Homework 1. Construct a martingale which is NOT a Markov chain.

Homework 2. For every $i = 1, \ldots, m$ let $\left\{M_n^{(i)}\right\}_{n=1}^{\infty}$ be a sequence of martingales w.r.t. $\{X_n\}_{n=1}^{\infty}$. Show that

$$M_n := \max_{1 \le i \le n} M_n^{(i)}$$

is a submartingal w.r.t. $\{X_n\}$.

Homework 3. Let $F : \mathbb{R} \to [0,1]$ be a function which satisfies the conditions (1)-(3) of Theorem **??** of File "Some basic facts from probability theory". Let $\Omega := (0,1)$, $\mathcal{F}$ be the Borel σ-algebra on the interval $(0,1)$ and let $\mathbb{P}$ be the Lebesgue measure on the interval $(0,1)$. For an $\omega \in (0,1)$ let

$$X(\omega) := \sup \{y : F(y) < \omega\}. \tag{1}$$

(a) Show that $\{\omega : X(\omega) \le x\} = \{\omega : \omega \le F(x)\}$.

(b) Prove that F is the CDF for the r.v. X.

(c) Although F is not a bijection we can define $F^{-1} := X$. Let U be the uniform distribution on $(0,1)$. Prove that the CDF of $F^{-1}(U)$ is F.

