Homework 4 Solution

1. **(Ross 4.3)**
In this case, the state of the system is determined by the weather conditions in the last three days. Letting D indicate a dry day and R indicate a rainy day, the possible weather patterns that can be observed over a sequence of three days are as follows: RRR,RRD, RDR, RDD, DRR, DRD, DDR, DDD. Furthermore, based on the provided information, the possible transitions between these eight patterns are characterized by the following transition probability matrix:

$$P = \begin{pmatrix} .8 & .2 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & .4 & .6 & & & & \\ & & & & .6 & .4 & & \\ & & & & & & .4 & .6 \\ .6 & .4 & & & & & & \\ & & .4 & .6 & & & & \\ & & & & .6 & .4 & & \\ & & & & & & .2 & .8 \end{pmatrix}$$

See also Example 4.4 in your textbook.

2. **(Ross 4.5)**
Cubing the transition probability matrix, we obtain P^3:

$$P = \begin{pmatrix} 13/36 & 11/54 & 47/108 \\ 4/9 & 4/27 & 11/27 \\ 5/12 & 2/9 & 13/36 \end{pmatrix}$$

Thus,

$$E[X_3] = \sum_{i=0}^{2} E[X_3|X_0 = i]P(X_0 = i)$$

where

$$E[X_3|X_0 = 0] = 0+1*P[X_3 = 1|X_0 = 0]+2*P[X_3 = 2|X_0 = 0] = \frac{11}{54}+2*\frac{47}{108}$$

$$E[X_3|X_0 = 1] = \frac{4}{27} + 2 * \frac{11}{27}$$

$$E[X_3|X_0 = 2] = \frac{2}{9} + 2 * \frac{13}{36}$$

so

$$E[X_3] = \frac{53}{54}$$

3. **(Ross 4.13)**
Consider $n > r$. Then, from the Chapman-Kolmogorov equations,

$$P_{ij}^n = \sum_k P_{ik}^{n-r} P_{kj}^r$$

But the sum on the right-hand-side of the above equation is strictly greater than zero because $P_{kj}^r > 0$ for all k and P_{ik}^{n-r} must be positive for at least one k.

4. **(Ross 4.14)**
The best way to resolve this problem is by drawing the corresponding state transition diagrams. Then, you should be able to see that:
(i) This MC is irreducible and therefore, all its states are recurrent.
(ii) This MC is irreducible as well.
(iii) This MC has three classes: Numbering the states from 1 to 4, the first class consists of $\{1\}$ and it is transient, the second class consists of $\{0, 2\}$ and it is recurrent, and the last class consists of $\{3, 4\}$ and it is recurrent.
(iv) In this case, the various classes and their characterization are as follows: $\{0, 1\}$ recurrent, $\{2\}$ recurrent, $\{3\}$ transient, $\{4\}$ transient.

5. **(Ross 4.18)**
If the state i at time n is the coin to be flipped, where $i = 1, 2$, then sequence of consecutive states constitute a two state Markov chain with transition probabilities

$$P_{1,1} = .6 = 1 - P_{1,2}, P_{2,1} = .5 = P_{2,2}$$

(a) The stationary probabilities satisfy

$$\pi_1 = .6\pi_1 + .5\pi_2$$

$$\pi_1 + \pi_2 = 1$$

Solving yields that $\pi_1 = 5/9, \pi_2 = 4/9$. So the proportion of flips that use coin 1 is $5/9$.
(b)$P_{1,2}^4 = .44440$

6. **(Ross 4.20)**
We know that for a finite-state, irreducible, aperiodic CTMC, the limiting probabilities are given by the *unique* solution of the following system of Equations:

$$\pi_j = \sum_{i=0}^{m} \pi_i P_{ij}$$

and

$$\sum_{j=0}^{m} \pi_j = 1$$

Hence, it suffices to check that the suggested values for π_j satisfy the above equations.

