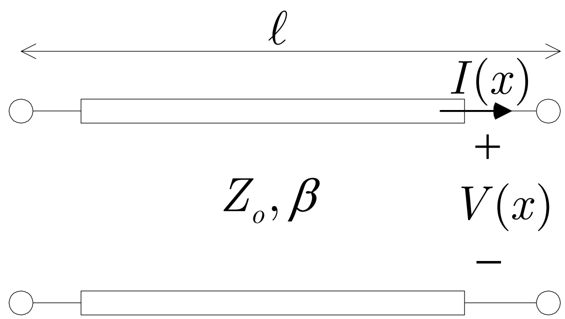
Transmission Line Review



$$Z_o = \frac{V^+(x)}{I^+(x)} = \sqrt{\frac{L}{C}} = \text{Characteristic Impedance}$$

$$\beta = \frac{2\pi}{\lambda} = \omega\sqrt{LC} = \text{the wave number}$$

$$V(x) = V^+(x) + V^-(x) = \text{the line voltage}, \quad I(x) = I^+(x) + I^-(x) = \text{the line current}$$

$V(x)$ and $I(x)$ satisfy the ***Transmission Line equations*:**

$$1)\ \frac{\partial V(x,t)}{\partial x} = -L\frac{\partial I(x,t)}{\partial t}, \quad 2)\ \frac{\partial I(x,t)}{\partial x} = -C\frac{\partial V(x,t)}{\partial t}$$

where:

$L =$ the inductance per unit length of the line (H/m)

$C =$ the capacitance per unit length of the line (F/m)

Assuming a CW excitation ($\cos(\omega t)$), the line voltages and currents are expressed as phasors:

$$V(x,t) = \mathrm{Re}\left\{\tilde{V}(x)e^{j\omega t}\right\}, \quad I(x,t) = \mathrm{Re}\left\{\tilde{I}(x)e^{j\omega t}\right\}$$

Subsequently, the Transmission line equations can be expressed in the phasor domain as:

$$3)\ \frac{\partial V(x)}{\partial x} = -j\omega L I(x), \quad 4)\ \frac{\partial I(x)}{\partial x} = -j\omega C V(x)$$

From these equations, we can formulate the Wave equation. That is, differentiate 3) with respect to *x*:

$$\frac{\partial^2 V(x)}{\partial x^2} = -j\omega L \frac{\partial I(x)}{\partial x}$$

Then, from 4):

$$5)\ \frac{\partial^2 V(x)}{\partial x^2} + \omega^2 LC\, V(x) = 0 \quad \rightarrow \textit{Scalar Wave Equation}$$

Similarly, if we started with 4):

$$6)\ \frac{\partial^2 I(x)}{\partial x^2} + \omega^2 LC\, I(x) = 0$$

$$\frac{\partial^2 V(x)}{\partial x^2} + \omega^2 LC\,V(x) = 0$$

Solutions to the Scalar Wave Equation:

7) $V(x) = V_o^+ e^{-j\beta x} + V_o^- e^{+j\beta x}$

where,

$$\beta = \omega\sqrt{LC} = \frac{\omega}{c},\ c = \frac{1}{\sqrt{LC}} = \text{ wave speed}$$

$$V_o^+ e^{-j\beta x} \rightarrow \text{ Forward traveling wave}$$

$$V_o^- e^{+j\beta x} \rightarrow \text{ Backward traveling wave}$$

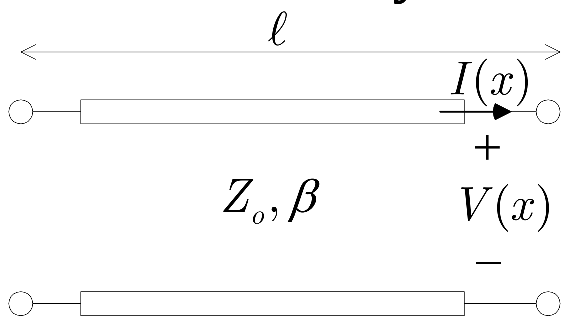
Once the solution to *V(x)* is known, *I(x)* can be determined from 4). To this end:

$$I(x) = \sqrt{\frac{C}{L}} V_o^+ e^{-j\beta x} - \sqrt{\frac{C}{L}} V_o^- e^{+j\beta x}$$