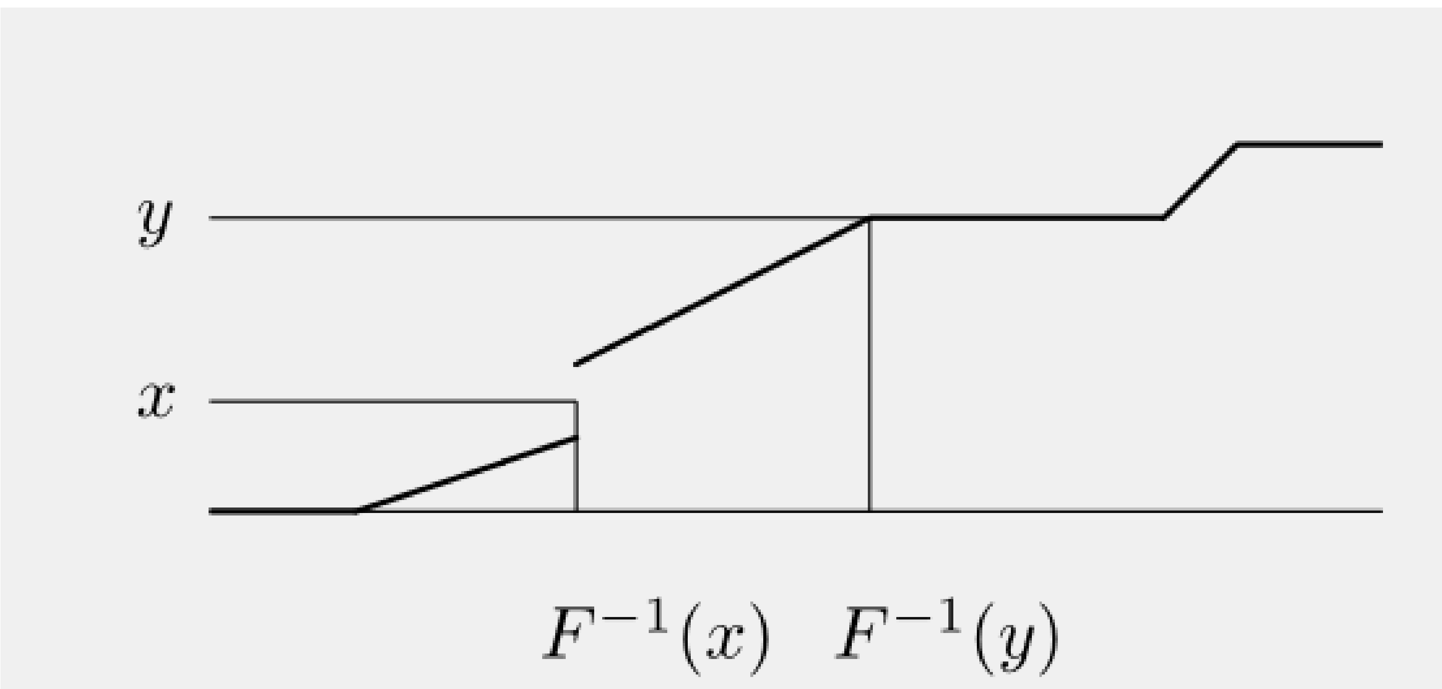

Figure 1: F^{-1} in Homework 3.The figure is from Durrett's book

Homework 4. Let (X, d) be a metric space and let $h : \mathbb{R}^+ \to \mathbb{R}^+$ be a function satisfying:

1. $h(0) = 0$
2. h is differentiable
3. $h'(x)$ decreasing on $[0, \infty)$,
4. $h'(x) > 0$ for all $x > 0$

Prove that $h(d(x, y))$ is a metric.

Homework 5. Let d be the following distance between random variables X, Y defined on the same probability space:

$$d(X,Y) := \mathbb{E}\left[\frac{|X-Y|}{1+|X-Y|}\right]$$

Prove that: $d(X_n, X) \to 0$ iff $X_n \xrightarrow{P} X$.

The following 5 homeworks are due at 27 September 2014.

Homework 6. Let $X, Y \in L^1(\Omega\mathcal{F}, \mathbb{P})$ satisfying

$$\mathbb{E}[X|Y] = Y \text{ and } \mathbb{E}[Y|X] = X$$

Show that $\mathbb{P}(X = Y) = 1$.

Homework 7. Let $\xi_1, \xi_2, \dots$ standard normal variables. (Recall that in this case the moment generating function $M(\theta) = \mathbb{E}\left[e^{\theta\xi_i}\right] e^{\theta^2/2}$.) Let $a, b \in \mathbb{R}$ and

$$S_n := \sum_{k=1}^{n} \xi_k \text{ and } X_n := e^{aS_n - bn}$$

Prove that

(a) $X_n \to 0$ a.s. iff $b > 0$

(b) $X_n \to 0$ in L^r iff $r < \frac{2b}{a^2}$.

Homework 8. Consider the simple symmetric random walk $\{S_n\}_{n=1}^{\infty}$ on $\mathbb{Z}^2$. That is $S_n = X_1 + \cdots + X_n$, where $X_1, X_2, \dots$ are $\mathbb{Z}^2$-valued iid r.v. with

$$\mathbb{P}(X_1 = (i,j)) = \begin{cases} \frac{1}{4}, & \text{if } (i,j) \in \{(0,1),(0,-1),(1,0),(-1,0)\}; \\ 0, & \text{otherwise.} \end{cases}$$

Let $D_n := \|S_n\|$ and we write $\nu_r := \inf\{n : S_n > r\}$. Prove that

(a) $D_n^2 - n$ is a martingale,

(b) $\lim_{r\to\infty} r^{-2}\mathbb{E}[\nu_r] = 1$

Homework 9. The same as Homework 8 with the difference that now X_1 is a random unit vector whose angle is chosen uniformly from $[0, 2\pi]$.

Homework 10. Let $S_n := X_1 + \cdots + X_n$, where $X_1, X_2, \dots$ are iid with $X_1 \sim \text{Exp}(1)$. Verify that

$$M_n := \frac{n!}{(1+S_n)^{n+1}} \cdot e^{S_n}$$

is a martingale w.r.t. the natural filtration $\mathcal{F}_n$.

The following 5 homeworks are due at 10 October 2014.

Homework 11. Let $Y = \mathbb{P}(C|\mathcal{F})$. Then for every $B \in \mathcal{F}$, $\mathbb{P}(B) \neq 0$ we have

$$\frac{\mathbb{E}[Y; B]}{\mathbb{P}(B)} = \mathbb{P}(C|B).$$

Using this, complete the proof of File A Theorem 6.9.

