Continuous time Markov chains (week 7)
Solutions

1 Poisson process.

A Write a Matlab function to generate an arrival process. Use $T = 10$ minutes, $\lambda = 1$ customer per minute, and $n = 10^3$. Compare the histogram of $N(T)$ obtained from 10^4 experiments with the Poisson pmf with parameter λT. Compare also the histogram of $N(T)/2$ with the Poisson pmf with parameter $\lambda T/2$.
The Matlab code follows. The results are shown in fig. 1

```
clc; clear all; close all
T=10;       %minutes
lambda= 1; %customers per minute;
nr_experiments=10^4;
n=1000;

h=T/n;
p = lambda*h;

% Generate arrivals for all times and experiments
arrival = binornd(1,p,n,nr_experiments);

% Compare with Poisson pmfs
x=0:30;
pdf_approx = hist(sum(arrival),x)/nr_experiments;
bar(x,pdf_approx,'r')
hold on
plot(x,poisspdf(x,lambda*T),'b','Linewidth', 2)
xlabel('t','Fontsize',12)
ylabel('pmf','Fontsize',12)
title('pmf of number of arrivals for a Poisson Process of \lambda=1, and for T=10'
legend('Estimated','Calculated','Location','Best')

figure
pdf_approx = hist(sum(arrival(1:n/2,:)),x)/nr_experiments;
bar(x,pdf_approx,'r')
hold on
plot(x,poisspdf(x,lambda*T/2),'b','Linewidth', 2)
xlabel('t','Fontsize',12)
ylabel('pmf','Fontsize',12)
title('pmf of number of arrivals for a Poisson Process of \lambda=1, and for T=5',
legend('Estimated','Calculated','Location','Best')
```

B The comparisons in Part A should have yielded accurate fits. Use the Poisson approximation of the binomial distribution to justify why this is true. Argue that this implies that the pmf of $N(t)$ is Poisson with parameter λt for all t, i,.e., We want to show:

$$\mathrm{P}\left\{N(t)=k\right\} = e^{-\lambda t}\frac{(\lambda t)^k}{k!} \tag{1}$$

The probability distribution of N(T) is binomial with parameter $(n, \lambda T/n)$, and can be approximated by the Poisson distribution with parameter $(n)(\lambda T/n) = \lambda T$. Therefore, for a large enough n,

$$P\{N(t)=k\} = e^{-\lambda t}\frac{(\lambda t)^k}{k!} \approx \binom{n}{k}(\frac{\lambda t}{n})^k(1-\lambda t/n)^{(n-k)}$$

We need to prove that the limit of the above binomial distribution as n$\rightarrow \infty$ equals the the above Poisson distribution. Now, taking the limit as $n \rightarrow \infty$:

$$\begin{aligned}
\mathrm{P}\left\{N(t)=k\right\} &= \lim_{n\to\infty}\frac{n!}{k!(n-k)!}\frac{(\lambda t)^k}{n^k}(1-\lambda t/n)^{n-k} \\
&= \lim_{n\to\infty}\frac{n!}{(n-k)!n^k} \times \frac{(\lambda t)^k}{k!} \times \lim_{n\to\infty}(1-\frac{\lambda t}{n})^{n-k} \\
&= 1 \times \frac{(\lambda t)^k}{k!} \times \lim_{n\to\infty}\left[\left[\left(1-\frac{\lambda t}{n}\right)^{\frac{n}{\lambda t}}\right]^{\frac{n-k}{n}}\right]^{\lambda t} \\
&= \frac{(\lambda t)^k}{k!} \times \left[\lim_{n\to\infty}\left(1-\frac{\lambda t}{n}\right)^{\frac{n}{\lambda t}}\right]^{\lambda t} \\
&= \frac{(\lambda t)^k}{k!} \times (e^{-1})^{\lambda t} = \boxed{e^{-\lambda t}\frac{(\lambda t)^k}{k!}}
\end{aligned}$$

C Compute a cumulative histogram of the time T_1 until the first arrival to estimate the cdf of T_1. Compare with the exponential pdf with parameter λ. A good fit should be observed.
The Matlab code follows (to be appended to the code of part A)

```
% Compute time of first arrival : Method 1
first_arrival_times=n*ones(1,nr_experiments);
nr_arrived=cumsum(arrival);
for i=1:nr_experiments
    temp=find(nr_arrived(:,i),1);
    if ~isempty(temp)
        first_arrival_times(1,i)=temp;
    end
end
hist_firs_arrival_times=hist(first_arrival_times,1:n);

% Compute time of first arrival : Method 2
time=0;
experiment=1;
time_histogram = zeros(n,1);
while (experiment <= nr_experiments) && (time < n)
    time = time+1;
    if arrival(time, experiment)
        time_histogram(time)=time_histogram(time)+1;
        experiment = experiment+1;
```

```
        time=0;
    end
end

%Compare with exponential pdf
figure
plot((1:n)*h,time_histogram/nr_experiments/h,'r')
hold on
plot((1:n)*h,exppdf((1:n)*h,lambda),'b','Linewidth', 2)
xlabel('t','Fontsize',12)
ylabel('pdf','Fontsize',12)
title('pdf of first arrival time for a Poisson Process of \lambda=1, Method 1','Fo
legend('Estimated','Calculated','Location','Best')


figure
plot((1:n)*h,hist_firs_arrival_times/nr_experiments/h,'r')
hold on
plot((1:n)*h,exppdf((1:n)*h,lambda),'b','Linewidth', 2)
xlabel('t','Fontsize',12)
ylabel('pdf','Fontsize',12)
title('pdf of first arrival time for a Poisson Process of \lambda=1, Method 2','Fo
legend('Estimated','Calculated','Location','Best')


figure
plot((1:n)*h,cumsum(time_histogram/nr_experiments),'r')
hold on
plot((1:n)*h,expcdf((1:n)*h,lambda),'b','Linewidth', 1)
xlabel('t','Fontsize',12)
ylabel('cdf','Fontsize',12)
title('cdf of first arrival time for a Poisson Process of \lambda=1','Fontsize',12
legend('Estimated','Calculated','Location','Best')
```

Figures 2(a) and 2(b) reveal a close fit with the exponential distribution.

