Advanced Engineering Mathematics

School of Emerging Technologies, IUST   Homework 1

Prob. 1: Find the function $y(x)$ for extermal (minimum of maximum) of

$$J(y, y') = \int_{x_0}^{x_1} \frac{1+y^2}{y'^2}\,dx$$

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Prob. 2: Fermat's principle of optics states that a light ray will follow the path $y(x)$ for which

$$\int_{(x_1,y_1)}^{(x_2,y_2)} n(y,x)\,ds$$

is a minimum when n is the index of refraction an ds is differential path element.

For $y_2 = y_1 = 1$ and $-x_1 = x_2 = 1$, find the ray path if $n = e^{y(x)}$.

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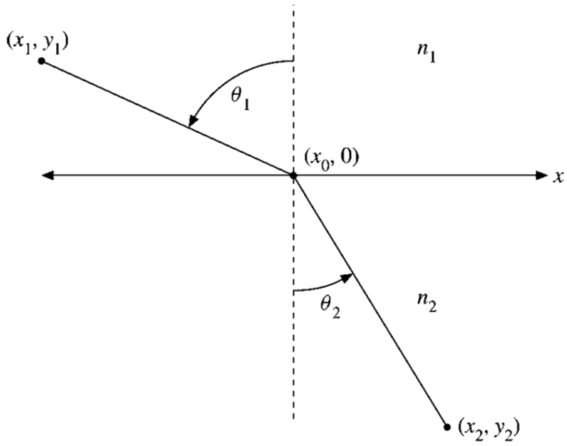
Prob. 3: A ray of light follows a straight-line path in a first homogeneous medium, is refracted at an interface, and then follows a new straight-line path in the second medium. Use Fermat's principle of optics to derive Snell's law of refraction:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

Hint. Keep the points (x_1, y_1) and (x_2, y_2) fixed and vary x_0 to satisfy Fermat (Fig. 1). This is not an Euler equation problem. (The light path is not differentiable at x_0.)



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Prob. 4: Find the function u(x, y, z) for extermal (minimum of maximum) of

$$J\left(u(x,y,z)\right) = \iiint_D \left[u_x^{\,2} + u_y^{\,2} + u_z^{\,2} + 2u\rho(x,y,z)\right] dxdydz$$

where $\rho(x, y, z)$ is a given (known) function.

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Prob. 5: The Lagrangian for a moving particle with mass m, charge q, and velocity vector $\boldsymbol{v}$ in an electromagnetic field described by scalar potential φ and vector potential $\boldsymbol{A}$ is

$$L = \frac{1}{2}mv^2 - q\varphi + q\boldsymbol{A}.\boldsymbol{v}$$

Find the equation of motion of the charged particle.

Hint. $\dfrac{d}{dt}A_j = \dfrac{\partial A_j}{\partial t} + \sum_i \dfrac{\partial A_j}{\partial x_i}\dot{x}_i$. The dependence of the force fields $\boldsymbol{E}$ and $\boldsymbol{B}$ upon the potentials φ and $\boldsymbol{A}$ is developed in Section 1.13 of the text book (compare Exercise 1.13.10).

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Prob. 6: Show that requiring J, given by

$$J = \int_a^b \int_a^b K(x,t)\,\varphi(x)\,\varphi(t)\,dx\,dt$$

to have a stationary value subject to the normalizing condition

$$\int_a^b \varphi^2(x)\,dx = 1$$

leads to the Hilbert–Schmidt integral equation,

$$\varphi(x) = \lambda \int_a^b K(x,t)\,\varphi(t)\,dt$$

Note. The kernel $K(x, t)$ is symmetric.

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Prob. 6: Find the functional associated with the homogeneous wave equation,

$$\nabla^2 \varphi + k^2 \varphi = 0$$

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Prob. 7: Using the Rayleigh-Ritz method, solve Laplace's equation:

$$\nabla^2 V(x, y) = 0$$

in a square $-1 \le x, y \le 1$, subject to the boundary conditions

$V(\pm 1, y) = 0$, $V(x, -1) = 0$, and $V(x, 1) = 1$.

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Vahid Nayyeri