**Practice Problems for Homework #8. Markov Chains.**

- Read Sections 7.1-7.3
- Solve the practice problems below.
- Open Homework Assignment #8 and solve the problems.

1. **(10 marks)** A computer system can operate in two different modes. Every hour, it remains in the same mode or switches to a different mode according to the transition probability matrix

$$P = \begin{bmatrix} 0.4 & 0.6 \\ 0.6 & 0.4 \end{bmatrix}$$

   **a)** Compute the 2-step transition probability matrix.

   **b)** If the system is in Mode I at 5:30 pm, what is the probability that it will be in Mode I at 8:30 pm on the same day?

2. **(10 marks)** The pattern of sunny and rainy days on the planet Rainbow is a homogeneous Markov chain with two states. Every sunny day is followed by another sunny day with probability 0.8. Every rainy day is followed by another rainy day with probability 0.6.

   **a)** Today is sunny on Rainbow. What is the chance of rain the day after tomorrow?

   **b)** Compute the probability that April 1 next year is rainy on Rainbow.

3. **(10 marks)** A computer device can be either in a busy mode (state 1) processing a task, or in an idle mode (state 2), when there are no tasks to process. Being in a busy mode, it can finish a task and enter an idle mode any minute with the probability 0.2. Thus, with the probability 0.8 it stays another minute in a busy mode. Being in an idle mode, it receives a new task any minute with the probability 0.1 and enters a busy mode. Thus, it stays another minute in an idle mode with the probability 0.9. The initial state is idle. Let $X_n$ be the state of the device after $n$ minutes.

   **a)** Find the distribution of $X_2$.

   **b)** Find the steady-state distribution of $X_n$.

4. **(10 marks)** A system has three possible states, 0, 1 and 2. Every hour it makes a transition to a different state, which is determined by a coin flip. For example, from state 0, it makes a transition to state 1 or state 2 with probabilities 0.5 and 0.5.

**a)** Find the transition probability matrix.

**b)** Find the three-step transition probability matrix.

**c)** Find the steady-state distribution of the Markov chain.

5. **(10 marks)** A certain machine is in one of four possible states: 0 = working without a problem; 1 = working but in need of minor repair; 2 = working but in need of major repair; 3 = out-of-order. The corresponding transition probability matrix is

$$\begin{bmatrix} 0.80 & 0.14 & 0.04 & 0.02 \\ 0 & 0.6 & 0.3 & 0.10 \\ 0 & 0 & 0.65 & 0.35 \\ 0.90 & 0 & 0 & 0.10 \end{bmatrix}$$

**a)** Verify that it is a regular Markov chain.

**b)** Find the steady-state distribution.

6. **(10 marks)** Consider a game of "ladder climbing". There are 5 levels in the game, level 1 is the lowest (bottom) and level 5 is the highest (top). A player starts at the bottom. Each time, a fair coin is tossed. If it turns up heads, the player moves up one rung. If tails, the player moves down to the very bottom. Once at the top level, the player moves to the very bottom if a tail turns up, and stays at the top if head turns up.

**a)** Find the transition probability matrix.

**b)** Find the two-step transition probability matrix.

**c)** Find the steady-state distribution of the Markov chain.

7. **(10 marks)** (Sec 7.2, page 346, #2) We observe the state of a system at discrete points in time. The system is observed only when it changes state. Define $X_n$ as the state of the system after nth state change, so that $X_n = 0$, if the system is running; $X_n = 1$ if the system is under repair; and $X_n = 2$ if the system is idle. Assume that the matrix $P$ is:

$$\begin{bmatrix} 0 & 0.5 & 0.5 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Compute the matrix $P^n$ for all possible $n$.

8. **(10 marks)** (Sec 7.3, page 355, #1) Consider a system with two components. We observe the state of the system every hour. A given component operating at time $n$ has probability $p$ of failing before the next observation at time $n+1$. A component that was in a failed condition at time $n$ has a probability $r$ of being repaired by time $n+1$, independent of how long the component has been in a failed state. The component failures and repairs are mutually independent events. Let $X_n$ be the number of components in operation at time $n$. The process $\{X_n, n = 0, 1, \ldots\}$ is a discrete time homogeneous Markov chain with state space $I = 0, 1, 2$.

   **a)** Determine its transition probability matrix, and draw the state diagram.

   **b)** Obtain the steady state probability vector, if it exists.

# Solutions:

1.

**a)** $P^{(2)} = P \cdot P = \begin{bmatrix} 0.52 & 0.48 \\ 0.48 & 0.52 \end{bmatrix}$

**b)** There are 3 transitions between 5:30 and 8:30, thus we need to compute $p_{11}^{(3)}$. The 3-step transition probability matrix is

$$P^{(3)} = P^{(2)} P = \begin{bmatrix} 0.496 & \dots \\ \dots & \dots \end{bmatrix}$$

(there is no need to compute the entire matrix). Hence, $p_{11}^{(3)} = 0.496$.