D Use the fact that the probability of having no arrivals by time t is approximately given – according to (1) – by $e^{-\lambda t}$ to explain the good fit observed in part C. Notice that we have $T_1 > t$ if and only if there are no arrivals by time t.

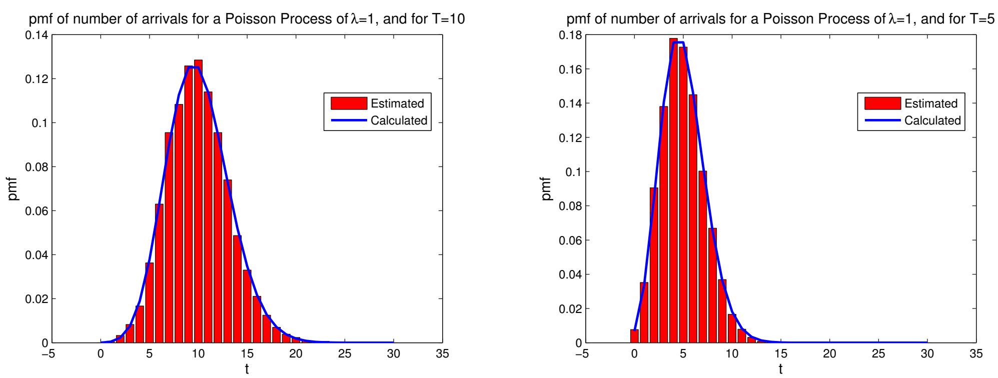
The probability of having no arrivals by time t ($P\{N(t) = 0\}$), is approximately $e^{-\lambda t}$, and $P\{N(t) = 0\}) = P\{T_1 > t\}$ because N(t) is only 0 if t is shorter than the time needed for the first arrival. We also know the cdf of T_1 represents the probability that at time t (along the x-axis), the first arrival has already happened. Using these facts we can explain the close fit observed in Part C by proving $P\{N(t) = 0\} = e^{-\lambda t}$ using the binomial distribution and the exponential distribution.

Using the binomial distribution:

$$
\begin{aligned}
P\{N(t)=0\} &= \lim_{n\to\infty} \frac{n!}{0!(n-0)!}(\frac{\lambda t}{n})^0(1-\frac{\lambda t}{n})^{(n-0)} \\
&= \lim_{n\to\infty} 1 \times 1 \times (1-\frac{\lambda t}{n})^{(n)} \\
P\{N(t)=0\} &= e^{-\lambda t}
\end{aligned}
$$

Using the exponential distribution:

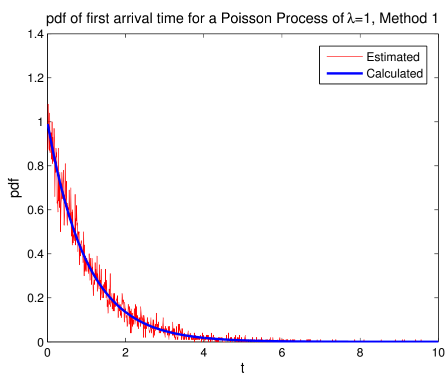
$$
\begin{aligned}
P\{N(t)=0\} &= P\{T_1 > t\} \\
&= 1 - P\{T_1 \leq t\} \\
&= 1 - P\{\text{by time } t\text{, the first arrival has occurred}\} \\
&= 1 - F(t;\lambda) \\
&= 1 - (1-e^{-\lambda t}) \\
P\{N(t)=0\} &= e^{-\lambda t}
\end{aligned}
$$



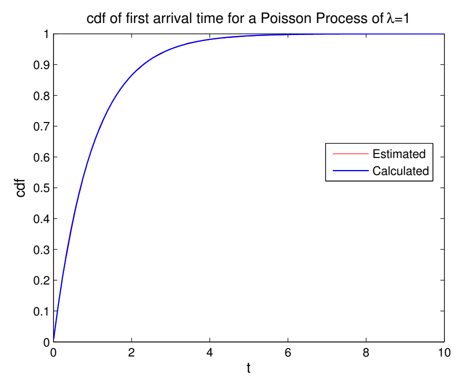
(a) $T = 10$

(b) $T = 10/2$

Fig. 1. Compararison between the histogram of $N(T)$ obtained from 10^4 experiments with the Poisson pmf with parameter λT for $T = 10$ minutes, $\lambda = 1$ customer per minute, and $n = 10^3$ (subfig. (a)). Comparison of histogram of $N(T)/2$ with the Poisson pmf with parameter $\lambda T/2$ (subfig. (b)).



(a) pdf



(b) cdf

Fig. 2. Comparison between the pdf (subfig. (a)) and cdf (subfig. (b)) of an exponential r.v. and the corresponding values based on estimation.