Basic stochastic processes
Problems from old examinations with solutions

Problem 1. Taxis are waiting in a queue for passengers to come. Passengers for those taxis arrive according to a Poisson process with an average of 60 passengers per hour. A taxi departs as soon as two passengers have been collected or 3 minutes have expired since the first passenger has got in the taxi. Suppose you get in the taxi as first passenger. What is your average waiting time for the departure?

Hint: Condition on the first arrival after you get in the taxi.

Solution

Let X denote your waiting time in minutes, and let $N(t)$ be the process counting the arrivals of passenger from the moment you get in the taxi. $N(t)$ is a Poisson process of parameter $\lambda = 1$ passenger per minute. Let S_1 denote the first arrival time of the process. We have

$$E[X] = E[X|S_1 \geq 3]P\{S_1 \geq 3\} + E[X|S_1 < 3]P\{S_1 < 3\}$$

$$E[X|S_1 \geq 3]P\{S_1 \geq 3\} = 3e^{-3}$$

$$E[X|S_1 < 3]P\{S_1 < 3\} = E[S_1|S_1 < 3]P\{S_1 < 3\} = \int_0^3 x f_{S_1}(x)dx$$

$$= \int_0^3 xe^{-x}dx = -xe^{-x}|_0^3 + \int_0^3 e^{-x}dx = 1 - 4e^{-3}$$

$$E[X] = 1 - e^{-3} = 0.95 \approx 57sec.$$

Problem 2. Oil tankers with destination Rotterdam leave from harbours in the Middle East according to a Poisson process with an average of two tankers per day. The sailing time to Rotterdam has a Gamma distribution with an expected value of 10 days and a standart deviation of 4 days. Estimate the probability that the number of oil tunkers that are under way from the Middle East to Rotterdam at an arbitrary point of time exceeds 30.

Solution. We recognize the $M/G/\infty$ system. The number of busy servers in the system at time t corresponds to the number $L(t)$ of oils tunkers that are under way at this time point. In the long run the distribution of $L(t)$ is approximately $Poisson(20)$ and the distribution of $\dfrac{L(t)-20}{\sqrt{20}}$ is then approximately $N(0,1)$. This gives

$$P\{L(t) > 30\} = P\left\{\frac{L(t)-20}{\sqrt{20}} > \frac{10}{\sqrt{20}}\right\} \approx 1 - \Phi(2.236) = 0.0094$$

Exact computations give 0.0135.

Problem 3. The bus that takes you home from Chalmers arrives at the nearest bus station from early morning till late in the evening according to a renewal process with interarrival times that are uniformly distributed between 5 and 10 minutes. You come to the bus station at 5 pm. Estimate your average waiting time for the bus to arrive.

Solution. Set the time unit to be a minute. Your waiting time for the bus to come is the excess variable γ associated with the time you come to the bus station. Since at this time the process has reached statistical equilibrium we can use the approximation

$$E[\gamma] \approx \frac{\mu_2}{2\mu_1}$$

where

$$\mu_1 = \frac{5+10}{2} = 7.5 \quad \mu_2 = \frac{1}{5}\int_5^{10} x^2 dx = \frac{1}{15}x^3|_5^{10} = 58.33$$

$$E[\gamma] \approx \frac{58.33}{15} \approx 3.89\ min.$$

Problem 4. The diffusion of electrons and holes across a potential barrier in an electronic devise is modelled as follows. There are m black balls (electrons) in urn A and m white balls (holes) in urn B. We perform independent trials, in each of which a ball is selected at random from each urn and the selected ball from urn A is placed in urn B, while that from urn B is placed in A. Consider the Markov chain representing the number of black balls in urn A immediately after the $n-th$ trial.

(a) Describe the one-step transition probabilities of the process.

(b) Suppose $m = 3$. Compute the long-run fraction of time when urn A does not contain a black ball.

3p

Solution. Let X_n=the # of black balls in urn A just after the $n-th$ trial. $\{X_n,\ n \geq 0\}$ is a MC with state space $I = \{0,\ 1,\ \ldots\ m\}$.

(a) The one-step transition probabilities of the MC are

$$p_{i,i-1} = P\{X_{n+1} = i-1 | X_n = i\} = \left(\frac{i}{m}\right)^2, \quad i \geq 1$$
$$p_{i,i+1} = P\{X_{n+1} = i | X_n = i\} = \left(\frac{m-i}{m}\right)^2, \quad i \leq m-1$$
$$p_{i,i} = P\{X_{n+1} = i-1 | X_n = i\} = 1 - \left(\frac{i}{m}\right)^2 - \left(\frac{m-i}{m}\right)^2$$
$$p_{i,j} = 0 \quad otherwise.$$

(b) Let $m = 3$. The one-step transition probability matrix is

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{9} & \frac{4}{9} & \frac{4}{9} & 0 \\ 0 & \frac{4}{9} & \frac{4}{9} & \frac{1}{9} \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The equilibrium equations are

$$\begin{aligned} \pi_0 &= \frac{1}{9}\pi_1 \\ \pi_1 &= \pi_0 + \frac{4}{9}\pi_1 + \frac{4}{9}\pi_2 \\ \pi_2 &= \frac{4}{9}\pi_1 + \frac{4}{9}\pi_2 + \pi_3 \\ \pi_3 &= \frac{1}{9}\pi_2 \\ \sum_{j=0}^{3} \pi_j &= 1 \end{aligned}$$

The solution is

$$\pi_0 = \frac{1}{20} \quad \pi_1 = \frac{9}{20} \quad \pi_2 = \frac{9}{20} \quad \pi_3 = \frac{1}{20}$$

The long-run fraction of time when urn A does not contain a black ball is 1/20.