7. **(Ross 4.45)**
(a) Since this chain is finite and irreducible, all states are positive recurrent and therefore the considered probability is equal to 1.
(b) One way to address this problem is by making both states N and 0 *absorbing* states, and noticing that, in the modified chain, the considered probabilities x_i are equal to the probabilities that the chain will be absorbed in state N rather than 0, when starting from the remaining states $i = 1, 2, \ldots, N-1$. Then, according to the relevant theory presented in class, the requested x_i can be obtained from the following system of equations:

$$x_i = \sum_{j=1}^{N-1} P_{ij} x_j + P_{iN}, \ i = 1, ..., N-1$$

$$x_0 = 0, \ x_N = 1.$$

(c) We must show that

$$\frac{i}{N} = \sum_{j=1}^{N-1} \frac{j}{N} P_{ij} + P_{iN} = \sum_{j=0}^{N} \frac{j}{N} P_{ij}$$

which follows from the working hypothesis.

8. **(Ross 4.52)**
Let the state be the successive zonal pickup locations. Then $P_{A,A} = 0.6, P_{A,B} = 0.4, P_{B,A} = 0.3, P_{B,B} = 0.7$, and the long run proportions of pickups at each zone are obtained by

$$\pi_A = .6\pi_A + .3\pi_B = .6\pi_A + .3(1 - \pi_A)$$

Therefore, $\pi_A = 3/7, \ \pi_B = 4/7$. Let X denote the profit in a trip. Conditioning on the location of the pickup gives

$$E[X] = \frac{3}{7} E[X|A] + \frac{4}{7} E[X|B] = \frac{3}{7}[.6(6) + .4(12)] + \frac{4}{7}[.3(12) + .7(8)] = \frac{62}{7}$$

9. **(Ross 4.63)**

$$P_T = \begin{pmatrix} .4 & .2 & .1 \\ .1 & .5 & .2 \\ .3 & .4 & .2 \end{pmatrix}$$

$$S = (I - P_T)^{-1} = \begin{pmatrix} 0.6 & -0.2 & -0.1 \\ -0.1 & 0.5 & -0.2 \\ -0.3 & -0.4 & 0.8 \end{pmatrix}^{-1} = \begin{pmatrix} 2.2069 & 1.3793 & 0.6207 \\ 0.9655 & 3.1034 & 0.8966 \\ 1.3103 & 2.0690 & 1.9310 \end{pmatrix}$$

So

$$f_{13} = \frac{S_{13}}{S_{33}} = 0.3214$$

$$f_{23} = \frac{S_{23}}{S_{33}} = 0.4643$$

$$f_{33} = \frac{S_{33} - 1}{S_{33}} = 0.4821$$

10. **(Electronic reserves #15)**

(a) For states 1 pint, 2 pints, ..., 7 pints, the transition matrix is

$$P = \begin{pmatrix} 0.6 & 0.4 & & & & & \\ 0.3 & 0.3 & 0.4 & & & & \\ 0.1 & 0.2 & 0.3 & 0.4 & & & \\ & 0.1 & 0.2 & 0.3 & 0.4 & & \\ & & 0.1 & 0.2 & 0.3 & 0.4 & \\ & & & 0.1 & 0.2 & 0.3 & 0.4 \\ & & & & 0.1 & 0.2 & 0.7 \end{pmatrix}$$

(b) From

$$\pi_1 = 0.6\pi_1 + 0.3\pi_2 + 0.1\pi_3;$$

$$\pi_2 = 0.4\pi_1 + 0.3\pi_2 + 0.2\pi_3 + 0.1\pi_4;$$

$$\vdots$$

$$\pi_7 = 0.4\pi_6 + 0.7\pi_7;$$

$$\pi_1 + \pi_2 + \cdots + \pi_7 = 1.$$

we get

$\pi_1 = 0.139; \pi_2 = 0.139; \pi_3 = 0.139; \pi_4 = 0.138; \pi_5 = 0.141; \pi_6 = 0.130; \pi_7 = 0.174.$

11. **(Electronic reserves #16)**
(a)Not to advertise:

$$P_1 = \begin{pmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{pmatrix}$$

Always advertise:

$$P_2 = \begin{pmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{pmatrix}$$