Homework 12. Prove the general version of Bayes's formula: Given the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$. Let $G \in \mathcal{G}$. Show that

$$\mathbb{P}(G|A) = \frac{\int_G \mathbb{P}(A|\mathcal{G})\, d\mathbb{P}}{\int_\Omega \mathbb{P}(A|\mathcal{G})} d\mathbb{P} \tag{2}$$

Homework 13. Prove the conditional variance formula

$$\mathrm{Var}(X) = \mathbb{E}\left[\mathrm{Var}(X|Y)\right] + \mathrm{Var}\left(\mathbb{E}\left[\mathrm{X|Y}\right]\right) \tag{3}$$

Homework 14. Let $X_1, X_2, \dots$ iid r.v. and N is a non-negative integer valued r.v. that is independent of X_i, $i \geq 1$. Prove that

$$\mathrm{Var}\left(\sum_{i=1}^{N} X_i\right) = \mathbb{E}\left[N\right]\mathrm{Var}(X) + (\mathbb{E}\left[X\right])^2 \mathrm{Var}(N). \tag{4}$$

Homework 15. Let X, Y be two independent $\mathrm{Exp}(\lambda)$ r.v. and $Z := X+Y$. Show that for any non-negative measurable h we have $\mathbb{E}\left[h(X)|Z\right] = \frac{1}{Z}\int_0^Z h(t)dt$.

Homework 16. Let $X_1, X_2, \dots$ be iid r.v. with $\mathbb{P}(X_n = -n^2) = \frac{1}{n^2}$ and $\mathbb{P}\left(X_n = \frac{n^2}{n^2-1}\right) = 1 - \frac{1}{n^2}$. Let $S_n := X_1 + \cdots + X_n$. Show that

(a) $\lim\limits_{n\to\infty} S_n/n = \infty$.

(b) $\{S_n\}$ is a martingale which converges to ∞ a.s..

The following 4 homeworks are due at 17 October 2014.

Homework 17. Let X, Z be r.v. defined on the $(\Omega, \mathcal{F}, \mathbb{P})$. Assume that $\mathbb{E}\left[\mathrm{e}^{i\mathbf{t}\cdot\mathbf{X}+i\mathbf{s}\cdot\mathbf{Z}}\right] = \mathbb{E}\left[\mathrm{e}^{i\mathbf{t}\mathbf{X}}\right] \cdot \mathbb{E}\left[\mathrm{e}^{i\mathbf{s}\mathbf{Z}}\right], \quad \forall \mathbf{s}, \mathbf{t} \in \mathbb{R}^n$. Prove that X, Z are independent.

Homework 18. Prove that the following two definitions of λ-system $\mathcal{L}$ are equivalent:

Definition 1. **(a)** $\Omega \in \mathcal{L}$.

(b) If $A, B \in \mathcal{L}$ and $A \subset B$ then $B \setminus A \in \mathcal{L}$

(c) If $A_n \in \mathcal{L}$ and $A_n \uparrow A$ (that is $A_n \subset A_{n+1}$ and $A = \cup_{n=1}^{\infty} A_n$) then $A \in \mathcal{L}$.

Definition 2. **(i)** $\Omega \in \mathcal{L}$.

(ii) If $A \in \mathcal{L}$ then $A^c \in \mathcal{L}$.

(iii) If $A_i \cap A_j = \emptyset$, $A_i \in \mathcal{L}$ then $\cup_{i=1}^{\infty} A_i \in \mathcal{L}$.

Homework 19. There are n white and n black balls in an urn. We pull out all of them one-by-one without replacement. Whenever we pull:

- a black ball we have to pay 1\$,
- a white ball we receive 1\$.

Let $X_0 := 0$ and X_i be the amount we gained or lost after the i-th ball was pulled. We define

$$Y_i := \frac{X_i}{2n-i}, \text{ for } 1 \leq i \leq 2n-1, \quad \text{and } Z_i := \frac{X_i^2 - (2n-i)}{(2n-i)(2n-i-1)} \text{ for } 1 \leq i \leq 2n-2.$$

(a) Prove that $Y = (Y_i)$ and $Z = (Z_i)$ are martingales.

