

STAT217

HOMEWORK 4 SOLUTIONS

4.1.13 Using the definition of conditional probability, we can rearrange the required probability to get

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbf{P}(X_{n-1} = 2 | X_n = 1) &= \lim_{n \rightarrow \infty} \frac{\mathbf{P}(X_{n-1} = 2, X_n = 1)}{\mathbf{P}(X_n = 1)} \\ &= \lim_{n \rightarrow \infty} \frac{\mathbf{P}(X_n = 1 | X_{n-1} = 2) \mathbf{P}(X_{n-1} = 2)}{\mathbf{P}(X_n = 1)} \\ &= \frac{\mathbf{P}_{21} \times \pi_2}{\pi_1} \end{aligned}$$

Now all we need to find is π_1 and π_2 . We do this by solving the follow equations

$$\begin{aligned} 0.4\pi_0 + 0.6\pi_1 + 0.4\pi_2 &= \pi_0 \\ 0.4\pi_0 + 0.2\pi_1 + 0.2\pi_2 &= \pi_1 \\ 0.2\pi_0 + 0.2\pi_1 + 0.4\pi_2 &= \pi_2 \\ \pi_0 + \pi_1 + \pi_2 &= 1 \end{aligned}$$

This yields $\pi_0 = \frac{11}{24}$, $\pi_1 = \frac{7}{24}$ and $\pi_2 = \frac{6}{24}$. Plugging these back into the expression we derived above, we get

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbf{P}(X_{n-1} = 2 | X_n = 1) &= \frac{\mathbf{P}_{21} \times \pi_2}{\pi_1} \\ &= \frac{0.2 \times \frac{6}{24}}{\frac{7}{24}} \\ &= \frac{12}{70} \\ &= 0.1714 \end{aligned}$$

4.2.5 (a) Letting X_n denote the state of the n th day, we want to find

$$\mathbf{P}(X_5 = (S, S) | X_1 = (S, S)) + \mathbf{P}(X_5 = (C, S) | X_1 = (S, S))$$

We can work this out by calculating the four-step transition matrix $\mathbf{P}^{(4)}$ and reading off the corresponding entries. Therefore, we have

$$\mathbf{P}_{(S,S),(S,S)}^{(4)} + \mathbf{P}_{(S,S),(C,S)}^{(4)} = 0.3421 + 0.1368 = 0.4789$$

(b) We solve the following equations

$$\begin{aligned} 0.7\pi_{(S,S)} + 0.5\pi_{(C,S)} &= \pi_{(S,S)} \\ 0.3\pi_{(S,S)} + 0.5\pi_{(C,S)} &= \pi_{(S,C)} \\ 0.4\pi_{(S,C)} + 0.2\pi_{(C,C)} &= \pi_{(C,S)} \\ 0.6\pi_{(S,C)} + 0.8\pi_{(C,C)} &= \pi_{(C,C)} \\ \pi_{(S,S)} + \pi_{(S,C)} + \pi_{(C,S)} + \pi_{(C,C)} &= 1 \end{aligned}$$

to obtain $\pi_{(S,S)} + \pi_{(C,S)} = 0.25 + 0.15 = 0.40$ as the fraction of sunny days in the long run.

4.3.2 Since recurrence and transience are class properties, an irreducible Markov chain is either recurrent or transient. If the state space is finite, the Markov chain must be recurrent. To see this, suppose that the Markov chain is transient. From the definition of a transient state, starting from any state i , the process will only hit state i a finite number of times. However, since there are only a finite number of states, it is not possible for the process to hit each state only a finite number of times (the process would run out of states to hit). Hence a finite state irreducible Markov chain is recurrent. From Theorem 4.1 of the textbook, we know that for a recurrent aperiodic irreducible Markov chain, $\mathbf{P}_{ij}^{(n)} \rightarrow \pi_j > 0$ as $n \rightarrow \infty$ for all i, j . Thus for each i, j , there exists N_{ij} such that $\mathbf{P}_{ij}^{(n)} > 0$ for all $n > N_{ij}$. Because there are only a finite number of states, $N = \max_{ij} N_{ij}$ is finite, and for $n > N$, we have $\mathbf{P}_{ij}^{(n)} > 0$ for all i, j . This shows that \mathbf{P} is regular, and therefore we have shown that a finite state aperiodic irreducible Markov chain is regular and recurrent.