where

$$\sqrt{\frac{L}{C}} = Z_o = \text{ characteristic wave impedance } (\Omega)$$

Summary

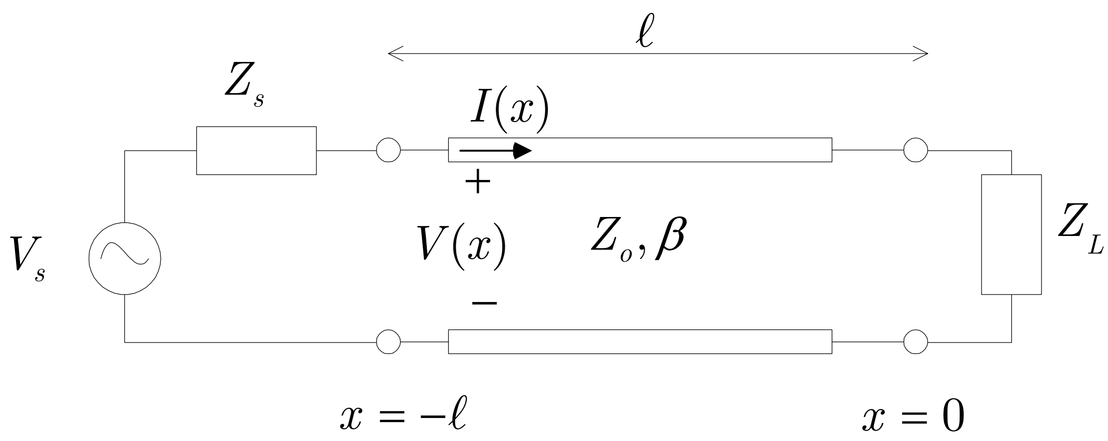


Given *L* and *C* determined by the physics of the Tx-line:

$$V(x) = V_o^+ e^{-j\beta x} + V_o^- e^{+j\beta x}$$
$$I(x) = \frac{V_o^+}{Z_o} e^{-j\beta x} - \frac{V_o^-}{Z_o} e^{+j\beta x}$$
$$Z_o = \sqrt{\frac{L}{C}}, \quad \beta = \omega\sqrt{LC} = \frac{\omega}{c} = \frac{2\pi}{\lambda}$$

- The line voltage and currents are a superposition of *forward and backward traveling waves*
- The ratio of the forward traveling voltage and current is the characteristic impedance
- The amplitudes of the forward and backward waves are determined by the source and load conditions of the line

Source and Loading of Tx-Lines



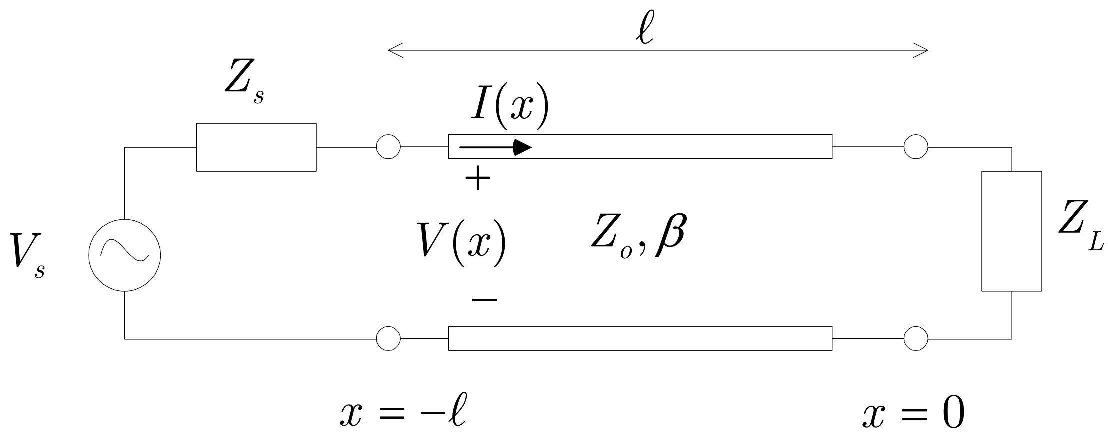
Tx-line excited by voltage source V_s operating at frequency ω and internal impedance Z_s. The line is terminated by the load Z_g. The source will excite the line voltage. The load will result in reflection of the voltage and current waves, leading to standing waves on the line in the steady state condition.

