1. _____

A Markov chain with state space $\{1, 2, 3\}$ has transition probability matrix

$$P = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 0 & 1/2 & 1/2 \\ 0 & 0 & 1 \end{pmatrix}$$

Show that state 3 is absorbing and, starting from state 1, find the expected time until absorption occurs.

Solution. Let $\psi(i)$ be the expected time to reach state 3 starting from state i, where $i \in \{1, 2, 3\}$. We have

$$\psi(3) = 0$$
$$\psi(2) = 1 + \frac{1}{2}\psi(2) + \frac{1}{2}\psi(3)$$
$$\psi(1) = 1 + \frac{1}{3}\psi(1) + \frac{1}{3}\psi(2) + \frac{1}{3}\psi(3).$$

We solve and find $\psi(3) = 0$, $\quad \psi(2) = 2$, $\quad \psi(1) = 5/2$.

2. _____

Smith is in jail and has 3 dollars; he can get out on bail if he has 8 dollars. A guard agrees to make a series of bets with him. If Smith bets A dollars, he wins A dollars with probability 0.4 and loses A dollars with probability 0.6. Find the probability that he wins 8 dollars before losing all of his money if
(a) he bets 1 dollar each time (timid strategy).
(b) he bets, each time, as much as possible but not more than necessary to bring his fortune up to 8 dollars (bold strategy).
(c) Which strategy gives Smith the better chance of getting out of jail?

Solution. (a) The Markov chain $(X_n, n = 0, 1, \ldots)$ representing the evolution of Smith's money has diagram
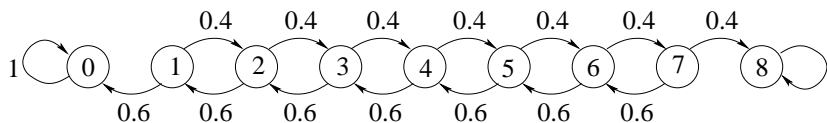
Let $\varphi(i)$ be the probability that the chain reaches state 8 before reaching state 0, starting from state i. In other words, if S_j is the first $n \geq 0$ such that $X_n = j$,

$$\varphi(i) = P_i(S_8 < S_0) = P(S_8 < S_0 | X_0 = i).$$

Using first-step analysis (viz. the Markov property at time $n = 1$), we have

$$\varphi(i) = 0.4\varphi(i+1) + 0.6\varphi(i-1), \quad i = 1, 2, 3, 4, 5, 6, 7$$
$$\varphi(0) = 0$$
$$\varphi(8) = 1.$$

We solve this system of linear equations and find

$$\varphi = (\varphi(1),\ \varphi(2),\ \varphi(3),\ \varphi(4),\ \varphi(5),\ \varphi(6),\ \varphi(7))$$
$$= (0.0203,\ 0.0508,\ 0.0964,\ 0.1649,\ 0.2677,\ 0.4219,\ 0.6531,\ 1).$$

E.g., the probability that the chain reaches state 8 before reaching state 0, starting from state 3 is the third component of this vector and is equal to 0.0964. Note that $\varphi(i)$ is increasing in i, which was expected.

(b) Now the chain is
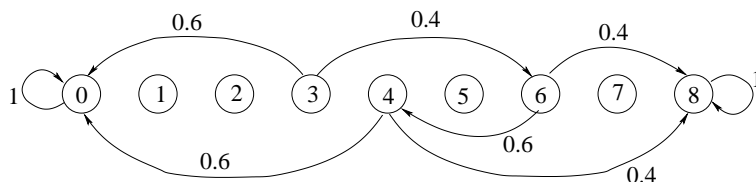
and the equations are:

$$\varphi(3) = 0.4\varphi(6)$$
$$\varphi(6) = 0.4\varphi(8) + 0.6\varphi(4)$$
$$\varphi(4) = 0.4\varphi(8)$$
$$\varphi(0) = 0$$
$$\varphi(8) = 1.$$

We solve and find
$$\varphi = (0,\ \ 0,\ \ 0.256,\ \ 0.4,\ \ 0,\ \ 0.64,\ \ 0).$$

(c) By comparing the third components of the vector φ we find that the bold strategy gives Smith a better chance to get out jail.

3.

A fair coin is tossed repeatedly and independently. Find the expected number of tosses till the pattern HTH appears.

