Math 450 - Homework 5
Solutions

1. **Exercise 1.3.2, textbook.** The stochastic matrix for the gambler problem has the following form, where the states are ordered as $(0, 2, 4, 6, 8, 10)$:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The corresponding diagram is shown below, with the absorbing states boxed.

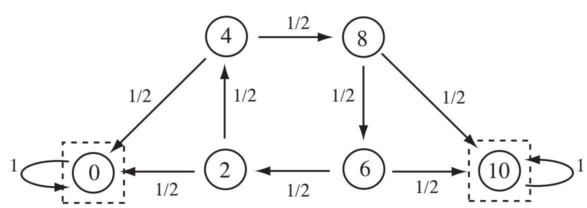

Figure 1: Digraph for the gambler problem.

To find the probability of reaching state 10 before state 0, we need to solve the following system of equations, where h_i is the hitting probability of state 10 starting at i.

$$\begin{aligned} h_2 &= \frac{1}{2}h_4 + \frac{1}{2}h_0 \\ h_4 &= \frac{1}{2}h_8 + \frac{1}{2}h_0 \\ h_6 &= \frac{1}{2}h_2 + \frac{1}{2}h_{10} \\ h_8 &= \frac{1}{2}h_6 + \frac{1}{2}h_{10}. \end{aligned}$$

Using the values $h_0 = 0$, $h_{10} = 1$ the system has a unique solution, which is easily found by simple substitution. We obtain $h_2 = 1/5$.

Now denote by μ_i the expected time before reaching either 0 or 10 given that the process starts at state i. Then $\mu_0 = 0$, $\mu_{10} = 0$, and the following system holds:

$$\begin{aligned}
\mu_2 &= 1 + \frac{1}{2}\mu_4 + \frac{1}{2}\mu_0 \\
\mu_4 &= 1 + \frac{1}{2}\mu_8 + \frac{1}{2}\mu_0 \\
\mu_6 &= 1 + \frac{1}{2}\mu_2 + \frac{1}{2}\mu_{10} \\
\mu_8 &= 1 + \frac{1}{2}\mu_6 + \frac{1}{2}\mu_{10}.
\end{aligned}$$

This is also easily solved for μ_2. The result is $\mu_2 = 2$.

The program below can be used for the simulation. (See program in the cat-and-mouse problem of homework 4.)

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%I will enumerate the states in the following
%order: 0 2 6 10 4 8, and use the program
%stop_at_set of homework 4.
rand('seed',324)
p=[0 1 0 0 0 0];
P=zeros(6,6);
P(2,[1,5])=1/2;
P(3,[2,4])=1/2;
P(5,[1,6])=1/2;
P(6,[3,4])=1/2;
P(1,1)=1;
P(4,4)=1;
A=[1 4];
a=0;
for j=1:1000
    x=stop_at_set(p,P,A,100000);
    a=a+length(x)-1;
end
a=a/1000
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

The above gave the simulated value $\mu_2 = 1.9760$.

2. The linear system for finding the h_i in the cat and mouse problem is (using

notation as in the previous problem): $h_7 = 0, h_9 = 1$ and

$$\begin{pmatrix} 2 & -1 & 0 & -1 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 3 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & -1 \\ 0 & 0 & -1 & 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 3 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \\ h_6 \\ h_8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

This can be solved numerically. The result is

$$h_1 = 0.4,\ h_3 = 0.6,\ h_4 = 0.3,\ h_2 = h_5 = h_8 = 0.5,\ h_6 = 0.7.$$

The value we want is $h_3 = 0.6$.

3. Let $X_0, X_1, X_2, \dots$ be a Markov chain with state space S. Decide whether the random variable T in each case below is a stopping time or not. Explain your answer.

   (a) *T is the time of the rth visit to state $i \in S$, where r is a positive integer.* This is a stopping time. The event $\{T = n\}$ can be written as the union of events of the type: $E_0 \cap E_1 \cap \dots \cap E_n$, where $E_n = \{X_n = i\}$, and each E_k for $k < n$ is either $\{X_k \neq i\}$ or $\{X_k = i\}$, where the latter occurs exactly $r - 1$ times. Therefore, $\{T = n\}$ is an event expressible in terms of the X_k for k up to time n.

   (b) *$T = T_A + n_0$, where A is a subset of S, T_A is the hitting time at A, and n_0 is a positive integer.* This is a stopping time. In fact

   $$\{T = n\} = \{T_A = n - n_0\}.$$

   As T_A is a stopping time, the event $\{T = n\}$ only depends on X_k for $k \leq n - n_0$.

   (c) *$T = T_A - n_0$, where A is a subset of S, T_A is the hitting time at A, and n_0 is a positive integer.* Essentially the same argument used in the previous item shows that this T is not a stopping time since the event $\{T = n\}$ depends on X_k for k up to $n + n_0$.

   (d) *T is the first nonnegative integer n such that $X_{n+1} = 1$, and $T = \infty$ if $X_{n+1} \neq i$ for all n.* This T is not a stopping time since the event $\{T = n\}$ depends on knowledge of X_{n+1}.