Question: How do we solve for the amplitudes of the forward and backward traveling waves?
Boundary conditions:

1) Impedance Match: $(x=0): \quad \frac{V(0)}{I(0)} = Z_L$

$(x=-\ell): \quad \frac{V(-\ell)}{I(-\ell)} = Z_{in}$

2) Voltage continuity: $V(-\ell^-) = V(-\ell^+)$



Pose Line voltage and current as:

$$V(x) = V_o^+ e^{-j\beta x}\left(1 + \Gamma_L e^{j2\beta x}\right),$$

$$I(x) = \frac{V_o^+}{Z_o} e^{-j\beta x}\left(1 - \Gamma_L e^{j2\beta x}\right)$$

where:

$$\Gamma_L = \frac{V_o^-}{V_o^+} = \textit{reflection coefficient due to the load}$$

Then, enforce the impedance match condition at the load:

$$Z_L = \frac{V(0)}{I(0)} = \left.\frac{V_o^+ e^{-j\beta x}\left(1 + \Gamma_L e^{j2\beta x}\right)}{\frac{V_o^+}{Z_o} e^{-j\beta x}\left(1 - \Gamma_L e^{j2\beta x}\right)}\right|_{x=0} = Z_o \frac{\left(1 + \Gamma_L\right)}{\left(1 - \Gamma_L\right)} \Rightarrow \Gamma_L = \frac{Z_L - Z_o}{Z_L + Z_o}$$

To determine V_o^+,use an impedance match at $x = -\ell$ and enforce voltage continuity.

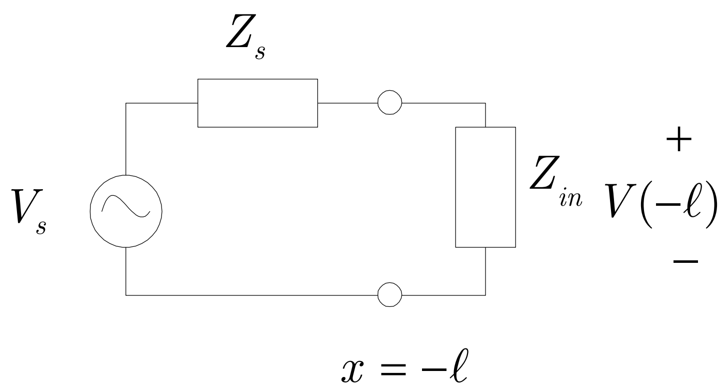
Input impedance:

$$Z_{in}=\frac{V(-\ell)}{I(-\ell)}=\left.\frac{V_o^+e^{-j\beta x}\left(1+\Gamma_L e^{j2\beta x}\right)}{\frac{V_o^+}{Z_o}e^{-j\beta x}\left(1-\Gamma_L e^{j2\beta x}\right)}\right|_{x=-\ell}=Z_o\frac{\left(1+\Gamma_L e^{-j2\beta\ell}\right)}{\left(1-\Gamma_L e^{-j2\beta\ell}\right)}$$

Substituting in for Γ_L, and applying some simple trigonometric identities:

$$Z_{in}=Z_o\frac{\left(1+\Gamma_L e^{-j2\beta\ell}\right)}{\left(1-\Gamma_L e^{-j2\beta\ell}\right)}=Z_o\frac{Z_L+jZ_o\tan(\beta\ell)}{Z_o+jZ_L\tan(\beta\ell)}$$

Given Z_{in}, one can now determine the voltage $V(-\ell)$ using an equivalent circuit:



Applying voltage division and then voltage continuity:

$$V(-\ell)=V_s\frac{Z_{in}}{Z_s+Z_{in}},\qquad V(-\ell)=V_s\frac{Z_{in}}{Z_s+Z_{in}}=V_o^+e^{j\beta\ell}\left(1+\Gamma_L e^{-j2\beta\ell}\right)$$

$$\Rightarrow V_o^+=\frac{V_s}{e^{j\beta\ell}\left(1+\Gamma_L e^{-j2\beta\ell}\right)}\frac{Z_{in}}{Z_s+Z_{in}}$$

Definitions:

Define the generalized reflection coefficient as:

$$\Gamma(x) = \Gamma_L e^{2j\beta x}, \text{ where } \quad \Gamma_L = \frac{Z_L - Z_o}{Z_L + Z_o}$$

