Homework 7 (Stats 620, Winter 2015)

Due Tuesday March 24, in class

1. Show that a continuous-time Markov chain is regular, given (a) that $\nu_i < M < \infty$ for all i or (b) that the corresponding embedded discrete-time Markov chain with transition probabilities P_{ij} is irreducible and recurrent.

   Hint: For (a), you may follow the method suggested in the book solution (p. 491).

   Solution:

   (a) Let X_n denote the duration of the nth transition and $N(t)$ the number of transitions up to time t. Define, $\widetilde{X}_n \equiv \nu_i X_n/M$. Thus $\widetilde{X}_n \leq X_n$ and $\widetilde{X}_n \sim \exp(M)$. Let

$$\widetilde{N}(t) = \max\{n : \sum_{i=0}^{n} \widetilde{X}_i \leq t\}\,.$$

   Then $\widetilde{N}(t)$ is a Poisson random variable with mean Mt and $\widetilde{N}(t) \geq N(t)$ for all t. Thus

$$\begin{aligned}\mathbb{P}[N(t) = \infty] &\leq \mathbb{P}[\widetilde{N}(t) = \infty] \\ &= \lim_{n\to\infty} \sum_{k=n}^{\infty} e^{-M} M^k/k! \\ &= 0\end{aligned}$$

   (b) Let $N_i(t)$ count visits to i by time t for some i. Recurrence and irreducibility ensure that w.p. 1 there are a finite number of transitions between visits to i. These conditions also imply that $N_i(t)$ is a renewal process, due to which we know that $\mathbb{P}[N_i(t) = \infty] = 0$ for any $t < \infty$. Thus, the number of transitions of the Markov chain, $N(t)$, is also finite.

2. Let $\{X(t), t \geq 0\}$ be a continuous-time Markov chain on the non-negative integers, having transition rates q_{ij}. Let $P(t) = P_{00}(t)$.

   (a) Find $\lim_{t\to 0} \frac{1-P(t)}{t}$.

   (b) Show that $P(t)P(s) \leq P(t+s) \leq 1 - P(s) + P(s)P(t)$.

   (c) Show $|P(t) - P(s)| \leq 1 - P(t-s)$, $s < t$ and conclude that P is continuous.

   Hint: For (a) you should justify your answer but need not prove the necessary limit theorem, so your answer could be quite short! One way to obtain (b) is through two applications of the Chapman-Kolmogorov identity. One way to solve (c) is by algebraic manipulation of (b).

   Solution:

   (a) By Lemma 5.4.1, which is proved as the solution to exercise 5.8 in Ross.

   (b) Note that

$$\begin{aligned}P(t)P(s) &= P_{00}(t)P_{00}(s) \leq \sum_{k=0}^{\infty} P_{0k}(t)P_{k0}(s) = P_{00}(t+s) = P(t+s) \\ &= \sum_{k=1}^{\infty} P_{0k}(t)P_{k0}(s) + P_{00}(t)P_{00}(s) \leq \sum_{k=1}^{\infty} P_{0k}(s) + P_{00}(t)P_{00}(s) \\ &= (1 - P_{00}(s)) + P(t)P(s)\,.\end{aligned}$$

(c) By part (b), we have,

$$P(s)P(t-s) \leq P(t) \leq 1 - P(t-s) + P(t-s)P(s)\,.$$

Thus, substracting $P(s)$ on the inequality above, we obtain

$$P(s)(P(t-s)-1) \leq P(t) - P(s) \leq 1 - P(t-s) + (P(t-s)-1)P(s)\,. \tag{1}$$

Note that

$$P(s)(P(t-s)-1) \geq P(t-s) - 1$$

and

$$(P(t-s)-1)P(s) \leq 0\,,$$

it follows from (1) that

$$|P(t) - P(s)| \leq 1 - P(t-s)\,.$$

Finally,

$$\begin{aligned} \lim_{t\to s} |P(t) - P(s)| &\leq \lim_{t\to s} 1 - P(t-s) \\ &= 1 - P(0) \\ &= 0 \end{aligned}$$

Thus $P(t)$ is continuous.

3. Suppose that the "state" of a system can be modeled as a two-state continuous-time Markov chain with transition rates $\nu_0 = \lambda, \nu_1 = \mu$. When the state of the system is i, "events" occur in accordance with a Poisson process with rate α_i for $i = 0, 1$. Let $N(t)$ denote the number of events in $(0, t)$.

   (a) Find $\lim_{t\to\infty} N(t)/t$.

   (b) If the initial state is state 0, find $\mathbb{E}[N(t)]$.

