E3106, Solutions to Homework 6

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Exercise 1. The state space of the Markov chain is $\{(i,j)|i \geq 1, j \geq 1\}$. Suppose that the Markov chain is currently in the state (i,j). Let T denote the time the Markov chain stays in state (i,j), and let T_{kl} denote the time until the k-th male mates with the l-th female. Since they mate in any time interval of length h with probability $\lambda h + o(h)$, it follows that T_{kl} is exponentially distributed with rate λ. Since

$$T = \min_{1 \leq k \leq i, 1 \leq l \leq j} T_{kl}$$

and the ij random variables T_{kl} are independent with each other, it follows that T is exponentially distributed with rate $ij\lambda$, namely

$$v_{(i,j)} = ij\lambda.$$

Since each offspring is equally likely to be male or female, we conclude

$$P_{(i,j)(i,j+1)} = P_{(i,j)(i+1,j)} = \frac{1}{2}.$$

Exercise 3. We can't analyze the problem as a birth and death process, since the states of the system depend on not only the number of working machines, but also which machine is working. But we can analyze it as a continuous time Markov chain with the following 5 states:

0: both machines are working
1: machine 1 is working, machine 2 is down
2: machine 2 is working, machine 1 is down
3: both machines are down, machine 1 is being repaired
4: both machines are down, machine 2 is being repaired

Since the functioning time of both machines and the repair times are independent exponential random variables, it follows that

$$v_0 = \mu_1 + \mu_2, v_1 = \mu_1 + \mu, v_2 = \mu_2 + \mu, v_3 = v_4 = \mu$$

and

$$P_{01} = 1 - P_{02} = \frac{\mu_2}{\mu_1 + \mu_2}$$
$$P_{10} = 1 - P_{14} = \frac{\mu}{\mu + \mu_1}$$
$$P_{20} = 1 - P_{23} = \frac{\mu}{\mu + \mu_2}$$
$$P_{31} = P_{42} = 1.$$

Exercise 5. (a) $\{X(t), t \geq 0\}$ is a continuous-time Markov chain, since given the current number of infected individuals, the infected number of individuals in the future is independent with the number of infected individuals in the past.

(b) It is a pure birth process with birth rate $\{\lambda_n\}_{n=1}^{N}$.

(c) Let T_n denote the time $X(t)$ stay in state n, then T_n is exponentially distributed with rate λ_n. Suppose currently there are n infected members in the population. Since the probability that a contact involves an infected and an uninfected individuals is $\frac{n(N-n)}{\binom{N}{2}}$, it follows that the birth rates are

$$\lambda_n = \frac{pn(N-n)\lambda}{\binom{N}{2}}, \quad n = 1, \ldots, N-1.$$

Since it is a pure birth process and there is no downside transition, we have

$$\begin{aligned} & E[\text{time until all members are infected}] \\ = \; & E[\sum_{n=1}^{N-1} T_n] \\ = \; & \sum_{n=1}^{N-1} \frac{\binom{N}{2}}{pn(N-n)\lambda} = \frac{\binom{N}{2}}{p\lambda} \sum_{n=1}^{N-1} \frac{1}{n(N-n)} \end{aligned}$$