Define the generalized line impedance:

$$Z(x) = \frac{V(x)}{I(x)} = Z_o \frac{1 + \Gamma_L e^{2j\beta x}}{1 - \Gamma_L e^{2j\beta x}} = Z_o \frac{1 + \Gamma(x)}{1 - \Gamma(x)}$$

Note that:

$$\Gamma(0) = \Gamma_L \;\; \& \; Z(0) = Z_L, \;\; Z(-\ell) = Z_{in}$$

Power:

Input:

$$P_{in} = \frac{1}{2}\text{Re}\{V_{in} I_{in}^*\} = \frac{1}{2}\text{Re}\left\{V_{in}\left(\frac{V_{in}}{Z_{in}}\right)^*\right\} = \frac{1}{2}\text{Re}\{Z_{in}\}\left|\frac{V_s}{Z_s + Z_{in}}\right|^2$$

Delivered to the load:

$$P_L = \frac{1}{2}\text{Re}\{V_L I_L^*\} = \frac{1}{2}\text{Re}\left\{V_L\left(\frac{V_L}{Z_L}\right)^*\right\} = \frac{1}{2}\text{Re}\{Z_L\}\left|\frac{V_L}{Z_L}\right|^2$$

For a lossless line: $P_{in} = P_L$.

Example

Given a 50 Ω transmission line that is 0.25 λ long excited by a 1 V voltage source at 300 MHz frequency with an internal impedance of 100 Ω, and the line is terminated by a load $Z_L = 100 - j40\,\Omega$, determine $\Gamma_L, Z_{in}, V_{in}, V_o^+$, the power delivered by the source, and the power delivered to the load.

Solution:

$Zo := 50 \quad L := 0.25 \quad ZL := 100 - 40j \quad Zs := 100 \quad Vs := 1 \quad \beta L := 2\cdot\pi\cdot L$

$\Gamma := \frac{ZL - Zo}{ZL + Zo} \quad \Gamma = 0.378 - 0.166i \quad |\Gamma| = 0.412 \quad \arg(\Gamma) = -0.414 \quad Zin := Zo\cdot\frac{\left(1 + \Gamma\cdot e^{2j\cdot\beta L}\right)}{\left(1 - \Gamma\cdot e^{2j\cdot\beta L}\right)} \quad Zin = 21.552 + 8.621i$

$Zin2 := Zo\cdot\frac{(ZL + j\cdot Zo\cdot\tan(\beta L))}{(Zo + j\cdot ZL\cdot\tan(\beta L))} \quad Zin2 = 21.552 + 8.621i$

$Vin := Vs\cdot\frac{Zin}{Zs + Zin} \quad Vin = 0.181 + 0.058i \quad Vop := \frac{Vin}{e^{j\cdot\beta L}\cdot\left(1 + \Gamma\cdot e^{-2j\cdot\beta L}\right)} \quad Vop = 0.015 - 0.295i \quad Pin := \frac{1}{2}\cdot \mathrm{Re}\left(Vin\cdot\overline{\left(\frac{Vin}{Zin}\right)}\right) \quad Pin = 7.257\times10^{-4}$

$VL := Vop\cdot(1 + \Gamma) \quad VL = -0.029 - 0.409i \quad PL := \frac{1}{2}\cdot \mathrm{Re}\left(VL\cdot\overline{\left(\frac{VL}{ZL}\right)}\right) \quad PL = 7.257\times10^{-4}$

Note that here it is observed that *Pin* = *PL*. Is this expected? Why?

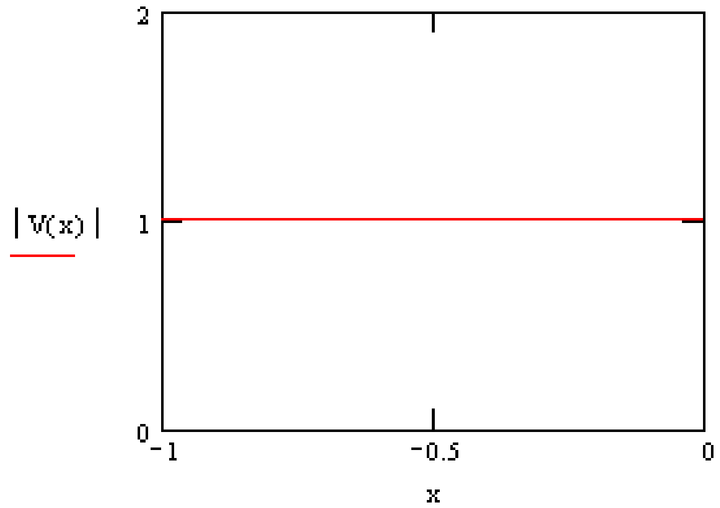
Standing Waves

Consider a uniform line terminated by load Z_L. Consider the voltage distribution over the line:

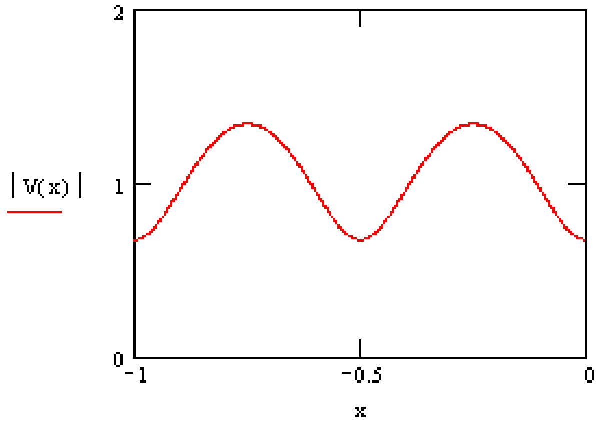
$$|V(x)| = \left|V_o^+ e^{-j\beta x}\left(1+\Gamma e^{j2\beta x}\right)\right| = |V_o^+|\left|1+\Gamma e^{j2\beta x}\right|$$

where: $\Gamma = \dfrac{Z_L - Z_o}{Z_L + Z_o}$

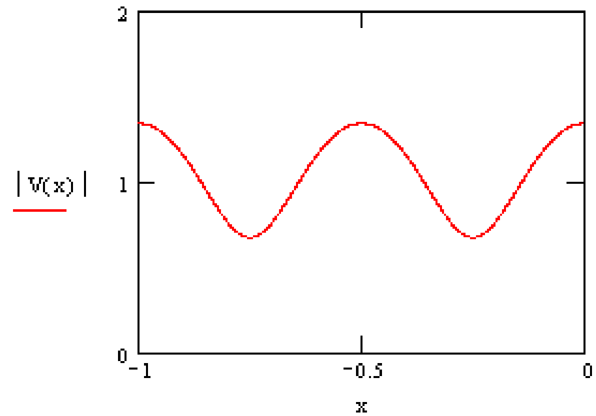
$Z_L = Z_o$:



$Z_L = Z_o / 2$

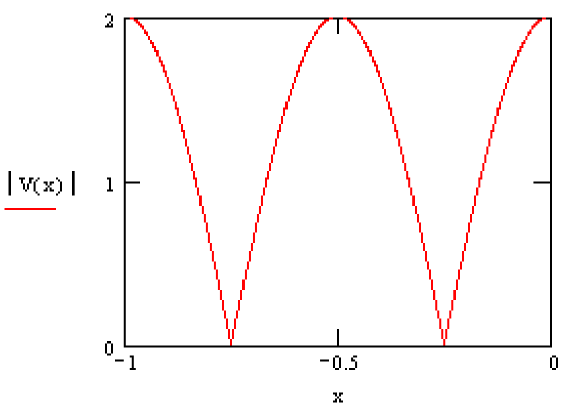


$Z_L = 2Z_o$

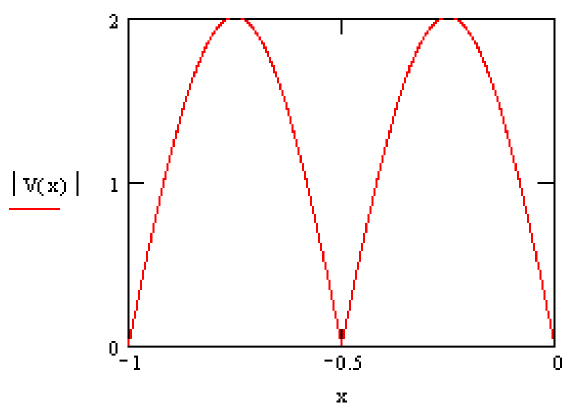


Where do the maximum and minimum voltages occur?