   Hint For (a), one approach is to let return times into state 0 form a renewal process, and consider a reward to be the number of "events" in the renewal period. For (b), you are asked to find the exact result for finite t, rather than a limiting result as $t \to \infty$.

   Solution:

   (a) Define a renewal reward process as follows. A renewal occurs when the process enters state 0 and reward in a cycle equals the number of events in that cycle. Let the length of nth cycle X_n is the sum of time spent in states 0 and 1, say X_{0n} and X_{1n} respectively. Thus $\mathbb{E}[X_n] = \mathbb{E}[X_{0n}] + \mathbb{E}[X_{1n}] = \lambda^{-1} + \mu^{-1}$. Further if R_n is the reward in the nth cycle, with R_{0n} and R_{1n} earned in state 0 and 1 respectively

$$\begin{aligned} \mathbb{E}[R_n] &= \mathbb{E}[R_{0n}] + \mathbb{E}[R_{1n}] = \mathbb{E}[\mathbb{E}[R_{0n}|X_{0n}]] + \mathbb{E}[\mathbb{E}[R_{0n}|X_{0n}]] \\ &= \mathbb{E}[\alpha_0 X_{0n}] + \mathbb{E}[\alpha_1 X_{1n}] \\ &= \alpha_0/\lambda + \alpha_1/\mu\,. \end{aligned} \tag{2}$$

   Thus

$$\lim_{t\to\infty} \frac{N(t)}{t} = \lim_{t\to\infty} \frac{R(t)}{t} = \frac{\mathbb{E}[R]}{\mathbb{E}[X]} = \frac{\alpha_0\mu + \alpha_1\lambda}{\lambda + \mu}\,.$$

(b) By similar argument as in Equation (2), it is clear that $\mathbb{E}[N(t)] = \alpha_0\mathbb{E}[T_0(t)]+\alpha_1\mathbb{E}[T_1(t)]$, where $T_i(t)$ is the time spent in state i up to time t. Thus

$$\begin{aligned}\mathbb{E}[N(t)] &= \alpha_0\mathbb{E}[T_0(t)] + \alpha_1(t - \mathbb{E}[T_0(t)]) \\ &= (\alpha_0 - \alpha_1)\mathbb{E}[T_0(t)] + \alpha_1 t \\ &= \alpha_1 t + (\alpha_0 - \alpha_1)\int_0^t P_{00}(s)\, ds\,.\end{aligned}$$

By the forward equation,

$$P'_{00}(t) = -(\lambda+\mu)P_{00}(t) + \mu\,.$$

With the boundary condition $P_{00}(0) = 1$, we have

$$P_{00}(t) = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu}e^{-(\lambda+\mu)t}$$

and

$$\mathbb{E}[T_0(t)] = \frac{\mu t}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu}\int_0^t e^{-(\lambda+\mu)s}\, ds\,.$$

Finally one can show that

$$\mathbb{E}[N(t)] = \frac{\alpha_0\mu+\alpha_1\lambda}{\lambda+\mu}t + \frac{\alpha_1-\alpha_0}{(\lambda+\mu)^2}\lambda(e^{-(\lambda+\mu)t} - 1)\,.$$

4. Consider a population in which each individual independently gives birth at an exponential rate λ and dies at an exponential rate μ. In addition, new members enter the population in accordance with a Poisson process with rate θ. Let $X(t)$ denote the population size at time t.

   (a) Explain why $\{X(t), t \geq 0\}$ is a birth/death process. What are its parameters?

   (b) Set up and solve a differential equation to find $\mathbb{E}\big[X(T)|X(0) = i\big]$.

   <u>Solution</u>:

   (a) The Markov property comes from the memorylessness of the exponential distribution for event times. This is a linear birth/death process with immigration, having parameters $\mu_n = n\mu$ and $\lambda_n = n\lambda + \theta$.

   (b) Note that

$$\mathbb{E}[X(t+h)|X(0)] = \mathbb{E}[X(t)|X(0)] + (\lambda-\mu)\mathbb{E}[X(t)|X(0)]h + \theta h + o(h)$$

Thus defining $M(t) \equiv \mathbb{E}[X(t)|X(0)]$ we get the differential equation

$$M'(t) = (\lambda-\mu)M(t) + \theta\,.$$

With the initial condition $M(0) = i$, we solve

$$M(t) = \begin{cases} \theta t + i & \text{if } \lambda = \mu \\ (i + \frac{\theta}{\lambda-\mu})e^{(\lambda-\mu)t} - \frac{\theta}{\lambda-\mu} & \text{otherwise} \end{cases}$$

Recommended reading:
Sections 5.3, 5.4, 5.5.

Supplementary exercise: 5.14
Optional, but recommended. Do not turn in a solution—it is in the back of the book.