E3106, Solutions to Homework 3

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Problem 4.18. Define

$$X_n = \begin{cases} 0 & \text{if coin 1 is flipped on the } n\text{-th day} \\ 1 & \text{if coin 2 is flipped on the } n\text{-th day} \end{cases}$$

then $\{X_n, n \geq 0\}$ is an irreducible ergodic Markov chain with transition probability matrix

$$P = \begin{bmatrix} 0.6 & 0.4 \\ 0.5 & 0.5 \end{bmatrix}.$$

The limiting probabilities satisfy

$$\pi_0 + \pi_1 = 1$$
$$\pi_0 = 0.6\pi_0 + 0.5\pi_1$$

These solve to yield

$$\pi_0 = \frac{5}{9}, \pi_1 = \frac{4}{9}.$$

(a) The desired proportion is equal to $\pi_0 = \frac{5}{9}$.

(b)

$$P^4 = \begin{bmatrix} 0.6 & 0.4 \\ 0.5 & 0.5 \end{bmatrix}^4 = \begin{bmatrix} 0.5556 & 0.4444 \\ 0.5555 & 0.4445 \end{bmatrix}.$$

The desired probability is equal to $P^4_{01} = 0.4444$.

Problem 4.20 We have an irreducible and aperiodic Markov chain with a finite number of states $\{0, ..., M\}$ such that

$$\sum_{i=0}^{M} P_{ij} = 1 : \text{ for all } j \in \{0, ..., M\}$$

Since it has only one class with finite number of states, the Markov chain is recurrent (remark (ii) page 193). Thus, it also positive recurrent (see page 200) as it has only finite number of states. Hence, the limiting probabilities exist and are unqiue. Therefore, we only need to show that as

$$\pi_i = \frac{1}{M+1}, : \text{for all } j \in \{0, ..., M\}$$

solves (4.7) on page 201. This is true as

$$\pi_i = \frac{1}{M+1} = \frac{1}{M+1}\sum_{i=0}^{M} P_{ij} = \sum_{i=0}^{M} \frac{1}{M+1} P_{ij} = \sum_{i=0}^{M} \pi_i P_{ij}$$

$$\sum_{j=0}^{M} \pi_i = \sum_{i=0}^{M} \frac{1}{M+1} = \frac{1}{M+1}\sum_{i=0}^{M} 1 = \frac{M+1}{M+1} = 1.$$

In summary we have that $\pi_j = \frac{1}{M+1}$ are the limiting probabilities.

Problem 4.22 Let X_n denote the value of Y_n modulo 13 (that is, X_n is the remainder when Y_n is divided by 13). Then X_n is a Markov chain with states in $\{0, \ldots, 12\}$.

If we want to use Problem 4.20, we need to show that $\sum_{i=0}^{12} P_{ij} = 1$ for every $j = 0, 1, ..., 12$. To see this, take, for example, $j = 3$. In this case, there are only six possible transitions to state 3, namely from states 0, 1, 2, 10, 11, 12, each with probability 1/6

$$\sum_{i=0}^{12} P_{ij} = P_{0,3} + P_{1,3} + P_{2,3} + P_{12,3} + P_{11,3} + P_{10,3} = \frac{1}{6} * 6 = 1.$$

Similarly, this holds for all $j \in \{0, \ldots, 12\}$.

Since $P(X_n = X_{n+1}) = 0$ but $P(X_n = X_{n+i}) > 0$ for $i > 1$, the Markov chain $\{X_i\}$ is aperiodic and irreducible. By Problem 4.20,

$$\lim_{n\to\infty} P(Y_n \text{ is a multiple of 13 }) = \lim_{n\to\infty} P(X_n = 0) = \frac{1}{13}.$$

Problem 4.23. Define

$$X_n = \begin{cases} 0 & \text{if the } n\text{-th and } n\text{-1-th trials both succeeded (SS)} \\ 1 & \text{if the } n\text{-th trial succeeded but the } n\text{-1-th failed (FS)} \\ 2 & \text{if the } n\text{-th trial failed but the } n\text{-1-th successed (SF)} \\ 3 & \text{both the } n\text{-th and } n\text{-1-th trials failed (FF)} \end{cases}$$

then $\{X_n, n \geq 0\}$ is a Markov chain with transition probability matrix

$$P = \begin{bmatrix} 0.8 & 0 & 0.2 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.5 & 0 & 0.5 \\ 0 & 0.5 & 0 & 0.5 \end{bmatrix},$$

The limiting probabilities satisfy

$$\begin{aligned} \pi_0 &= 0.8\pi_0 + 0.5\pi_1 \\ \pi_1 &= 0.5\pi_2 + 0.5\pi_3 \\ \pi_2 &= 0.2\pi_0 + 0.5\pi_1 \\ \pi_0 + \pi_1 + \pi_2 + \pi_3 &= 1 \end{aligned}$$

These solve to yield

$$\pi_0 = \frac{5}{11}, \pi_1 = \frac{2}{11}, \pi_2 = \frac{2}{11}, \pi_3 = \frac{2}{11}.$$

Add one sucess trail will change

$$SS \to SS,\ FS \to SS,\ SF \to FS,\ FF \to FS$$

Consequently, the desired proportion is

$$0.8\pi_0 + 0.5\pi_1 + 0.5\pi_2 + 0.5\pi_3 = \frac{7}{11}.$$

Problem 4.36.