$Z_L = \infty (open)$



$Z_L = 0 (short)$



locate the voltage maximum and minimums? What are their values and separations? What are the distances between maxs and mins?

Standing Wave Ration (SWR)

Definition: $|V(x)|$ is characterized by the *Standing Wave Ratio:*

$$SWR = \frac{V_{\max}}{V_{\min}}$$

$$V_{\max} = \max\left(\left|V_o^+\left(1+\Gamma_L e^{2j\beta x}\right)\right|\right) = \left|V_o^+\right|\left(1+\left|\Gamma_L\right|\right)$$

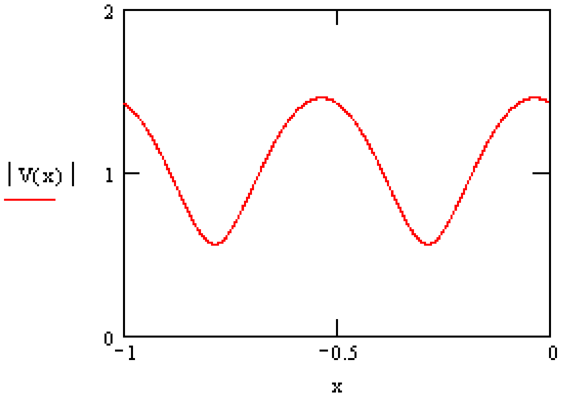
$$V_{\min} = \min\left(\left|V_o^+\left(1+\Gamma_L e^{2j\beta x}\right)\right|\right) = \left|V_o^+\right|\left(1-\left|\Gamma_L\right|\right)$$

$$\therefore SWR = \frac{1+\left|\Gamma_L\right|}{1-\left|\Gamma_L\right|}$$

Z_L / Z_o	Γ	SWR
0	-1	∞
0.25	-0.6	4
0.5	-0.3333	2
0.75	-0.1429	1.333333
1	0	1
2	0.33333	2
4	0.6	4
∞	+1	∞

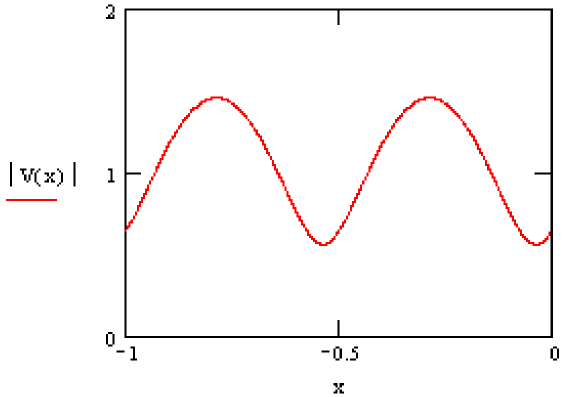
Complex Loads:

$$Z_L = (2 + j)\,Z_o$$



$$\Gamma_L = 0.4 + 0.2j$$

$$Z_L = (2 - j)Z_o / 5$$

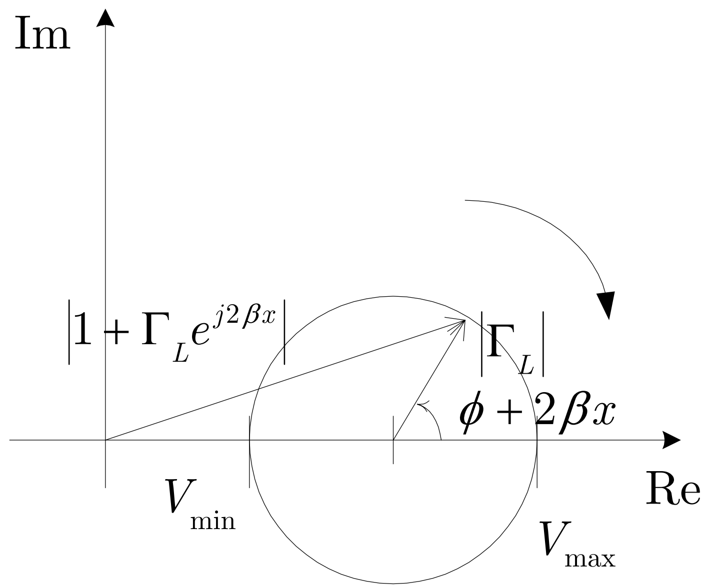


$$\Gamma_L = 0.4 - 0.2j$$

Note that the maximum or minimum no longer occurs at the load.