(b) Find $\mathrm{Var}(X_i) = ?$

Homework 20 (Extension of part (iii) of Doob's optional stopping Theorem File A 6.6)**.** Let X be a supermartingale. Let T be a stopping time with $\mathbb{E}[T] < \infty$.(Like in part (iii) of Doob's optional stopping Theorem.) Assume that there is a C such that

$$\mathbb{E}[|X_k - X_{k-1}||\mathcal{F}_{k-1}](\omega) \leq C, \quad \forall k > 0 \text{ and for a.e. } \omega.$$

Prove that $\mathbb{E}[X_T] \leq \mathbb{E}[X_0]$.

The following 5 homeworks are due at 29 October 2014.

Homework 21. Let $\{\varepsilon_n\}$ be an iid sequence of real numbers satisfying $\mathbb{P}(\varepsilon_n = \pm 1) = \frac{1}{2}$. Show that $\sum_n \varepsilon_n a_n$ converges alsmost surely iff $\sum_{k=1}^{\infty} a_n^2 < \infty$.

Homework 22. Let $X = (X_n)$ be an L^2 random walk that is a martingale. Let σ^2 be the variance of the k-th increment $Z_k := X_k - X_{k-1}$ for all k. Prove that the quadratic variance is $A_n = n\sigma^2$.

Homework 23. Prove the assertion of Remark **??**.

Homework 24. Let $M = (M_n)$ be a martingale with $M_0 = 0$ and $|M_k - M_{k-1}| < C$ for a $C \in \mathbb{R}$. Let $T \geq 0$ be a stopping time and we assume that $\mathbb{E}[T] \leq \infty$. Let

$U_n := \sum_{k=1}^{n} (M_k - M_{k-1})^2 \cdot \mathbb{1}_{T \geq k}, \quad V_n := 2 \sum_{1 \leq i < j \leq n} (M_i - M_{i-1}) \cdot (M_j - M_{j-1}) \cdot \mathbb{1}_{T \geq j}.$

$U_\infty := \sum_{k=1}^{\infty} (M_k - M_{k-1})^2 \cdot \mathbb{1}_{T \geq k}, \quad V_\infty := 2 \sum_{1 \leq i < j} (M_i - M_{i-1}) \cdot (M_j - M_{j-1}) \cdot \mathbb{1}_{T \geq j}.$ Prove that

(a) $M_{T \wedge n}^2 = U_n + V_n$ and $M_T^2 = U_\infty + V_\infty$.

(b) Further, if $\mathbb{E}[T^2] < \infty$ then $\lim_{n \to \infty} U_n = U_\infty$ a.s. and $\mathbb{E}[U_\infty] < \infty$ and $\mathbb{E}[V_n] = \mathbb{E}[V_\infty] = 0$.

(c) Conclude that $\lim_{n \to \infty} \mathbb{E}[M_{T \wedge n}^2] = \mathbb{E}[M_T^2]$.

Homework 25 (Wald inequalities)**.** Let $Y_1, Y_2, \dots$ be iid r.v. with $Y_i \in L^1$. Let $S_n := Y_1 + \cdots + Y_n$ and we write $\mu := \mathbb{E}[Y_i]$. Given a stopping time $T \geq 1$ satisfying: $\mathbb{E}[T] < \infty$. Prove that

(a)

$$\mathbb{E}[S_T] = \mu \cdot \mathbb{E}[T]. \tag{5}$$

(b) Further, assume that Y_i are bounded ($\exists C_i \in \mathbb{R}$ s.t. $|Y_i| < C_i$) and $\mathbb{E}[T^2] < \infty$. We write $\sigma^2 := \mathrm{Var}(Y_i)$. Then

$$\mathbb{E}\left[(S_T - \mu T)^2\right] = \sigma^2 \cdot \mathbb{E}[T]. \tag{6}$$

Hint: Introduce an appropriate martingale and apply the result of the previous exercise.

The following homeworks are due at 5 November 2014.