Problem 5. Peter takes the course Basic Stochastic Processes this quarter on Tuesday, Thursday, and Friday. The classes start at 10:00 am. Peter is used to work until late in the night and consequently, he sometimes misses the class. His attendance behaviour is such that he attends class depending only on whether or not he went to the latest class. If he attended class one day, then he will go to class next time it meets with probability 1/2. If he did not go to one class, then he will go to the next class with probability 3/4.

(a) Describe the Markov chain that models Peter's attendance. What is the probability that he will attend class on Thursday if he went to class on Friday?

(b) Suppose the course has 30 classes altogether. Give an estimate of the number of classes attended by Peter and explain it.

3p

Solution.

(a) Let $X_n = 0$ if Peter goes to the $n-th$ class meeting and $X_n = 1$ if he skips it. The process $\{X_n,\, n \geq 1\}$ is a MC with state space $I = \{0, 1\}$ and one-step transition probability matrix

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & \frac{1}{4} \end{bmatrix}.$$

$$P\{\text{Peter will attend on Thursday}|\text{he went on Friday}\} = p_{11}^{(2)} = \frac{5}{8},$$

where $p_{11}^{(2)}$ is taken from the two-step transition matrix

$$P^2 = \begin{bmatrix} \frac{5}{8} & \frac{3}{8} \\ \\ \frac{9}{16} & \frac{7}{16} \end{bmatrix}$$

(b) The state space has two states which communicate, so the equilibrium probabilities exist. From

$$\pi_0 = \pi_0 \frac{1}{2} + \pi_1 \frac{3}{4}$$
$$\pi_0 + \pi_1 = 1$$

we obtain $\pi_0 = 0.6$, $\pi_1 = 0.4$. The process reaches equilibrium before the end of the course, since

$$P^8 = \begin{bmatrix} 0.5998 & 0.3999 \\ \\ 0.5998 & 0.3999 \end{bmatrix}.$$

Thus the long-run fraction of classes attended is ≈ 0.6, so the estimation for the number of classes attended is $0.6 \times 30 = 18$.

Problem 6. In an inventory system for a single product the depletion of stock is due to demand and deterioration. The demand process for the product is the Poisson process with rate λ. The lifetime of each product is exponentially distributed with mean $1/\mu$. The stock control is exercised as follows. Each time the stock drops to zero an order for Q is placed. The lead time of each order is negligible. We are interested in the long run average number of orders placed per time unit.

(a) Introduce an appropriate continuous-time Markov chain to analyse the system and compute the infinitesimal transition rates.

(b) Give a recursive algorithm for computing the equilibrium probabilities and a formula for the long run average number of orders placed per time unit.

4p

Solution. Let $X(t)$ be the stock on hand at time t. The process $X(t)$, $t \geq 0$ is a continuous-time MC with state space $I = \{1, 2, \ldots Q\}$ and transitions rates

(a)

$$q_{i,i-1} = \lambda + i\mu, \quad i = 2, \ldots Q$$
$$q_{1Q} = \lambda + \mu$$
$$q_{ij} = 0 \quad \textit{otherwise.}$$

(b) By equating the rate out of state i to the rate into state i we obtain

$$
\begin{aligned}
(\lambda + i\mu)p_i &= [\lambda + (i+1)\mu]p_{i+1}, \quad 1 \le i \le Q-1 \\
(\lambda + Q\mu)p_Q &= [\lambda + \mu]p_1.
\end{aligned}
$$

Starting from $\overline{p}_q = 1$ we can recursively compute $\overline{p}_i$, $1 = q-1, \ldots 1$. The normalizing equation then gives

$$p_i = \frac{\overline{p}_i}{\sum_{j=1}^{Q} \overline{p}_j}.$$

The long run average number of orders placed per time unit equals the long run average number of transitions from state 1 to state Q per time unit, which is $p_1(\lambda + \mu)$.

Computation of the equilibrium probabilities (not required in the problem).

We have

$$
\begin{aligned}
p_2 &= p_1 \frac{\lambda + \mu}{\lambda + 2\mu} \\
p_3 &= p_2 \frac{\lambda + 2\mu}{\lambda + 3\mu} = p_1 \frac{\lambda + \mu}{\lambda + 3\mu} \\
&. \\
&. \\
&. \\
p_Q &= p_1 \frac{\lambda + \mu}{\lambda + Q\mu} \\
1 &= \sum_1^Q p_1 \frac{\lambda + \mu}{\lambda + i\mu} \\
p_1 &= \frac{1}{(\lambda + \mu) \sum_1^Q \frac{1}{\lambda + i\mu}} \\
p_i &= \frac{1}{(\lambda + i\mu) \sum_1^Q \frac{1}{\lambda + i\mu}}, \quad i \le Q
\end{aligned}
$$

The long-run average number of orders placed per time unit is thus

$$\frac{1}{\sum_1^Q \frac{1}{\lambda + i\mu}} \to 0 \; as \; Q \to \infty,$$

as expected.