1. Let (X_n) be a Markov chain with state space $\{0, 1\}$ and transition matrix $\begin{bmatrix} p & q \\ q & p \end{bmatrix}$. Find
   (a) $\mathbb{P}(X_1 = 0 \mid X_0 = 0 \text{ and } X_2 = 0)$,
   (b) $\mathbb{P}(X_1 \neq X_2)$.

Proof. (a) First we compute

$$\begin{aligned}
&\mathbb{P}(X_1 = 0, X_0 = 0 \text{ and } X_2 = 0) \\
=&\mathbb{P}(X_0 = 0, X_1 = 0 \text{ and } X_2 = 0) \\
=&\mathbb{P}(X_2 = 0 \mid X_1 = 0 \text{ and } X_0 = 0)\mathbb{P}(X_1 = 0 \mid X_0 = 0)\mathbb{P}(X_0 = 0) \\
=&\mathbb{P}(X_2 = 0 \mid X_1 = 0)\mathbb{P}(X_1 = 0 \mid X_0 = 0)\mathbb{P}(X_0 = 0) \qquad \text{[Markov Property]} \\
=&P_{00}P_{00}\pi_0(0) \\
=&p^2\pi_0(0),
\end{aligned}$$

and similarly

$$\begin{aligned}
&\mathbb{P}(X_0 = 0 \text{ and } X_2 = 0) \\
=&\mathbb{P}(X_0 = 0, X_1 = 0 \text{ and } X_2 = 0) + \mathbb{P}(X_0 = 0, X_1 = 1 \text{ and } X_2 = 0) \\
=&(p^2 + q^2)\pi_0(0).
\end{aligned}$$

By the definition of conditional probability,

$$\begin{aligned}
\mathbb{P}(X_1 = 0 \mid X_0 = 0 \text{ and } X_2 = 0) &= \frac{\mathbb{P}(X_1 = 0, X_0 = 0 \text{ and } X_2 = 0)}{\mathbb{P}(X_0 = 0 \text{ and } X_2 = 0)} \\
&= \frac{p^2\pi_0(0)}{(p^2 + q^2)\pi_0(0)} \\
&= \frac{p^2}{p^2 + q^2}.
\end{aligned}$$

(b)

$$\mathbb{P}(X_1 \neq X_2) = \mathbb{P}(X_1 = 0, X_2 = 1) + \mathbb{P}(X_1 = 1, X_2 = 0) = q\pi_1(0) + q\pi_1(1) = q.$$

□

2. Suppose we have two urns (a left urn and a right urn). The left urn contains n black balls and the right urn contains n red balls. Every time step you take one ball (chosen randomly) from each urn, swap the balls, and place them back in the urns. Let X_m be the number of black balls in the left urn after n time steps. Find the transition function of the Markov chain (X_m).

Proof. When $x \neq 0, d$, the transition function is

$$p(x, y) = \begin{cases} \frac{(n-x)^2}{n^2} & y = x + 1, \\ \frac{2x(n-x)}{n^2} & y = x, \\ \frac{x^2}{n^2} & y = x - 1, \\ 0 & \text{otherwise.} \end{cases}$$

Take the first case as an example, for the number of black ball in the left box increase by 1, we will have choose a white ball from the left box, which has probability $(n - x)/n$, and choose a black ball from the right box, which also has probability $(n - x)/n$. By the counting rule, the probability of this case is the product of the two.

When $x = 0$, the transition function is $p(0, 1) = 1$ and 0 otherwise. When $x = n$, the transition function is $p(n, n-1) = 1$ and 0 otherwise. These two cases agree with the formula given above. Therefore the transition function is simply

$$p(x,y) = \begin{cases} \frac{(n-x)^2}{n^2} & y = x+1, \\ \frac{2x(n-x)}{n^2} & y = x, \\ \frac{x^2}{n^2} & y = x-1, \\ 0 & \text{otherwise.} \end{cases}$$

□

3. Consider the Ehrenfest gas model. That is, you have two urns (a left urn and a right urn) and balls labeled $1, \ldots, d$. At each time step a number is chosen uniformly from $1, \ldots, d$ and the ball of that number is removed from its urn and placed in the other urn. Let X_n be the number of balls in the left urn at the nth time step. Assuming that the distribution of X_0 is given by

$$\mathbb{P}(X_0 = i) = 2^{-d}\binom{d}{i},$$

compute $\mathbb{P}(X_1 = i)$.