Homework 26 (Branching Processes)**.** You might want to recall what you have learned about Bransching Processes. (See was Section **??** in File C of the course "Stochastic Processes".) A Branching Process $Z = (Z_n)_{n=0}^{\infty}$ is defined recursively by a given family of $\mathbb{Z}^+$ valued iid rv. $\left\{X_k^{(n)}\right\}_{k,n=1}^{\infty}$ as follows:

$$Z_0 := 1, \quad Z_{n+1} := X_1^{(n+1)} + \cdots + X_{Z_n}^{(n+1)}, \quad n \geq 0.$$

Let $\mu = \mathbb{E}\left[X_k^{(n)}\right]$ and $\mathcal{F}_n = \sigma(Z_0, Z_1, \dots Z_n)$. We write $f(s)$ for the generating function. that is

$$f(s) = \sum_{\ell=0}^{\infty} \underbrace{\mathbb{P}\left(X_k^{(n)} = \ell\right)}_{p_\ell} \cdot s^\ell \text{ for any } k, n.$$

Further, let

$$\{extinction\} := \{Z_n \to 0\} = \{\exists n, Z_n = 0\} \quad \{explosion\} = \{Z_n \to \infty\}.$$

let $q := \mathbb{P}([extinction]) :=$. Recall that we learned that q is the smaller (if there are two) fixed point of $f(s)$. That is q is the smallest solution of $f(q) = q$. Prove that

(a) $\mathbb{E}[Z_n] = \mu^n$. Hint: Use induction.

(b) For every $s \geq 0$ we have $\mathbb{E}\left[s^{Z_{n+1}} | \mathcal{F}_n\right] = f(s)^{Z_n}$. Explain why it is true that q^{Z_n} is a martingale and $\lim_{n\to\infty} Z_n = Z_\infty$ exists a.s.

(c) Let $T := \min\{n : Z_n = 0\}$. ($T = \infty$ if $Z_n > 0$ always.)

(d) Prove that $q = \mathbb{E}\left[q^{Z_T}\right] = \mathbb{E}\left[q^{Z_\infty} \cdot \mathbb{1}_{T=\infty}\right] + \mathbb{E}\left[q^{Z_T} \cdot \mathbb{1}_{T<\infty}\right]$.

(e) Prove that $\mathbb{E}\left[q^{Z_\infty} \cdot \mathbb{1}_{T=\infty}\right] = 0$.

(f) Conclude that if $T(\omega) = \infty$ then $Z_\infty = \infty$.

(g) Prove that

$$\mathbb{P}(\text{extinction}) + \mathbb{P}(\text{explosion}) = 1. \tag{7}$$

Homework 27 (Branching Processes cont.)**.** Here we assume that

$$\mu = \mathbb{E}\left[X_k^{(n)}\right] < \infty \text{ and } 0 < \sigma^2 := \mathrm{Var}(X_k^{(n)}) < \infty.$$

Prove that

(a) $M_n = Z_n/\mu^n$ is a martingale for the natural filtration $\mathcal{F}_n$

(b) $\mathbb{E}\left[Z_{n+1}^2 | \mathcal{F}_n\right] = \mu^2 Z_n + \sigma^2 Z_n$. Conclude that

$$M \text{ is bounded in } L^2 \iff \mu > 1.$$

(c) If $\mu > 1$ then $M_\infty := \lim_{n\to\infty} M_n$ exists (in L^2 and a.s.) and

$$\mathrm{Var}(M_\infty) = \frac{\sigma^2}{\mu(m-1)}.$$

Homework 28 (Branching Processes cont.)**.** Assume that $q = 1$. Prove that $M_n = Z_n = \mu^n$ is NOT a UI martingale.

Homework 29. Let $X_1, X_2, \ldots$ be iid rv. with **continuous** distribution distribution function. Let E_i be the event that a record occurs at time n. That is $E_1 := \Omega$ and $E_n := \{X_n > X_m, \forall m < n\}$. Prove that $\{E_i\}_{i=1}^\infty$ independent and $\mathbb{P}((E_i)) = \frac{1}{i}$.

Homework 30 (**Continuation**)**.** Let $E_1, E_2, \ldots$ be independent with $\mathbb{P}(E_i) = 1/i$. Let $Y_i := \mathbb{1}_{E_i}$ and $N_n := Y_1 + \cdots + Y_n$. (In the special case of the previous homework, N_n is the number of records until time n.) Prove that

(a) $\sum_{k=1}^{\infty} \frac{Y_k - 1/k}{\log k}$ converges almost surely.