2.

**a)** Let "sunny" be state 1 and "rainy" be state 2. Write the transition probability matrix,

$$\begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{bmatrix}$$

We need to find the 2-step transition probability $p_{12}^{(2)}$. The 2-step transition probability matrix is

$$P^2 = \begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{bmatrix} \cdot \begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{bmatrix} = \begin{bmatrix} \cdots & 0.28 \\ \dots & \dots \end{bmatrix}$$

Only one element $p_{12}^{(2)}$ needs to be computed. The probability that the day after tomorrow is rainy is 0.28.

b. April 1 next year is so many transitions away that we can use the steady-state distribution. To find it, we solve the system of equations $\pi P = \pi, \;\; \pi_1 + \pi_2 = 1$. These equations are:

$$\begin{array}{rcl} 0.8\pi_1 + 0.4\pi_2 & = & \pi_1 \\ 0.2\pi_1 + 0.6\pi_2 & = & \pi_2 \\ \pi_1 + \pi_2 & = & 1 \end{array}$$

From the first equation, $\pi_1 = 2\pi_2$. From the second equation, again $\pi_1 = 2\pi_2$. We know that one equation will follow from the others. Substituting $\pi_1 = 2\pi_2$ into the last equation, we get $2\pi_2 + \pi_2 = 1$. From here, $\pi_2 = 1/3$ and $\pi_1 = 2/3$.

Hence, the probability that April 1 next year is rainy is $\pi_2 = 1/3$.

3.

The transition probability matrix is given as

$$P = \begin{bmatrix} 0.8 & 0.2 \\ 0.1 & 0.9 \end{bmatrix}.$$

Also, we have $X(0) = 2$ (idle mode) with probability 1, i.e., $P_0 = (0,\ 1)$.

**a)**

$$P_2 = P_0 P^2 = (0,1) \begin{bmatrix} 0.8 & 0.2 \\ 0.1 & 0.9 \end{bmatrix}^2 = (0.17, 0.83).$$

Thus, $X(2)$, the state after 2 transitions, is busy with probability 0.17 and idle with probability 0.83.

**b)** To find the steady-state distribution, we solve the system of equations $\pi P = \pi$, $\pi_1 + \pi_2 = 1$. These equations are:

$$\begin{aligned} 0.8\pi_1 + 0.1\pi_2 &= \pi_1 \\ 0.2\pi_1 + 0.9\pi_2 &= \pi_2 \\ \pi_1 + \pi_2 = 1 & \end{aligned}$$

From the first equation, $\pi_2 = 2\pi_1$. From the second equation, again $\pi_2 = 2\pi_1$. Substituting $\pi_2 = 2\pi_1$ into the last equation, we get $\pi_1 + 2\pi_1 = 1$. From here, $\pi_1 = 1/3$ and $\pi_2 = 2/3$. In a steady state,

$$P\{X_n = busy\} = 1/3, \quad P\{X_n = idle\} = 2/3.$$

4.

**a)**

$$P = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{bmatrix}$$

**b)**

$$P^3 = \begin{bmatrix} 0.250 & 0.375 & 0.375 \\ 0.375 & 0.250 & 0.375 \\ 0.375 & 0.375 & 0.250 \end{bmatrix}$$

**c)** Solve the system $\pi P = \pi$ along with the normalizing condition $\pi_1 + \pi_2 + \pi_3 = 1$.

$$\left\{ \begin{array}{lcl} 0.5\pi_2 + 0.5\pi_3 & = & \pi_1 \\ 0.5\pi_1 + 0.5\pi_3 & = & \pi_2 \\ 0.5\pi_1 + 0.5\pi_2 & = & \pi_3 \\ \pi_1 + \pi_2 + \pi_3 & = & 1 \end{array} \right. \Longrightarrow \left\{ \begin{array}{lcl} \pi_2 + \pi_3 & = & 2\pi_1 \\ \pi_1 + \pi_3 & = & 2\pi_2 \\ \pi_1 + \pi_2 & = & 2\pi_3 \\ \pi_1 + \pi_2 + \pi_3 & = & 1 \end{array} \right. \Longrightarrow$$

$$\Longrightarrow \left\{ \begin{array}{lcl} 1 - \pi_1 & = & 2\pi_1 \\ 1 - \pi_2 & = & 2\pi_2 \\ 1 - \pi_3 & = & 2\pi_3 \end{array} \right. \Longrightarrow \left\{ \begin{array}{lcl} \pi_1 & = & 1/3 \\ \pi_2 & = & 1/3 \\ \pi_3 & = & 1/3 \end{array} \right.$$

The steady-state probability distribution is $\pi = (1/3, 1/3, 1/3)$.