Proof. By an analogous argument as in Q2, the transition function for the Ehrenfest chain is

$$p(x,y) = \begin{cases} \frac{x}{d} & y = x-1, \\ 1 - \frac{x}{d} & y = x+1, \\ 0 & \text{otherwise.} \end{cases}$$

Regardless of the value of x that we start with. Now we compute

$$\begin{aligned}
\mathbb{P}(X_1 = i) &= \mathbb{P}(X_1 = i, X_0 = i-1) + \mathbb{P}(X_1 = i, X_0 = i-1) \\
&= \mathbb{P}(X_1 = i \mid X_0 = i-1)\mathbb{P}(X_0 = i-1) + \mathbb{P}(X_1 = i \mid X_0 = i+1)\mathbb{P}(X_0 = i+1) \\
&= \left(1 - \frac{i-1}{d}\right)\frac{\binom{d}{i-1}}{2^d} + \left(\frac{i+1}{d}\right)\frac{\binom{d}{i+1}}{2^d} \\
&= \frac{1}{2^d}\left(\frac{d-i+1}{d}\frac{d!}{(i-1)!(d-i+1)!} + \frac{i+1}{d}\frac{d!}{(i+1)!(d-i-1)!}\right) \\
&= \frac{1}{2^d}\left(\frac{(d-1)!}{(i-1)!(d-i)!} + \frac{(d-1)!}{i!(d-i-1)!}\right) \\
&= \frac{1}{2^d}\left(\binom{d-1}{i-1} + \binom{d-1}{i}\right) \\
&= \frac{1}{2^d}\binom{d}{i}.
\end{aligned}$$

Observe that $\mathbb{P}(X_1 = i) = \mathbb{P}(X_0 = i)$, a.k.a. $\pi_1 = \pi_0$. This is an example of stationary distribution.

□

4. Let (X_n) be a Markov chain. Show that it has a Markov-type property backwards in time. That is, show

$$\mathbb{P}(X_0 = x_0 \mid X_1 = x_1) = \mathbb{P}(X_0 = x_0 \mid X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n).$$

Proof.

$$\begin{aligned}
&\mathbb{P}(X_0 = x_0 \mid X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n) \\
=&\frac{\mathbb{P}(X_0 = x_0, X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n)}{\mathbb{P}(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n)} \qquad \text{[Defintion of conditional probability]} \\
=&\frac{\mathbb{P}(X_n = x_n, X_{n-1} = x_{n-1}, \ldots, X_1 = x_1, X_0 = x_0)}{\mathbb{P}(X_n = x_n, X_{n-1} = x_{n-1}, \ldots, X_1 = x_1)} \\
=&\frac{\mathbb{P}(X_n = x_n \mid X_{n-1} = x_{n-1}) \ldots \mathbb{P}(X_2 = x_2 \mid X_1 = x_1)\mathbb{P}(X_1 = x_1 \mid X_0 = x_0)\mathbb{P}(X_0 = x_0)}{\mathbb{P}(X_n = x_n \mid X_{n-1} = x_{n-1}) \ldots \mathbb{P}(X_2 = x_2 \mid X_1 = x_1)\mathbb{P}(X_1 = x_1)} \\
=&\frac{\mathbb{P}(X_1 = x_1 \mid X_0 = x_0)\mathbb{P}(X_0 = x_0)}{\mathbb{P}(X_1 = x_1)} \qquad \text{[Cancellation]} \\
=&\mathbb{P}(X_0 = x_0 \mid X_1 = x_1) \qquad \text{[Bayes' theorem]}
\end{aligned}$$

□

5. Consider the Markov chain (X_n) with state space $\{0, 1, 2\}$ and transition matrix

$$T = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/4 & 1/3 & 5/12 \\ 0 & 1/2 & 1/2 \end{bmatrix}.$$

Assuming the chain starts in state 1, compute $\mathbb{P}(X_2 = 2 \text{ and } X_4 = 1 \text{ and } X_5 = 0)$.