Crank Diagram

$$\left|\frac{V(x)}{V_o^+}\right| = \left|1+\Gamma_L e^{j2\beta x}\right|$$



$\therefore V_{\max}$ occurs when $\phi + 2\beta x = -2\pi n, \quad (n = 0,1,2,\ldots)$

$\therefore V_{\min}$ occurs when $\phi + 2\beta x = -(2n+1)\pi, \quad (n = 0,1,2,\ldots)$

(negative $\because$ *x* is becoming increasingly negative)

Voltage Maximum:

$V_{\max}$ occurs when: $\Gamma_L e^{-2j\beta d_{\max}} = |\Gamma_L| e^{j\phi} e^{-2j\beta d_{\max}} = |\Gamma_L|$

This occurs when:

$$\phi - 2\beta d_{\max} = -2\pi n \quad (n = 0,1,2,\ldots) \Rightarrow d_{\max} = \frac{2\pi n + \phi}{2\beta} = \frac{\lambda}{2}\left(n + \frac{\phi}{2\pi}\right) \quad (n = 0,1,2,\ldots)$$

Therefore, if $\phi > 0$, a maximum will occur first. If $\phi \le 0$, a minimum

Voltage Minimum:

$V_{\min}$ occurs when: $\Gamma_L e^{-2j\beta d_{\max}} = |\Gamma_L| e^{j\phi} e^{-2j\beta d_{\max}} = -|\Gamma_L|$

This occurs when:

$$\phi - 2\beta d_{\min} = -\pi(2n+1) \quad (n = 0,1,2,\ldots)$$

$$\Rightarrow d_{\min} = \frac{\pi(2n+1)+\phi}{2\beta} = \frac{\lambda}{2}\left(\frac{2n+1}{2} + \frac{\phi}{2\pi}\right) \quad (n = 0,1,2,\ldots)$$

Therefore, if $\phi = -\pi$, the first minimum occurs at the load ($n = 0 \rightarrow d_{\min} = 0$). If $-\pi < \phi \leq 0$, a minimum will occur first. If, $0 < \phi < \pi$ a maximum will occur first.

Note that the separation between maximums and minimums are:

$$d_{\max} - d_{\min} = \frac{\lambda}{2}\left(n + \frac{\phi}{2\pi}\right) - \frac{\lambda}{2}\left(\frac{2n+1}{2} + \frac{\phi}{2\pi}\right) = \frac{\lambda}{2}\left(n - \frac{2n+1}{2}\right) = \frac{\lambda}{4}$$

This is always true. The separation between each maximum and minimum is always $\lambda/2$.

Question, given a real loads, where is $V_{\max}$ or $V_{\min}$ when

a) $R_L < Z_o$?

b) $R_L > Z_o$?

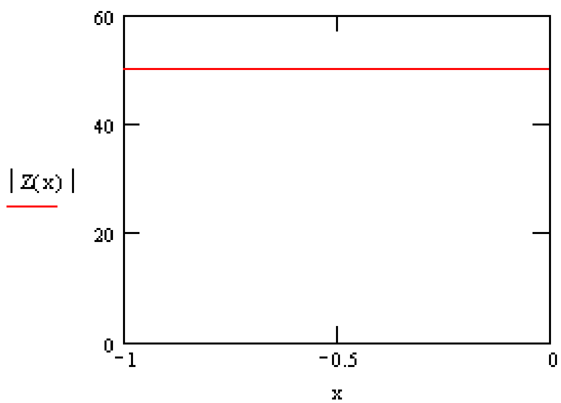
Line Impedance

It is seen that the line voltage (and hence line current) is periodic along the line length. The period is $\lambda/2$. That is, it repeats every half wavelength. Since the line impedance is a ratio of the line voltage and current, it also is periodic.