Manager's proposal:

$$P_3 = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

(b)Not to advertise:

$$\pi_0 = 3/4\pi_0 + 1/2\pi_1$$
$$\pi_1 = 1/4\pi_0 + 1/2\pi_1$$
$$\pi_0 + \pi_1 = 1$$

get

$$\pi_0 = 2/3, \pi_1 = 1/3;$$

Similarly for "Always advertise":

$$\pi_0 = 1/3, \pi_1 = 2/3;$$

Manager's proposal:

$$\pi_0 = 1/2, \pi_1 = 1/2;$$

(c) profit when never advertise: $2 * 2/3 + 4 * 1/3 = 8/3$ million;
profit when always advertise: $2 * 1/3 + 4 * 2/3 - 1 = 7/3$ million;
profit of manager's proposal: $2 * 1/2 + 4 * 1/2 - 1 * 1/2 = 5/2$ million.
Hence, the strategy of never advertising is the most profitable.

12. **(Electronic reserves #24)**
(a) State 0 means the recorder has been replaced, State 1 means the machine is new, State 2 means the machine works well through the whole first year, State 3 means the machine works well for the whole first 2 years, and does not need repair. The transition matrix is

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ .01 & 0 & .99 & 0 \\ .05 & 0 & 0 & .95 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(b) This is the probability that the recorder will fail either during the first year or the second year: $.01 + .99 * .05 = .0595$.

13. **(Extra Credit: Ross 3.26)**
Consider the process defined by the successive card orderings, so there are $n!$ states. The one-step transition probabilities of this process are

$$P_{(i_1,...,i_n),(i_j,i_1,...,i_{j-1},i_{j+1},...,i_n)} = \frac{1}{n}$$

Obviously, there are n equally possible resulting states in each transition, i.e., each row of the corresponding Transition Probability Matrix (TPM) has n entries equal to $1/n$. However, the suggested shuffling further implies that each ordering $(i_1, ..., i_n)$ can be reached in one transition from exactly n different orderings, specifically, those orderings obtained from $(i_1, ..., i_n)$ by reolcating card i_1 to the i^{th} position, for $i = 1, 2, \ldots, n$. Hence, each column of the considered TPM will have exactly n non-zero entries, and since each such entry is equal to $1/n$, the matrix is doubly stochastic.

But then, the result off Exercise 4.20 implies that, in the limit, all $n!$ possible states are equally likely.

14. **(Extra Credit: Ross 3.37)**
We need to show that

$$\pi_j = \sum_i \pi_i P^k_{i,j}$$

But this equation is an immediate consequence of the facts that (i) $P^k_{i,j}$ are the k-step transition probabilities for the original Markov chain defined by $[P_{i,j}]$, and (ii) $\{\pi_j\}$ is a stationary distribution for that chain.

15. **(Extra Credit: Ross 3.49)**
(a)This is not a Markov chain, because the process transition at the n^{th} epoch is not determined only by its current state but also by the result of the coin's flipping at the very beginning.

For the considered process,

$$\lim_{n\to\infty} P(X_n = i) = p\pi^1(i) + (1-p)\pi^2(i).$$

(b)This is a Markov chain, since for this case, we will flip the coin at the beginning of each epoch, and these trials are independent. Since the original stochastic processes are Markov chains, the composite process will also satisfy the Markovian property.

The one-step transition probability matrix for this process is given by

$$P_{ij} = pP^{(1)}_{ij} + (1-p)P^{(2)}_{ij}.$$

The following counter-example establishes that the limiting distributions for the two above cases may not be the same:

Let

$$P^{(1)} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

Then, its limiting distribution is $\pi^1(1) = 1/2, \pi^1(2) = 1/2$.

Also, let

$$P^{(1)} = \begin{pmatrix} 1/4 & 3/4 \\ 1/2 & 1/2 \end{pmatrix}$$

The corresponding limiting distribution is $\pi^2(1) = 2/3, \pi^2(2) = 1/3$.

Finally, let also $p = 1/2$.

Then, according to the above,

$$\lim_{n\to\infty} P(X_n = 1) = 1/2 \cdot 1/2 + 1/2 \cdot 2/3 = 7/12$$

On the other hand, the transition matrix for the second process is

$$P = \begin{pmatrix} 3/8 & 5/8 \\ 1/2 & 1/2 \end{pmatrix}$$

with limiting probability for State 1 equal to 4/9.