Proof. The two step transition matrix is

$$T^2 = \begin{bmatrix} 3/8 & 5/12 & 5/24 \\ 5/24 & 4/9 & 25/72 \\ 1/8 & 5/12 & 11/24 \end{bmatrix}.$$

The desired probability is

$$\begin{aligned}
&\mathbb{P}(X_2 = 2 \text{ and } X_4 = 1 \text{ and } X_5 = 0 \mid X_0 = 1) \\
=&\mathbb{P}(X_5 = 0 \mid X_4 = 1)\mathbb{P}(X_4 = 1 \mid X_2 = 2)\mathbb{P}(X_2 = 2 \mid X_0 = 1) \\
=&T_{10} \cdot (T^2)_{21} \cdot (T^2)_{12} \\
=&1/4 \cdot 5/12 \cdot 25/72, \\
=&125/3456.
\end{aligned}$$

□

6. The weather on the newly-colonized Mars is one of two types: rainy or dry. If it is rainy, there's a 25% chance it will rain the next day. If it is dry, there is a 10% chance it will rain the next day.

   (a) The past Monday and Wednesday were dry, but you were away and don't know what the weather was on Tuesday. What's the probability it was dry on Tuesday?

   (b) Predictions show that on next Monday there is a 2/3 chance of rain. What is the probability that the weather for the upcoming week will be (rain, rain, dry, rain, dry)?

   (c) Predictions show that on next Monday there is a 2/3 chance of rain. What is the chance that both the following Saturday and Sunday will be dry?

Proof. Consider the Markov chain X_n with $\mathcal{S} = \{D, R\}$. The transition matrix is

$$T = \begin{bmatrix} 0.9 & 0.1 \\ 0.75 & 0.25 \end{bmatrix}.$$

(a) The desire probability is

$$\mathbb{P}(X_1 = D \mid X_0 = X_2 = D).$$

The calculation is the same as Q1(a), which ends up with $0.81/0.895 = 0.92$.

(b) We have $\pi_0 = (1/3, 2/3)$. The desired probability is

$$\mathbb{P}(X_0 = R, X_1 = R, X_2 = D, X_3 = R, X_4 = D) = T_{RD}T_{DR}T_{RD}T_{RR}\pi_0(R) = 3/320.$$

(c) Still, $\pi_0 = (1/3, 2/3)$. The desired probability is

$$\begin{aligned} &\mathbb{P}(X_5 = D, X_6 = D) \\ =&\mathbb{P}(X_0 = R, X_5 = D, X_6 = D) + \mathbb{P}(X_0 = D, X_5 = D, X_6 = D) \\ =&T_{DD}(T^5)_{RD}\pi_0(R) + T_{DD}(T^5)_{DD}\pi_0(D) \\ =&0.53 + 0.26 = 0.79, \end{aligned}$$

where we calculated a priori an estimate for T^5:

$$T^5 = \begin{bmatrix} 0.8824 & 0.1176 \\ 0.8823 & 0.1177 \end{bmatrix}.$$

□

7. Let (X_n) be a Markov chain with state space $\{0, 1\}$ and initial distribution π_0.

(a) Give a transition matrix for (X_n) such that the limiting distribution is not independent of π_0. That is

$$\lim_{n\to\infty} \mathbb{P}(X_n = a \mid X_0 = 0) \neq \lim_{n\to\infty} \mathbb{P}(X_n = a \mid X_0 = 1).$$

(b) Give a transition matrix for (X_n) such that the limiting distribution is independent of π_0. That is

$$\lim_{n\to\infty} \mathbb{P}(X_n = a \mid X_0 = 0) = \lim_{n\to\infty} \mathbb{P}(X_n = a \mid X_0 = 1).$$

(c) Give a transition matrix for (X_n) such that the limiting distribution depends on parity. That is,

$$\lim_{n\to\infty} \mathbb{P}(X_{2n} = a \mid X_0 = 0) \neq \lim_{n\to\infty} \mathbb{P}(X_{2n+1} = a \mid X_0 = 0).$$

Proof. We consider Markov chains with two states $\mathcal{S} = \{0, 1\}$.

(a) Take

$$T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = Id.$$

Then $\pi_n = \pi_0$, which certainly depends on the initial distribution.

(b) Take

$$T = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}.$$

Then $\pi_n = (1/2, 1/2)$ as n approaches infinity.

(c) Take

$$T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then $T^{2n} = Id$ and $T^{2n+1} = T$. The limiting distribution depends on the parity.

□