Exercises on Markov chains

13. Recall the notation for a Markov chain X_n with state space S,

$$p_{ij}^{(n)} := \mathbb{P}(X_n = j \mid X_0 = i) = (P^n)_{ij} \qquad \text{for } i, j \in S.$$

Justifying your steps, simplify

$$\mathbb{P}(X_5 = j \mid X_2 = i), \quad \text{and} \quad \mathbb{P}(X_2 = j,\, X_6 = k \mid X_0 = i), \text{ for } i, j, k \in S.$$

14. A Markov chain X_n with state space $S = \{1, 2, 3\}$ and initial value $X_0 = 1$ makes two types of transitions: at even times n, the chain jumps with transition matrix P while at odd times n, the chain jumps with transition matrix P', where

$$P = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{3} & \frac{2}{3} \end{pmatrix}, \quad P' = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}.$$

This means, for example, that $\mathbb{P}(X_3 = 2 \mid X_2 = 1) = \frac{1}{2}$ but $\mathbb{P}(X_4 = 2 \mid X_3 = 1) = \frac{1}{3}$. (This is called a *time-inhomoheneous Markov chain.*) Then the subsequence $X_0, X_2, X_4, \ldots$ is a Markov chain (why?): what is its transition matrix?

15. Suppose that the probability of rain each day is $0.4 + 0.2x$, where $x \in \{0, 1, 2\}$ is the total number of rainy days over the previous two days. Represent the weather as a 4-state Markov chain, writing down the meaning of each state and the one-step transition matrix. Suppose that it rained yesterday but not the day before. What is the probability that it will rain tomorrow?

16. *The Ehrenfest diffusion model.* At time 0, an urn contains a mixture of red and black balls, with N balls in total. At each time $1, 2, \ldots$ a ball is picked at random from the urn and replaced by a ball of the other colour. The total number of balls in the urn therefore remains constant at N. Let the state X_n of the system be the *number of black balls* in the urn at time $n = 0, 1, 2, \ldots$.

    (i) Write down the state space and the one-step transition probabilities for the Markov chain X_n.

    (ii) Suppose that the initial distribution for the initial number of black balls X_0 is binomial $(N, 1/2)$. What is the distribution of X_1, the number of black balls after one step? How about X_{100}?

17. Let X_n be a Markov chain on $\{1, 2, 3\}$ with transition matrix

$$P = \begin{pmatrix} 1 & 0 & 0 \\ \alpha & \beta & \gamma \\ 0 & 0 & 1 \end{pmatrix}.$$

Suppose that $X_0 = 2$ and let $T = \min\{n : X_n \neq 2\}$. Find $\mathbb{P}(X_T = 1)$ and $\mathbb{P}(X_T = 3)$ and deduce $\lim_{n\to\infty} p_{2j}^{(n)}$ for $j = 1, 2, 3$.

18. A board has 9 squares arranged 3 by 3. A counter is moved on these in a Markov chain, the permissible moves being to a neighbouring square (horizontally, vertically or diagonally) with equal probabilities for each neighbour. You can exploit the symmetry of this situation to reduce the problem to a 3-state Markov chain by considering the 3 classes of square Centre, Corner or Other. Find the transition probabilities for the simplified 3-state chain. Find the equilibrium distribution for the 3-state chain, and hence deduce the limiting probabilities for the 9-state chain.

19. *Cat and mouse Markov chain.* A cat and a mouse move randomly among six locations arranged and connected as a hexagon, so each location has two neighbours, and each move is equally likely to take the animal to either neighbour. Label the locations $1, 2, 3, 4, 5, 6$ in clockwise order. The locations of the cat and mouse are C_n and M_n respectively, starting with $C_0 = 1$ and $M_0 = 3$. The cat and mouse move independently until they are in the same place (i.e., $C_n = M_n$), at which point the cat catches the mouse. How long on average does it take until the cat catches the mouse?

Hint: Set up a single Markov chain that contains enough information to answer the problem. Making full use of symmetry, it suffices to work with a chain with only 2 states. Then use a one-step analysis, conditioning on the first step.