4.3.3 Equation (3.2) is

$$\mathbf{P}_{ii}^{(n)} = \sum_{k=0}^n f_{ii}^{(k)} \mathbf{P}_{ii}^{(n-k)}$$

for $n \geq 1$. Since we are interested in $f_{00}^{(n)}$, setting $i = 0$, we can rearrange the above equation to get $f_{00}^{(n)}$ on the left-hand side. We'll make use of the fact that $f_{00}^{(0)} = 0$ and $\mathbf{P}_{00}^{(0)} = 1$.

$$\begin{aligned} \mathbf{P}_{00}^{(n)} &= \sum_{k=0}^n f_{00}^{(k)} \mathbf{P}_{00}^{(n-k)} \\ \implies \mathbf{P}_{00}^{(n)} &= f_{00}^{(0)} \mathbf{P}_{00}^{(n)} + \sum_{k=1}^{n-1} f_{00}^{(k)} \mathbf{P}_{00}^{(n-k)} + f_{00}^{(n)} \mathbf{P}_{00}^{(0)} \\ \implies \mathbf{P}_{00}^{(n)} &= \sum_{k=1}^{n-1} f_{00}^{(k)} \mathbf{P}_{00}^{(n-k)} + f_{00}^{(n)} \\ \implies f_{00}^{(n)} &= \mathbf{P}_{00}^{(n)} - \sum_{k=1}^{n-1} f_{00}^{(k)} \mathbf{P}_{00}^{(n-k)} \end{aligned}$$

To calculate $f_{00}^{(n)}$ for $n = 1, 2, 3, 4, 5$, we need to know the corresponding values of $\mathbf{P}_{00}^{(n)}$. Once we have these, we can calculate $f_{00}^{(n)}$ iteratively.

n	1	2	3	4	5
$\mathbf{P}_{00}^{(n)}$	0	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{7}{32}$

$$f_{00}^{(1)} = \mathbf{P}_{00}^{(1)} = 0$$

$$f_{00}^{(2)} = \mathbf{P}_{00}^{(2)} - f_{00}^{(1)} \mathbf{P}_{00}^{(1)} = \frac{1}{4}$$

$$f_{00}^{(3)} = \mathbf{P}_{00}^{(3)} - f_{00}^{(1)} \mathbf{P}_{00}^{(2)} - f_{00}^{(2)} \mathbf{P}_{00}^{(1)} = \frac{1}{8}$$

and so forth ...

$$f_{00}^{(4)} = \frac{5}{16}$$

$$f_{00}^{(5)} = \frac{5}{32}$$

4.4.1 (a) It is clear that $\pi_0 + \pi_1 = 1$, so to verify $\pi = (\pi_0, \pi_1) = (\frac{\beta}{\alpha+\beta}, \frac{\alpha}{\alpha+\beta})$ is a stationary distribution, we simply check that $\pi \mathbf{P} = \pi$.

$$\begin{aligned} \pi \mathbf{P} &= \frac{1}{\alpha + \beta} \begin{bmatrix} \beta & \alpha \end{bmatrix} \begin{bmatrix} 1 - \alpha & \alpha \\ \beta & 1 - \beta \end{bmatrix} \\ &= \frac{1}{\alpha + \beta} \begin{bmatrix} \beta & \alpha \end{bmatrix} \\ &= \pi \end{aligned}$$

(b)

$$\begin{aligned} f_{00}^{(1)} &= \mathbf{P}(\text{Go to 0 at first step}) \\ &= 1 - \alpha \end{aligned}$$

and for $n = 2, 3, \dots$

$$\begin{aligned} f_{00}^{(n)} &= \mathbf{P}(\text{Go to 1 at first step, stay there for } n - 2 \text{ steps, then go to 0}) \\ &= \alpha \times (1 - \beta)^{(n-2)} \times \beta \end{aligned}$$