5.

**a)** Compute

$$P^3 = \begin{pmatrix} 0.5678 & 0.2097 & 0.1501 & 0.0724 \\ 0.2295 & 0.2286 & 0.3554 & 0.1866 \\ 0.4882 & 0.0441 & 0.2872 & 0.1804 \\ 0.6732 & 0.1890 & 0.0936 & 0.0442 \end{pmatrix}$$

It has no zeros, so this Markov chain is regular.

No desire to compute this matrix? You can also prove regularity by looking at the state transition diagram. Possible transitions are

$1 \to 1$, $1 \to 2$, $1 \to 3$, $1 \to 4$;

$2 \to 2$, $2 \to 3$, $2 \to 4$;

$3 \to 3$, $3 \to 4$;

$4 \to 1$, $4 \to 4$.

So, we can follow the full circle $1 \to 2 \to 3 \to 4 \to 1$ and reach any state from any other state in $\leq 3$ transitions.

If we reach the needed state too early, we just make an extra step from this state back to the same state. So, in 3 steps, all transitions are possible. The Markov chain is regular.

b. Solve the system $\pi P = \pi$ along with the normalizing condition $\sum \pi_i = 1$. The steady-state probability vector is $\pi = (0.50, 0.18, 0.21, 0.11)$.

6.

**a)**

$$P = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 \\ 0.5 & 0 & 0 & 0.5 & 0 \\ 0.5 & 0 & 0 & 0 & 0.5 \\ 0.5 & 0 & 0 & 0 & 0.5 \end{bmatrix}$$

**b)**

$$P^2 = \begin{bmatrix} 0.5 & 0.25 & 0.25 & 0 & 0 \\ 0.5 & 0.25 & 0 & 0.25 & 0 \\ 0.5 & 0.25 & 0 & 0 & 0.25 \\ 0.5 & 0.25 & 0 & 0 & 0.25 \\ 0.5 & 0.25 & 0 & 0 & 0.25 \end{bmatrix}$$

**c)** Solve the system $\pi P = \pi$ along with the normalizing condition $\sum \pi_i = 1$.

$$\left\{ \begin{array}{lcl} 0.5(\pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5) & = & \pi_1 \\ 0.5\pi_1 & = & \pi_2 \\ 0.5\pi_2 & = & \pi_3 \\ 0.5\pi_3 & = & \pi_4 \\ 0.5\pi_4 + 0.5\pi_5 & = & \pi_5 \\ \pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5 & = & 1 \end{array} \right. \implies \left\{ \begin{array}{lcccl} \pi_1 & = & (0.5)(1) & = & 1/2 \\ \pi_2 & = & (0.5)(\pi_1) & = & 1/4 \\ \pi_3 & = & (0.5)(\pi_2) & = & 1/8 \\ \pi_4 & = & (0.5)(\pi_3) & = & 1/16 \\ \pi_5 & = & \pi_4 & = & 1/16 \end{array} \right.$$

$\pi = (1/2, 1/4, 1/18, 1/16, 1/16)$ is the steady-state probability distribution vector.

7.

Possible transitions are:

$0 \to 1$ (with probability 0.5)

$0 \to 2$ (with probability 0.5)

$1 \to 0$ (with probability 1)

$2 \to 0$ (with probability 1)

This means: from states 1 and 2, the system surely goes to state 0. From state 0, it is equally likely to go to either 1 or 2. Thus, starting from 0, after any odd number of steps, it is at 1 or 2 with probabilities .5 and .5, and after any even number of steps, it is back at 0 with probability 1. Therefore, for even $n$ we have,

$$P^n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix}$$

and for odd $n$ we have,

$$P^n = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

8.

**a)** The transition probability matrix is:

$$\begin{bmatrix} (1-r)^2 & 2r(1-r) & r^2 \\ p(1-r) & pr+(1-p)(1-r) & r(1-p) \\ p^2 & 2p(1-p) & (1-p)^2 \end{bmatrix}$$

**b)** To compute the steady-state probabilities, one can of course solve the linear system. By doing this, one can see that the probabilities are: $\pi_0 = [r/(p+r)]^2$, $\pi_1 = 2[r/(p+r)][p/(p+r)]$ and $\pi_2 = [p/(p+r)]^2$.

One may also get this answer intuitively. Each component spends the average time of $1/p$ functioning without repairs and the average time of $1/r$ for each repair. Thus, the proportion of time it is functioning is $(1/p)/(1/p+1/r) = r/(p+r)$. That is, each time, independently of each other, each component is functioning with probability $r/(p+r)$ and not functioning with probability $p/(p+r)$. Hence, the limiting distribution of $X_n$ is Binomial with parameters 2 and $r/(p+r)$. Check this! Take a vector of these Binomial probabilities and substitute into our linear system, see if it is correct.