Solution. Call HTH our target. Consider a chain that starts from a state called nothing $\emptyset$ and is eventually absorbed at HTH. If we first toss H then we move to state H because this is the first letter of our target. If we toss a T then we move back to $\emptyset$ having expended 1 unit of time. Being in state H we either move to a new state HT if we bring T and we are 1 step closer to the target or, if we bring H, we move back to H: we have expended 1 onit of time, but the new H can be the beginning of a target. When in state HT we either move to HTH and we are done or, if T occurs then we move to $\emptyset$. The transition diagram is

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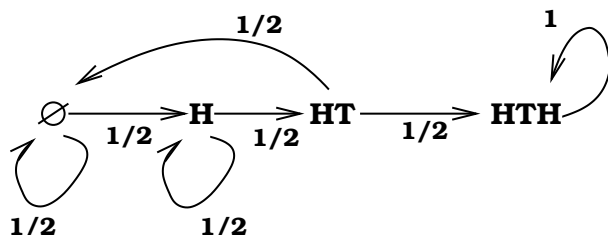
Rename the states $\emptyset$, H, HT, HTH as $0, 1, 2, 3$, respectively. Let $\psi(i)$ be the expected number of steps to reach HTH starting from i. We have

$$\psi(2) = 1 + \frac{1}{2}\psi(0)$$

$$\psi(1) = 1 + \frac{1}{2}\psi(1) + \frac{1}{2}\psi(2)$$

$$\psi(0) = 1 + \frac{1}{2}\psi(0) + \frac{1}{2}\psi(1).$$

We solve and find $\psi(0) = 10$.

4. _____

A. Assume that an experiment has m equally probable outcomes. Show that the expected number of independent trials before the first occurrence of k consecutive occurrences of one of these outcomes is

$$\frac{m^k - 1}{m - 1}.$$

Hint: Form an absorbing Markov chain with states $1, 2, \ldots, k$ with state i representing the length of the current run. The expected time until a run of k is 1 more than the expected time until absorption for the chain started in state 1.

B. How many tosses of a fair die are required, on the average, until 10 consecutive sixes are obtained?

C. It has been found that, in the decimal expansion of $\pi = 3.14159\ldots$, starting with the 24,658,601st digit, there is a run of nine 7's. What would your result say about the expected number of digits necessary to find such a run if the digits are produced randomly?

Solution. A. Let the outcomes be $a, b, c, \ldots$ (m of them in total). Suppose that a is the desirable outcome. We set up a chain as follows. Its states are

$$\emptyset, \quad (a), \quad (aa), \quad (aaa), \quad \ldots, \quad \underbrace{(aa\cdots a)}_{k \text{ times}}$$

Or, more simply, $0, 1, 2, \ldots, m$. State k means that you are currently at the end of a run of k a's. If you see an extra a (with probability $1/m$) you go to state $k + 1$. Otherwise, you go to $\emptyset$. Let $\psi(k)$ be the expected number of steps till state m is reached, starting from state k:

$$\psi(k) := E_k S_m.$$

We want to find $\psi(0)$. We have

$$\psi(k) = 1 + (1 - 1/m)\psi(0) + (1/m)\psi(k + 1).$$

Solving these, we find

$$\psi(0) = 1 + m + m^2 + \cdots + m^{k-1} = \frac{m^k - 1}{m - 1}.$$

B. So to get 10 consecutive sixes by rolling a die, you need more than 12 million rolls on the average ($12,093,235$ rolls to be exact).

C. They are not random. If they were, we expect to have to pick $(10^9 - 1)/9$ digits before we see nine consecutive sevens. That's about 100 million digits. The actual position (24 million digits) is one fourth of the expected one.

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5. _____

A rat runs through the maze shown below. At each step it leaves the room it is in by choosing at random one of the doors out of the room.
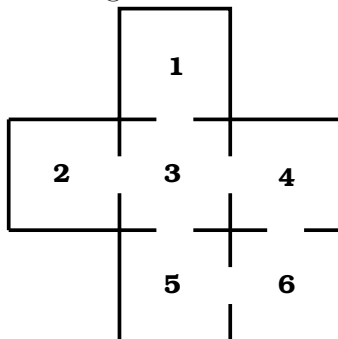
(a) Find the transition matrix for this Markov chain.
(b) Show that it is irreducible and find its period.
(c) Find the stationary distribution.
(d) Now suppose that a piece of mature cheddar is placed on a deadly trap in Room 5. The rat starts in Room 1. Find the expected number of steps before reaching Room 5 for the first time, starting in Room 1.
(e) Supppose again that the rat starts in Room 1 and moves until it dies in Room 5. Every time the rat is in room i it gnaws an amount of wood equal to i^2. Find the expected amount of wood gnawed by the rat before its death. (Assume when it enters room 5 it dies immediately before chewing any wood at all.)
(f) Supppose again that the rat starts in Room 1 but there is no trap in Room 5. Explain why it will, for sure, return to room 1. Find the average time until it returns to room 1.

Solution
(a) The transition matrix P for this Markov chain is as follows:

$$\mathsf{P} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1/4 & 1/4 & 0 & 1/4 & 1/4 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1/2 & 1/2 & 0 \end{pmatrix}.$$

(b) The chain is irreducible, because it is possible to go from any state to any other state. However, it is not aperiodic, because for any n even $p_{6,1}^{(n)}$ will be zero and for any n odd $p_{6,5}^{(n)}$ will also be zero (why?). This means that there is no power of P that would have all its entries strictly positive.
(c) The stationary distribution is

$$\pi = (\frac{1}{12}, \frac{1}{12}, \frac{4}{12}, \frac{2}{12}, \frac{2}{12}, \frac{2}{12}).$$

You should carry out the calculations and check that this is correct.
(d) Let
$$\psi(i) = E(\text{number of steps to reach state } 5 \mid X_0 = i).$$

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We have

$$\psi(5) = 0$$
$$\psi(6) = 1 + (1/2)\psi(5) + (1/2)\psi(4)$$
$$\psi(4) = 1 + (1/2)\psi(6) + (1/2)\psi(3)$$
$$\psi(3) = 1 + (1/4)\psi(1) + (1/4)\psi(2) + (1/4)\psi(4) + (1/4)\psi(5)$$
$$\psi(1) = 1 + \psi(3)$$
$$\psi(2) = 1 + \psi(3).$$

We solve and find $\psi(1) = 7$.