(c)

$$\begin{aligned}
m_0 &= \sum_{n=1}^{\infty} n f_{00}^{(n)} \\
&= 1f_{00}^{(1)} + 2f_{00}^{(2)} + 3f_{00}^{(3)} \\
&= \sum_{n=1}^{\infty} f_{00}^{(n)} + \sum_{n=2}^{\infty} f_{00}^{(n)} + \sum_{n=3}^{\infty} f_{00}^{(n)} + \dots \\
&= \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} f_{00}^{(n)} \\
&= 1 + \sum_{k=2}^{\infty} \sum_{n=k}^{\infty} f_{00}^{(n)} \\
&= 1 + \alpha\beta \sum_{k=2}^{\infty} \sum_{n=k}^{\infty} (1-\beta)^{n-2} \\
&= 1 + \alpha\beta \sum_{k=2}^{\infty} (1-\beta)^{k-2} \sum_{n=k}^{\infty} (1-\beta)^{n-k} \\
&= 1 + \alpha\beta \sum_{k=2}^{\infty} (1-\beta)^{k-2} \frac{1}{\beta} \\
&= 1 + \alpha \sum_{k=2}^{\infty} (1-\beta)^{k-2} \\
&= 1 + \frac{\alpha}{\beta} \\
&= \frac{\alpha + \beta}{\beta} \\
&= \frac{1}{\pi_0}
\end{aligned}$$

4.4.7 Measure time in *trips*, so there are two trips each day. Let $X_n = 1$ if the car and person are at the same location prior to the n th trip and $X_n = 0$ if not. Then the transition matrix becomes

$$\mathbf{P} = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1-p & p \end{array}$$

Since \mathbf{P} is regular, we can calculate the limiting distribution in the usual way. Therefore we conclude that, in the long run, he is *not* with the car for $\pi_0 = \frac{1-p}{2-p}$ fraction of trips, and walks in rain $\pi_0 p = \frac{p(1-p)}{2-p}$ fraction of trips. Note that on any given day, there can only be at most one trip in which he walks in the rain. Therefore, the number of trips in which he walks in the rain is equal to the number of days he walks in the rain. Also, the total number of days is half the number of trips. Therefore, the total fraction of *days* he walks in the rain is $\frac{2p(1-p)}{2-p}$.

If he owns two cars, let X_n be the number of cars at the location of the person. The corresponding

transition matrix is

$$\mathbf{P} = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0 & 0 & 1 \\ 1 & 0 & 1-p & p \\ 2 & 1-p & p & 0 \end{array}$$

In this situation we obtain $\pi_0 = \frac{1-p}{3-p}$ and therefore in the long run, the fraction of days he walks in the rain is $2p\pi_0 = \frac{2p(1-p)}{3-p}$.

4.5.2 There are two answers to this question, depending on whether you fixed the typo in the textbook.

(a) **Not Fixing Typo**

So there are two communication classes: $A = \{0, \dots, 4\}$, which is transient and $B = \{5, 6, 7\}$, which is recurrent. We can use the usual method to determining the stationary distribution in B which turns out to be $(\pi_5, \pi_6, \pi_7) = (0.45, 0.26, 0.29)$. The next step is to calculate the hitting probabilities from the transient states to the recurrent classes. This is where the typo makes this question easy, since the hitting probabilities from the transient states to the recurrent class will all be 1, since there is only one recurrent class. Therefore the limiting behavior of the Markov Chain is described in the following matrix.

