Advanced Engineering Mathematics

Homework 2

Prob. 1: Consider the following ODE with the given boundary values:

$$x^2 \frac{d^2\varphi}{dx^2} + x\frac{d\varphi}{dx} + \lambda\varphi = 0; \quad \varphi(1) = 0; \quad \varphi(b) = 0;$$

a) Transform the equation to Sturm-Liouville form.
b) Prove that $\lambda \geq 0$.
c) Find the first (minimum) eigenvalue.
d) Show that the eigenfunctions are orthogonal with respect to a weighting function w(x).
e) Determine the roots of eigenfunctions in [a, b]. How many roots does have each eigenfunction?

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Prob. 2: Using RQ (Rayleigh quotient), find an upper band (with reasonable accuracy) for the first eigenvalue of each following problems:

a) $\frac{d^2\varphi}{dx^2} + (\lambda - x^2)\varphi = 0; \quad \frac{d\varphi}{dx}(0) = 0; \quad \varphi(1) = 0.$

b) $\frac{d^2\varphi}{dx^2} + (\lambda - x)\varphi = 0; \quad \frac{d\varphi}{dx}(0) = 0; \quad \frac{d\varphi}{dx}(1) + 2\varphi(1) = 0.$

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Prob. 3: Consider Laguerre's ODE equation

$$x\frac{d^2\varphi}{dx^2} + (1 - x)\frac{d\varphi}{dx} + n\varphi = 0$$

in [0, +∞). Put the equation into the self-adjoint form.

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Prob. 4: Find the eigenvalues and normalized eigenfunctions for the following problem

$$\frac{d^2\varphi}{dx^2} + \lambda\varphi = 0; \quad \varphi(0) = 0; \quad \varphi(1) - \frac{d\varphi}{dx}(1) = 0;$$

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Prob. 5: Consider the fourth-order differential operator, $L = \frac{d^4}{dx^4}$.

a) Show

$$\int_0^1 (u\,Lv - v\,Lu)\,dx = 0$$

where u and v satisfy the boundary conditions: $\varphi(0) = \varphi(1) = \varphi'(0) = \varphi'(1) = 0$.

b) Show the eigenfunctions of equation $L\varphi + \lambda e^x\varphi = 0$ with the given boundary conditions are orthogonal.

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Prob. 6: The eigenvalue for operator ∇^2 (i.e. the eigenvalues of equation $\nabla^2\psi + \lambda\,\psi = 0$) in a multi-dimensions region is

$$\lambda = \frac{\int_V |\nabla \psi|^2 dv}{\int_V |\psi|^2 dv}$$

Using the calculus of variation and an appropriate function, find a reasonable approximation for the minimum (first) eigenvalue of ∇^2 with zero boundary values in 2-D region $0 \le \varphi \le \pi/3$, $1 \le r \le 2$.

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Prob. 7: Show that

$$y(x) = \int_0^1 g(x, x', \lambda)\, f(x')\, dx'$$

is the solution of inhomogeneous Sturm-Liouville equation, $[L + \lambda r]\, y(x) = f(x)$ where $g(x, x', \lambda)$ can be stated in terms of $y_1(x)$ and $y_2(x)$, solutions of homogeneous equation $[L + \lambda r]\, y(x) = 0$ which respectively satisfy the left and right boundary conditions, i.e. $\alpha_1 y_1(a) + \alpha_2 y_1{}'(a) = 0$ and $\beta_1 y_2(b) + \beta_2 y_2{}'(b) = 0$.

Hint: Substitute the given relation for y(x) in inhomogeneous Sturm-Liouville equation.

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Prob. 8: Show that

$$h(x) = \lim_{\varepsilon \to 0} \frac{\varepsilon}{\pi(\varepsilon^2 + x^2)}$$

has the properties of Dirac distribution.

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Prob. 9: Show that

$$h(x) = \lim_{\varepsilon \to 0} \frac{\sin(\alpha x)}{\pi x}$$

has the properties of Dirac distribution. Especially show that $\int_{-\infty}^{-\infty} h(x)dx = 1$.

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Prob. 10: Using a test function $f(x)$ prove that:

$$x\delta'(x) = -\delta(x)$$

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Prob. 11: Using a test function $f(x)$ prove that:

$$\nabla^2 \ln|\bar{x}| = 2\pi\delta(\bar{x}) \text{ where } \bar{x} = (x_1, x_2)$$