(b) Using Krocker's Lemma conclude that $\lim_{n\to\infty} \frac{N_n}{\log n} = 1$ a.s..

(c) Apply this to the situation of the previous exercise to get an estimate on the number of records until time n.

Homework 31. Let $\mathcal{C}$ be a class of rv on $(\Omega, \mathcal{F}, \mathbb{P})$. Prove that the following assertions (1) and (2) are equivalent:

1. $\mathcal{C}$ is UI.
2. Both of the following two conditions hold:

   (a) $\mathcal{C}$ is L^1-bounded. That is $A := \sup\{\mathbb{E}[|X|] : X \in \mathcal{C}\} < \infty$ AND

   (b) $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.
$$F \in \mathcal{F} \text{ and } \mathbb{P}(F) < \delta \Longrightarrow \mathbb{E}[|X|; F] < \varepsilon.$$

Homework 32. Let $\mathcal{C}$ and $\mathcal{D}$ be UI classes of rv.. Prove that $\mathcal{C} + \mathcal{D} := \{X + Y : X \in \mathcal{C} \text{ and } Y \in \mathcal{D}\}$ is also UI. Hint: use the previous exercise.

Homework 33. Given a $\mathcal{C}$ a UI family of rv.. Let us define $\mathcal{D} := \{Y : \exists X \in \mathcal{C},\ \exists \mathcal{G} \text{ sub-}\sigma\text{-algebra of } \mathcal{F} \text{ s.t. } Y = \mathbb{E}[X|\mathcal{G}]\}$. Prove that $\mathcal{D}$ is also UI.

Homework 34. Given a sequence of rv. X_n on $(\Omega, \mathcal{F}, \mathbb{P})$ s.t.

(i) $X := \lim\limits_{n\to\infty} X_n$ exists a.s..

(ii) X_n is dominated by a rv. $Y \in L^1$. That is $|X_n(\omega)| \leq Y(\omega)$ for all $n \in \mathbb{N}$ and $\omega \in \Omega$.

Further, given an **arbitrary** filtration $\{\mathcal{F}_n\}_{n=0}^{\infty}$. Prove that

(a) For $Z_m := \sup\limits_{r \geq m} |X_r - X|$ we have $Z_n \to 0$ both almost surely and in L^1.

(b) Prove that for $n \geq m$ almost surely:
$$|\mathbb{E}[X_n|\mathcal{F}_n] - \mathbb{E}[X|\mathcal{F}_\infty]| \leq |\mathbb{E}[X|\mathcal{F}_n] - \mathbb{E}[X|\mathcal{F}_\infty]| + \mathbb{E}[Z_m|\mathcal{F}_n].$$

(c) Conclude that $\lim\limits_{n\to\infty} \mathbb{E}[X_n|\mathcal{F}_n] = \mathbb{E}[X|\mathcal{F}_\infty]$.

The following homeworks are due at 3 December 2014.

Homework 35. Let $X_1, X_2, \dots$ be iid. rv. with $\mathbb{E}[X^+] = \infty$ and $\mathbb{E}[X^-] < \infty$. (Recall $X = X^+ - X^-$ and $X^+, X^- \geq 0$.) Use SLLN to prove that $S_n/n \to \infty$ a.s., where $S_n := X_1 + \cdots + X_n$. Hint: For an $M > 0$ let $X_i^M := X_i \wedge M$ and $S_n^M := X_n^M + \cdots + X_n^M$. Explain why $\lim\limits_{n\to\infty} S_n^M/n \to \mathbb{E}\left[X_i^M\right]$ and $\liminf\limits_{n\to\infty} S_n/n \geq \lim\limits_{n\to\infty} S_n^M/n$.

Homework 36. Let $X_1, X_2, \dots$ be iid rv with $\mathbb{E}[|X_i|] < \infty$. Prove that $\mathbb{E}[X_1|S_n] = S_n/n$. (This is trivial intuitively from symmetry, but prove it with formulas.)