(e) Let $h(i)$ be the total amount of wood gnawed by the rat before its death if the rat starts its rat-race in room i. We have

$$h(5) = 0$$
$$h(6) = 36 + (1/2)h(5) + (1/2)h(4)$$
$$h(4) = 16 + (1/2)h(6) + (1/2)h(3)$$
$$h(3) = 9 + (1/4)h(1) + (1/4)h(2) + (1/4)h(4) + (1/4)h(5)$$
$$h(1) = 1 + h(3)$$
$$h(2) = 4 + h(3).$$

We solve and find

$$h(5) = 0, h(6) = \frac{321}{4}, h(2) = \frac{275}{4}, h(4) = \frac{177}{2}, h(1) = \frac{263}{4}, h(3) = \frac{259}{4}.$$

(f) The chain is irreducible with a finite number of states. Therefore all states are recurrent. This means that the rat will return to Room 1 with probability 1. We find, from π, that the mean recurrence time (i.e. the expected time to return) for the room 1 is $1/\pi(1) = 12$.

6. _____

Two players, A and B, play the game of matching pennies: at each time n, each player has a penny and must secretly turn the penny to heads or tails. The players then reveal their choices simultaneously. If the pennies match (both heads or both tails), Player A wins the penny. If the pennies do not match (one heads and one tails), Player B wins the penny. Suppose the players have between them a total of 5 pennies. If at any time one player has all of the pennies, to keep the game going, he gives one back to the other player and the game will continue.
(a) Show that this game can be formulated as a Markov chain.
(b) Is the chain irreducible? Is it aperiodic?
(c) If Player A starts with 3 pennies and Player B with 2, what is the probability that A will lose his pennies first?

Solution (a) The problem is easy: The probability that two pennies match is $1/2$. The probability they do not match is $1/2$. Let x be the number of pennies that A has. Then with probability $1/2$ he will next have $x + 1$ pennies or with probability $1/2$ he will next have $x - 1$ pennies. The exception is when $x = 0$, in which case, he gets, for free, a penny from B and he next has 1 penny. Also, if $x = 5$ he gives a penny to B and he next has 4 pennies. Thus:
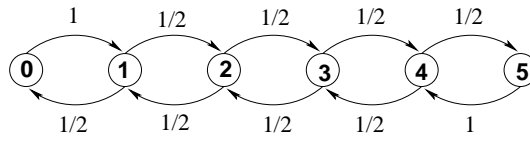
(b) The chain is clearly irreducible. But the period is 2. Hence it is not regular.

(c) To do this, modify the chain and make it stop once one of the players loses his pennies. After all, we are NOT interested in the behaviour of the chain after this time. The modification is an absorbing chain:
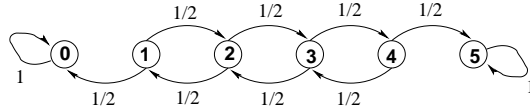
We then want to compute the absorbing probability $\varphi_{01}(3)$ where

$$\varphi_{01}(i) = P_i(\text{hit } 0 \text{ before } 1).$$

Write $\varphi(i) = \varphi_{01}(i)$, for brevity, and apply first-step analysis:

$$\varphi(0) = 1$$
$$\varphi(1) = \frac{1}{2}\varphi(0) + \frac{1}{2}\varphi(1)$$
$$\varphi(2) = \frac{1}{2}\varphi(1) + \frac{1}{2}\varphi(2)$$
$$\varphi(3) = \frac{1}{2}\varphi(2) + \frac{1}{2}\varphi(3)$$
$$\varphi(4) = \frac{1}{2}\varphi(3) + \frac{1}{2}\varphi(4)$$
$$\varphi(5) = 0.$$

Six equations with six unknowns. Solve and find: $\varphi(3) = 2/5$.
Alternatively, observe, from Thales' theorem,[1] that φ must be a straight line:

$$\varphi(x) = ax + b.$$

From $\varphi(0) = 1$, $\varphi(5) = 0$, we find $a = -1/5$, $b = 1$, i.e.

$$\varphi(i) \equiv 1 - (i/5),$$

which agrees with the above.

Thales' theorem says (proved around [1]the year 600 BCE) says that if the lines L, L' are parallel then $\frac{DE}{BC} = \frac{AE}{AC} = \frac{AD}{AB}$.