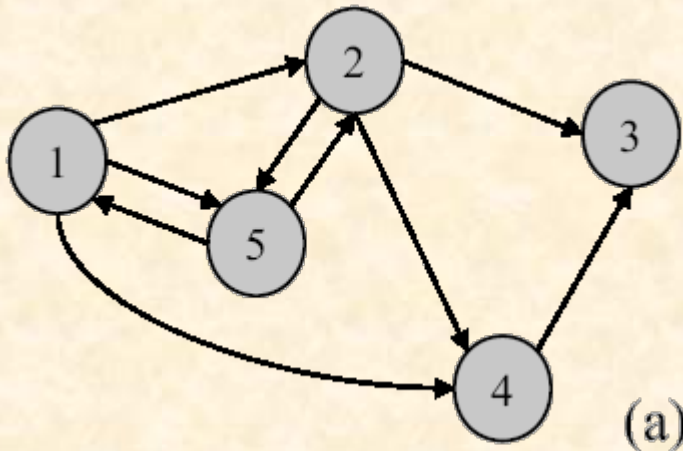
Ranking of webpages

Websearch engines apply a ranking criterion to sort the list of pages related to a query

PageRank (the hyperlink-based ranking system used by Google) exploits the power of discrete Markov theory

Markov model of the web: directed graph with N nodes

- Each node in the webgraph represents a webpage and
- the directed edges represent hyperlinks



$$P = \begin{bmatrix} 0 & P_{12} & 0 & P_{14} & P_{15} \\ 0 & 0 & P_{23} & P_{24} & P_{25} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & P_{43} & 0 & 0 \\ P_{51} & P_{52} & 0 & 0 & 0 \end{bmatrix}$$

Ranking of webpages

Assumption:

importance of a webpage \sim number of times that this page is visited

Consider a DTMC with transition probability matrix P that corresponds to the adjacency matrix of the webgraph

- P_{ij} is the probability of moving from webpage i (state i) to webpage j (state j) in one time step
- The component $s_i[k]$ of the state vector $s[k]$ denotes the probability that at time k the webpage i is visited

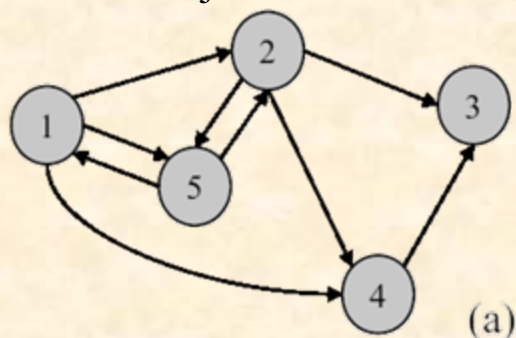
The long run mean fraction of time that webpage i is visited equals the steady-state probability π_i of the Markov chain

This probability π_i is the ranking measure of the importance of webpage i used in Google

Ranking of webpages

Uniformity assumption (if web usage information is not available):
 given we are on webpage i , any hyperlink on that webpage has equal probability to be clicked on

Thus, $P_{ij} = 1/d_i$ where the d_i is the number of hyperlinks on page i



$$P = \begin{bmatrix} 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \end{bmatrix} \quad \bar{P} = \begin{bmatrix} 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ 0 & 0 & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \end{bmatrix}$$

Problem: a node may not contain outlinks (dangling node)

\Rightarrow the corresponding row in P has only zero elements

Solution: each zero row is replaced by a non-zero row vector v^T

that obeys $\|v\|_1 = v^T u = 1$ where $u^T = [1 \ 1 \ \dots \ 1]$

simplest case: (uniformity) $v^T = \frac{u^T}{N}$

Ranking of webpages

The existence of a steady-state vector π must be ensured

If the Markov chain is irreducible, the steady-state vector exists

In an irreducible Markov chain any state is reachable from any other

By its very nature, the WWW leads almost surely to a reducible MC

Brin and Page have proposed

$$\bar{P} = \alpha \bar{P} + (1 - \alpha)uv^T$$

where $0 < \alpha < 1$ and v is a probability vector (each component of v is non-zero in order to guarantee reachability)

Brin and Page have called v^T the *personalization* vector

For $v^T = \left[\frac{1}{16} \quad \frac{4}{16} \quad \frac{6}{16} \quad \frac{4}{16} \quad \frac{1}{16} \right]$ and $\alpha = \frac{4}{5}$

$$\bar{P} = \begin{bmatrix} 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ 0 & 0 & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \end{bmatrix} \quad \bar{P} = \begin{bmatrix} \frac{1}{80} & \frac{19}{60} & \frac{3}{40} & \frac{19}{60} & \frac{67}{240} \\ \frac{1}{80} & \frac{1}{20} & \frac{41}{120} & \frac{19}{60} & \frac{67}{240} \\ \frac{1}{16} & \frac{1}{4} & \frac{3}{8} & \frac{1}{4} & \frac{1}{16} \\ \frac{1}{80} & \frac{1}{20} & \frac{7}{8} & \frac{1}{20} & \frac{1}{80} \\ \frac{33}{80} & \frac{9}{20} & \frac{3}{40} & \frac{1}{20} & \frac{1}{80} \end{bmatrix}$$

Computation of the PageRank steady-state vector

A more effective way to implement the described idea is to define a special vector r whose component $r_j = 1$ if row j in P is a zero-row or node j is dangling node

Then, $\bar{P} = P + rv^T$ is a rank-one update of P and so is $\bar{\bar{P}}$ because

$$\bar{\bar{P}} = \alpha (P + rv^T) + (1 - \alpha)u.v^T = \alpha P + (\alpha r + (1 - \alpha)u) v^T$$

Brin and Page propose to compute the steady-state vector from

$$\pi = \lim_{k \rightarrow \infty} s[k]$$

Specifically, for any starting vector $s[0]$ (usually $s[0] = \frac{u^T}{N}$), we iterate the equation $s[k+1] = s[k] P$ m -times and choose m sufficiently large such that $\|s[m] - \pi\| \leq \epsilon$ where ϵ is a prescribed tolerance

Computation of the PageRank steady-state vector

It holds $s[k+1] = s[k] \bar{P} = s[k] (\alpha P + (\alpha r + (1-\alpha)u) v^T)$

Since $s[k]u = 1$, we find

$$* s[k+1] = \alpha s[k]P + (\alpha s[k]r + (1-\alpha)) v^T$$

Thus, only the product of $s[k]$ with the (extremely) sparse matrix P needs to be computed and \bar{P} and \bar{P} are never formed nor stored

For any personalization vector, the second largest eigenvalue of \bar{P} is $\alpha \lambda_2$, where λ_2 is the second largest eigenvalue of \bar{P}

Brin and Page report that only 50 to 100 iterations of $*$ for $\alpha = 0.85$ are sufficient

A fast convergence is found for small α , but then the characteristics of the webgraph are suppressed

Langville and Meyer proposed to introduce one dummy node that is connected to all other nodes and to which all other nodes are connected to ensure overall reachability. Such approach changes the webgraph less