Homework 37. Williams book exercise E4.9 which is on the next page. In its hint $Y_0 \in \mathrm{m}\mathcal{L}$ means $Y_0 \in \mathcal{L}$

E4.9. Let $Y_0, Y_1, Y_2, \ldots$ be independent random variables with

$$\mathsf{P}(Y_n = +1) = \mathsf{P}(Y_n = -1) = \tfrac{1}{2}, \quad \forall n.$$

For $n \in \mathbb{N}$, define

$$X_n := Y_0 Y_1 \ldots Y_n.$$

Prove that the variables $X_1, X_2, \ldots$ are independent. Define

$$\mathcal{Y} := \sigma(Y_1, Y_2, \ldots), \quad \mathcal{T}_n := \sigma(X_r : r > n).$$

Prove that

$$\mathcal{L} := \bigcap_n \sigma(\mathcal{Y}, \mathcal{T}_n) \neq \sigma\left(\mathcal{Y}, \bigcap_n \mathcal{T}_n\right) =: \mathcal{R}.$$

Hint. Prove that $Y_0 \in \mathrm{m}\mathcal{L}$ and that Y_0 is independent of $\mathcal{R}$.

E4.10. *Star Trek*, 2

See E10.11, which you can do now.

Homework 38. Williams book exercise E4.9.

E14.2. Azuma-Hoeffding Inequality

(a) Show that if Y is a RV with values in $[-c, c]$ and with $\mathsf{E}(Y) = 0$, then, for $\theta \in \mathbb{R}$,

$$\mathsf{E}e^{\theta Y} \leq \cosh \theta c \leq \exp\left(\frac{1}{2}\theta^2 c^2\right).$$

(b) Prove that if M is a martingale null at 0 such that for some sequence $(c_n : n \in \mathbb{N})$ of positive constants,

$$|M_n - M_{n-1}| \leq c_n, \quad \forall n,$$

then, for $x > 0$,

$$\mathsf{P}\left(\sup_{k \leq n} M_k \geq x\right) \leq \exp\left(\frac{1}{2}x^2 \Big/ \sum_{k=1}^{n} c_k^2\right).$$

Hint for (a). Let $f(z) := \exp(\theta z)$, $z \in [-c, c]$. Then, since f is convex,

$$f(y) \leq \frac{c - y}{2c} f(-c) + \frac{c + y}{2c} f(c).$$

Hint for (b). See the proof of (14.7,a).

The last comments referes to the proof of LIL.

The following homeworks are due at 12 December 2014.

Homework 39. Prove that a measure μ is ergodic if for every $f \in L^1$ the fact that f is constant on μ-a.a. orbits $\{T^n(x)\}_{n=0}^{\infty}$ is equivalent to the fact that f is constant for μ-a.a. x.

Homework 40. Let $T : [0,1] \to [0,1]$, $T(x) = x^3$. Find all the invariant measures for T.

Homework 41. Prove that there are no invariant measures for the map Let $T : [0,1] \to [0,1]$,

$$T(x) := \begin{cases} \frac{x}{2}, & \text{if } 0 < x \leq 1; \\ 1, & \text{if } x = 0. \end{cases}$$

Homework 42. Let $T : [0,1] \to [0,1]$, $T(x) = \begin{cases} 2x, & \text{if } x \in \left[0, \frac{1}{2}\right]; \\ 2 - 2x, & \text{if } x \in \left[\frac{1}{2}, 1\right]. \end{cases}$. Construct infinitely many invariant measures for T.

Homework 43. Assume that $X := \{X_n\}_{n=0}^{\infty}$ is stationary, then X can be extended to a stationary process $\widetilde{X} := \{X_n\}_{n=-\infty}^{\infty}$.

Homework 44. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a measurable map $T : \Omega \to \Omega$. Let $\mathcal{I} := \{A \in \mathcal{F} : T^{-1}A = A\}$ Prove that

(a) A measurable map $(\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{R})$ is $\mathcal{I}$ measurable iff $f \circ T(x) = f(x)$

(b) $(\Omega, \mathcal{F}, \mathbb{P}, T)$ is ergodic iff for any $\mathcal{I}$ measurable real valued function f is almost surely constant.

Homework 45. Given an iid process $\{\xi_n\}$. Prove that the canonical dynamical system (for the definition see **??**) $(\Omega, \mathcal{F}, \mathbb{P}, \sigma)$ is ergodic.