(a) There are two possible transition probabilities from Monday to Tuesday, P_{00} and P_{01}. Thus, the answer is

$$p_0 P_{00} + p_1 P_{01} = 0.4p_0 + 0.6p_1.$$

(b) We need to compute

$$p_0 P_{00}^{(4)} + p_1 P_{01}^{(4)}.$$

Since

$$P = \begin{bmatrix} 0.4 & 0.6 \\ 0.2 & 0.8 \end{bmatrix},$$

we have

$$P^4 = \begin{bmatrix} 0.4 & 0.6 \\ 0.2 & 0.8 \end{bmatrix}^4 = \begin{bmatrix} 0.2512 & 0.7488 \\ 0.2496 & 0.7504 \end{bmatrix}.$$

Thus, the answer is

$$p_0(0.2512) + p_1(0.7488).$$

(c)

$$\begin{aligned} \pi_0 &= 0.4\pi_0 + 0.2\pi_1 \\ \pi_1 &= 0.6\pi_0 + 0.8\pi_1 \\ \pi_0 + \pi_1 &= 1 \end{aligned}$$

This yields

$$\pi_0 = 0.25,\ \pi_1 = 0.75.$$

The answer is

$$p_0\pi_0 + p_1\pi_1 = 0.25p_0 + 0.75p_1.$$

(d) This is not a Markov Chain, as the transition probability may depend on the time at which the transition happens not just the past state. For example, consider the conditional probability

$$P(Y_{n+1} = 2 | Y_n = 1) = \frac{P(Y_{n+1} = 2,\ Y_n = 1)}{P(Y_n = 1)} = \frac{P(Y_{n+1} = 2,\ Y_n = 1)}{p_0 P(X_n = 0) + p_1 P(X_n = 1)}.$$

If Y_n is a Markov chain, the the above conditional probability should not depend on n. However, the numerator above is equal to

$$\begin{aligned}
& P(Y_{n+1}=2,\ Y_n=1) \\
= \quad & P(Y_{n+1}=2,\ Y_n=1|\ X_n=0,\ X_{n+1}=0)P(X_n=0,\ X_{n+1}=0) \\
& \quad + P(Y_{n+1}=2,\ Y_n=1|\ X_n=0,\ X_{n+1}=1)P(X_n=0,\ X_{n+1}=1) \\
& \quad + P(Y_{n+1}=2,\ Y_n=1|\ X_n=1,\ X_{n+1}=0)P(X_n=1,\ X_{n+1}=0) \\
& \quad + P(Y_{n+1}=2,\ Y_n=1|X_n=1,\ X_{n+1}=1)P(X_n=1,\ X_{n+1}=1) \\
= \quad & p_0(1-p_0)P(X_n=0,\ X_{n+1}=0)+p_0(1-p_1)P(X_n=0,\ X_{n+1}=1) \\
& \quad + p_1(1-p_0)P(X_n=1,\ X_{n+1}=0)+p_1(1-p_1)P(X_n=1,\ X_{n+1}=1) \\
= \quad & p_0(1-p_0)P_{00}+p_0(1-p_1)P_{01}+p_1(1-p_0)P_{10}+p_1(1-p_1)P_{11}.
\end{aligned}$$

In summary, we have

$$P(Y_{n+1}=2|Y_n=1)=\frac{p_0(1-p_0)P_{00}+p_0(1-p_1)P_{01}+p_1(1-p_0)P_{10}+p_1(1-p_1)P_{11}}{p_0P(X_n=0)+p_1P(X_n=1)},$$

which in general depends on n, condicting to the definition of the Markov property.

Problem 4.42 For a Markov chain π_i is the long run proportion of time spent in state i, for all states $i \in 0, 1, ...$ We then partition the state spave into A and A^c.

(a) Let $i \in A$. Then $\pi_i P_{ij}$ is the long run proportion of being in state i and moving into state j in the next transition. Summing over all states j in A^c, $\sum_{j\in A^c} \pi_i P_{ij}$ is then the long run proportion of being in state i and transitioning into a state in A^c. Summing over all states i in A, $\sum_{i\in A}\sum_{j\in A^c} \pi_i P_{ij}$ is the long run proportion of transitions from a state in A to a state in A^c.

(b) As in a) $\sum_{i\in A^c}\sum_{j\in A} \pi_i P_{ij}$ is the long run proportion of crossing from A^c to A.

(c) For every two transitions going from A to A^c, there must be at least one transition going from A^c to A, and vice versa. Therefore, the total number of the transitions from from A to A^c and the total number of the transitions from from A to A^c can differ at most by one. Since $\lim_{N\to\infty}(1/N)=0$, where N is the total number of transitions, the long run proportions of moves from A to A^c and from A^c to A must be the same.