$$\mathbf{P}^\infty = \begin{array}{c|cccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0.45 & 0.26 & 0.29 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0.45 & 0.26 & 0.29 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0.45 & 0.26 & 0.29 \\ 3 & 0 & 0 & 0 & 0 & 0 & 0.45 & 0.26 & 0.29 \\ 4 & 0 & 0 & 0 & 0 & 0 & 0.45 & 0.26 & 0.29 \\ 5 & 0 & 0 & 0 & 0 & 0 & 0.45 & 0.26 & 0.29 \\ 6 & 0 & 0 & 0 & 0 & 0 & 0.45 & 0.26 & 0.29 \\ 7 & 0 & 0 & 0 & 0 & 0 & 0.45 & 0.26 & 0.29 \end{array}$$

(b) **Fixing Typo**

In this case, there are three communication classes: $A = \{0, 1, 2\}$, which is transient, $B = \{3, 4\}$ and $C = \{5, 6, 7\}$, both of which are recurrent. Again we can find the stationary distributions in B and C which turn out to be $(\pi_3, \pi_4) = (0.46, 0.54)$ and $(\pi_5, \pi_6, \pi_7) = (0.45, 0.26, 0.29)$ respectively. The next step is to calculate the hitting probabilities from the transient states to the recurrent classes. For the transient class A , let u_i^B , for $i = 0, 1, 2$ be the probability of ultimate absorption in class B . Using first step analysis, and following the example on page 262 of the text book, we obtain the following equations.

$$\begin{aligned} u_0^B &= 0.1u_0^B + 0.2u_1^B + 0.1u_2^B + 0.3 \\ u_1^B &= 0.1u_1^B + 0.2u_2^B + 0.1 \\ u_2^B &= 0.5u_0^B + 0.3 \end{aligned}$$

This yields $(u_0^B, u_1^B, u_2^B) = (0.44, 0.23, 0.52)$. Starting in a transient state, the only possibilities are that we get absorbed in class B or absorbed in class C . Therefore, $u_i^C = 1 - u_i^B$, and we obtain $(u_0^C, u_1^C, u_2^C) = (0.56, 0.77, 0.48)$. Thus we have

$$\mathbf{P}^\infty = \begin{array}{c|ccccccccc}
& 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
0 & 0 & 0 & 0 & (u_0^B)(\pi_3) & (u_0^B)(\pi_4) & (u_0^C)(\pi_5) & (u_0^C)(\pi_6) & (u_0^C)(\pi_7) \\
1 & 0 & 0 & 0 & (u_1^B)(\pi_3) & (u_1^B)(\pi_4) & (u_1^C)(\pi_5) & (u_1^C)(\pi_6) & (u_1^C)(\pi_7) \\
2 & 0 & 0 & 0 & (u_2^B)(\pi_3) & (u_2^B)(\pi_4) & (u_2^C)(\pi_5) & (u_2^C)(\pi_6) & (u_2^C)(\pi_7) \\
3 & 0 & 0 & 0 & 0.46 & 0.54 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 0.46 & 0.54 & 0 & 0 & 0 \\
5 & 0 & 0 & 0 & 0 & 0 & 0.45 & 0.26 & 0.29 \\
6 & 0 & 0 & 0 & 0 & 0 & 0.45 & 0.26 & 0.29 \\
7 & 0 & 0 & 0 & 0 & 0 & 0.45 & 0.26 & 0.29
\end{array}$$

That is,

$$\mathbf{P}^\infty = \begin{array}{c|ccccccccc}
& 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
0 & 0 & 0 & 0 & 0.20 & 0.24 & 0.25 & 0.15 & 0.16 \\
1 & 0 & 0 & 0 & 0.10 & 0.12 & 0.35 & 0.20 & 0.22 \\
2 & 0 & 0 & 0 & 0.24 & 0.28 & 0.21 & 0.12 & 0.14 \\
3 & 0 & 0 & 0 & 0.46 & 0.54 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 0.46 & 0.54 & 0 & 0 & 0 \\
5 & 0 & 0 & 0 & 0 & 0 & 0.45 & 0.26 & 0.29 \\
6 & 0 & 0 & 0 & 0 & 0 & 0.45 & 0.26 & 0.29 \\
7 & 0 & 0 & 0 & 0 & 0 & 0.45 & 0.26 & 0.29
\end{array}$$