

Homework 6 (Markov Chains)

For matrix multiplications, you'd better use a matrix based programming software such as Matlab or Octave. The problems marked with (*) are optional, however solving them will add to your grade.

1. Consider the Markov chain given in figure 1.

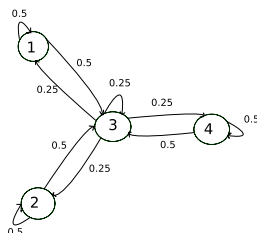


Figure 1: Problem 1

- (a) If the initial probability distribution is $\pi = [\frac{1}{2}, \frac{1}{4}, \frac{1}{12}, \frac{1}{6}]$, what is the probability distribution after two steps ($\pi^{(2)}$) ?
- (b) Find the steady state probability distribution of this process (i.e. π^* such that $\pi^* \mathbf{P} = \pi^*$).

2. Consider a Markov chain with transition matrix $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ and initial distribution (at time 0) $\pi = [\frac{1}{3} \quad \frac{1}{3} \quad \frac{1}{3}]$. For each n , $Y_n = 0$ if $X_n = 1$ and otherwise $Y_n = 1$. Is the process (Y_0, Y_1, \dots) a Markov chain?

3. Consider the Markov chain given in figure 2:

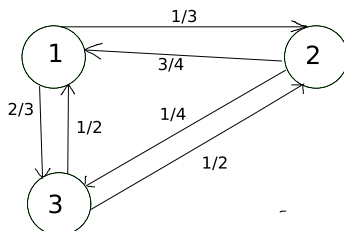


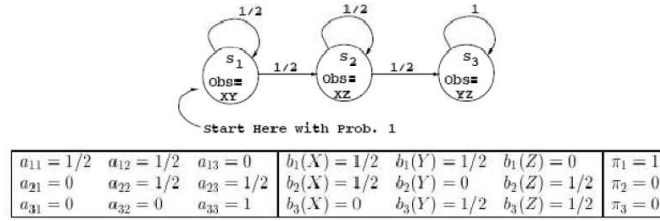
Figure 2: Problem 3

Let \mathbf{P} be the transition matrix of this chain.

- (a) $X[n]$ is a process generated from this chain, as follows: It starts from a state. In odd times, it moves one step according to \mathbf{P} , and in even times it moves two steps according to \mathbf{P} . Model $X[n]$ with a Markov chain.

- (b) $Y[n]$ is a process generated from this chain, as follows: It starts from a state. In each time, with probability of 0.5 it moves one step according to \mathbf{P} , and with probability of 0.5 it moves two steps according to \mathbf{P} . Model $Y[n]$ with a Markov chain.

4. Consider the following HMM:



Where

$$a_{ij} = P(q_{t+1} = S_j | q_t = S_i)$$

$$b_i(k) = P(O_t = k | q_t = S_i)$$

Suppose we have observed this sequence: XZXYYZYZZ (In long-hand: $O_1 = X, O_2 = Z, O_3 = X, O_4 = Y, O_5 = Y, O_6 = Z, O_7 = Y, O_8 = Z, O_9 = Z$).

- Find the most probable sequence of hidden states given the observations of the HMM.
- Compute the values of $\alpha_i(t)$ for $1 \leq i \leq 3, 1 \leq t \leq 9$, remembering the definition:

$$\alpha_i(t) = P(O_1, O_2, \dots, O_t, q_t = S_i)$$

Warning: this is a question that will take a few minutes if you really understand HMMs, but could take hours if you don't!

5. Consider the Markov chain in figure 3:

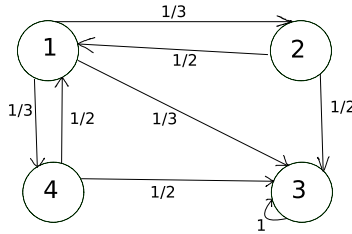


Figure 3: Problem 5

- Partition the states into equivalence classes.
- Classify each state as recurrent, positive recurrent or none.
- Let \mathbf{P} be the transition matrix of the Markov chain. Find $\lim_{n \rightarrow \infty} \mathbf{P}^n$.
- (*) Find expected number of steps the process will take before absorption if the process:
 - starts from state 1.
 - starts from state 2.
 - starts from state 3.

- iv. starts with the initial probability distribution $\pi = [\frac{1}{2}, \frac{1}{4}, \frac{1}{12}, \frac{1}{6}]$.
6. Let \mathbf{c} be a column vector with all elements equal to 1.
- (a) If \mathbf{P} is the transition matrix of a Markov chain, prove that \mathbf{c} is a fixed vector of \mathbf{P} .
- (b) (*) If \mathbf{P} is the transition matrix of an ergodic Markov chain and \mathbf{x} is a column vector, show that if \mathbf{x} is a fixed vector of \mathbf{P} then \mathbf{x} is a constant factor of \mathbf{c} , i.e. there exists a real number r such that $\mathbf{x} = r\mathbf{c}$.
7. (*) Some random walker will move on the states of the Markov chain given in figure 4, according to its transition probabilities, until he arrives an absorbing state.

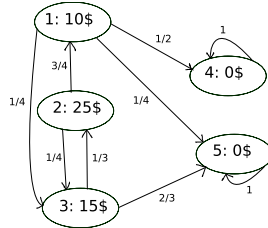


Figure 4: Problem 7

Each state has an award written on it. Each time he visits a state, he will gain its award (if he visits a state for n times, he will gain its award n times). He starts with initial property distribution $\pi = [\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}, 0]$.

- (a) What will be his expected total gain?
- (b) What will be his expected total gain, if he should pay 1\$ for each step?
- (c) What will be his expected total gain, if the awards are as following (he should pay nothing for each step)?
step 1: 0\$, step 2: 0\$, step 3: 0\$, step 4: 10\$, step 5: 25\$?
8. (*) Consider a Markov chain with following transition matrix:

$$P = \begin{pmatrix} 0 & 0.5 & 0 & 0.5 & 0 \\ 0.7 & 0 & 0.1 & 0 & 0.2 \\ 0 & 0.2 & 0 & 0.8 & 0 \\ 0.3 & 0 & 0.4 & 0 & 0.3 \\ 0 & 0.7 & 0 & 0.3 & 0 \end{pmatrix}$$

- (a) Prove that it's not a regular Markov chain.
- (b) Offer a vector as the initial probability distribution, which starting from it, the probability distribution vector will not converge to any fixed vector (prove your claim).
9. (*) Some random walker will move on the states of the Markov chain given in figure 5, according to its transition probabilities. Each state has an award written on it. Each time he visits a state, he will gain its award (if he visits a state for n times, he will gain its award n times). He starts with initial property distribution $\pi = [\frac{1}{3}, \frac{1}{3}, \frac{2}{6}, \frac{2}{6}]$. If he moves for a long time, what will be his average gain in each step?

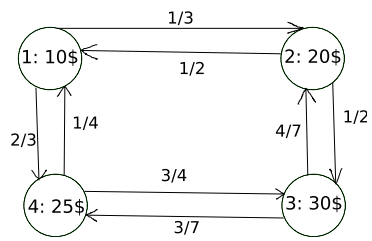


Figure 5: Random walker