4. Exercise 1.5.1. This problem refers to the diagram of figure 2.

5. Exercise 1.5.2. Let $(X_n)_{n\geq 0}$ be a Markov chain on $S = \{0, 1, 2, \dots\}$ with transition probabilities given by

$$p_{01} = 1,\quad p_{i,i+1} + p_{i,i-1} = 1,\quad p_{i,i+1} = \left(\frac{i+1}{i}\right)^{\alpha} p_{i,i-1}.$$

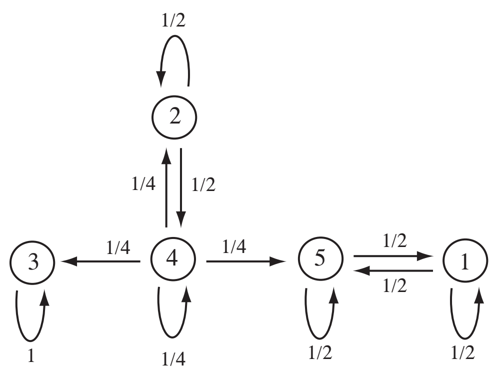

Figure 2: We know that for systems with finite state space, recurrent communicating classes are precisely the closed classes. So states 1, 5 and 3 are recurrent, whereas 2 and 4 are transient.

(In problem 1.3.4, $\alpha = 2$.) First note that all $p_{i,i-1}$ and $p_{i,i+1}$, for $i \geq 1$, are not equal to 0. This shows that the system is irreducible, i.e., there is only one communicating class. Therefore, either all states are recurrent or they are all transient. If all states are recurrent, the probability that $X_n \to \infty$ as $n \to \infty$ is 0, and if all states are transient, the same probability is one. In fact, transience means that with probability 1, after some time the system will never again return to a given state. That is, for ω in a set of probability 1, for each positive integer L (a state) we can find another positive integer N (a time step) such that $X_n(\omega) > L$ for all $n \geq N$. But this is just the calculus definition of the limit $X_n(\omega) \to \infty$ as $n \to \infty$.

Therefore, the probabilities we wish to find are either 1, if the system is transient, or 0 if it is recurrent. This is decided by using Theorem 1.5.3. Note that, as $p_{01} = 1$,

$$P_0(T_0 < \infty) = P_1(\text{hit } 0 \text{ in finite time}) = h_{1\{0\}}$$

where $h_{1\{0\}}$ is the probability of hitting 0 starting from state 1. This probability can be calculated just as in the birth-and-death example. It is equal to 1 if the series

$$C = \sum_{j=0}^{\infty} \gamma_j$$

is divergent (see page 16-17 of text), and if the series is convergent

$$h_{1\{0\}} = \frac{1}{C} \sum_{j=1}^{\infty} \gamma_j < 1.$$

(Recall that $\gamma_0 = 1$ by definition.) Thus, the probability we seek is 0 or 1 depending on whether the series A is divergent or not, respectively. A simple calculation shows that

$$\gamma_j = \frac{1}{(j+1)^\alpha}$$

so that

$$C = \sum_{j=0}^{\infty} \frac{1}{(j+1)^\alpha}.$$

We know that this series is divergent exactly when $\alpha \leq 1$. The conclusion is that

$$P(X_n \to \infty \text{ as } n \to \infty) = \begin{cases} 1 & \text{if } \alpha > 1 \\ 0 & \text{if } \alpha \leq 1. \end{cases}$$

6. Exercise 1.6.1. The rooted binary tree is the infinite graph indicated in figure 3. The graph on the right-hand side shows the transition probabilities of moving up or down along the tree. Notice that if we prove that the process on the right is transient, then the same is true for the process along the tree diagram. Thus the problem is reduced to showing transience for a birth-and-death chain. Since the chain is irreducible, we only need to show that the bottom state, denoted by 0, is transient.

   By theorem 1.5.3 it suffices to show that the probability of returning to 0 in finite time is less than 1. This probability is the same as the hitting probability, $h_{1,0}$, to 0 given that the system starts at the second state from the bottom. The calculation used in example 1.3.4 shows that this number is less than 1 if the series

$$C = \sum_{j=0}^{\infty} \gamma_j$$

   is convergent. But (with the notations of example 1.3.4) $q_i = q = 1/3$ and $p_i = p = 2/3$ for $i \geq 1$, so that $\gamma_j = (q/p)^j = (1/2)^j$. Therefore, the series is a convergent geometric series, and the chain is transient.

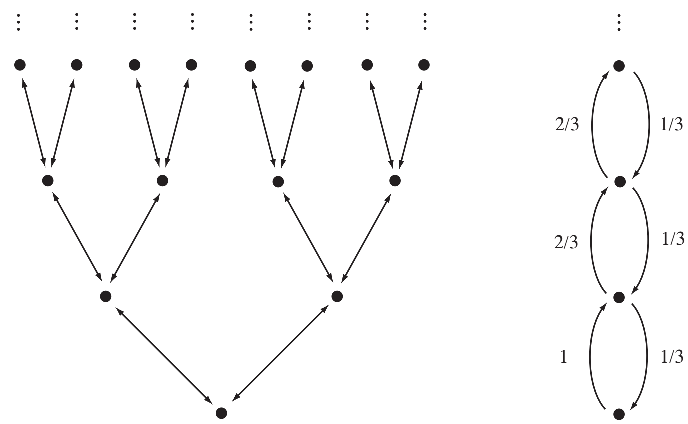

Figure 3: On the left: the rooted tree transition diagram. At each vertex the probabilities of moving to any of the neighboring vertices are equal. The transition diagram on the right shows the probabilities of moving up or down on the set of levels.