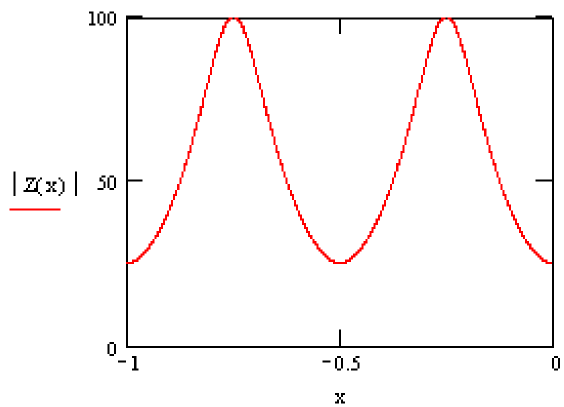
$$Z(x) = Z_o \frac{(1+\Gamma(x))}{(1-\Gamma(x))} = Z_o \frac{\left(1+\Gamma_L e^{j2\beta x}\right)}{\left(1-\Gamma_L e^{j2\beta x}\right)} = Z_o \frac{Z_L - jZ_o \tan(\beta x)}{Z_o - jZ_L \tan(\beta x)}$$

For example:

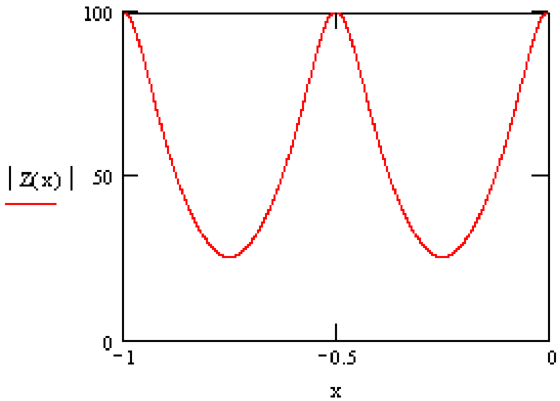
$Z_o = 50\,\Omega,\ Z_L = 50\,\Omega$



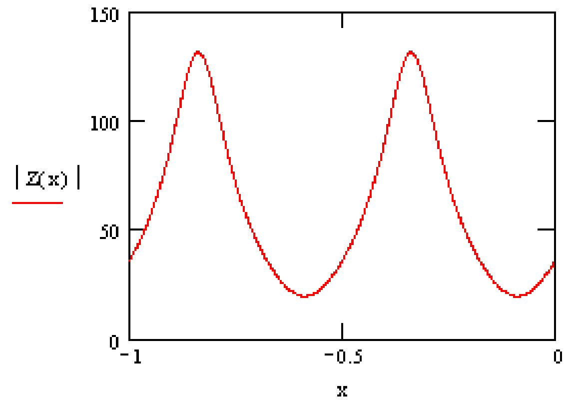
$Z_o = 50\,\Omega,\ Z_L = 25\,\Omega$



$$Z_o = 50\,\Omega,\ Z_L = 100\,\Omega$$

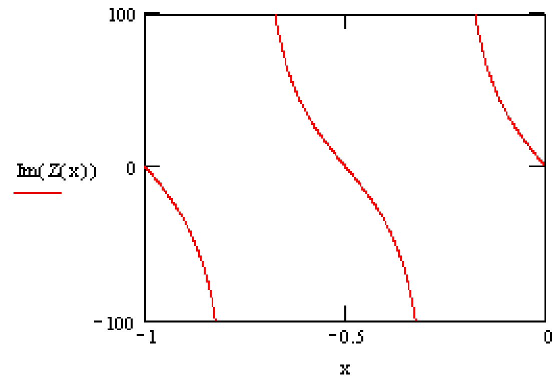


$$Z_o = 50\,\Omega,\ Z_L = 25 - j25\,\Omega$$



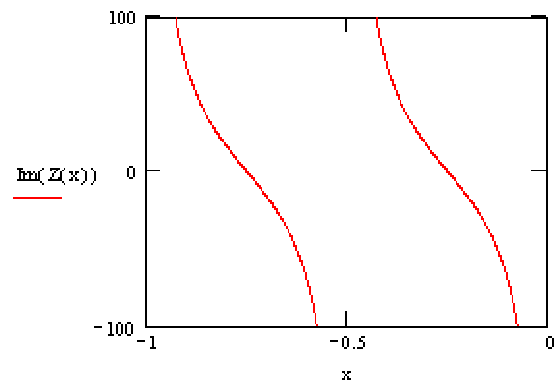
Short Circuit $Z_L = 0\,\Omega$**:**

$$Z_{sc}(x) = Z_o \frac{0 - jZ_o \tan(\beta x)}{Z_o - j0\tan(\beta x)} = -jZ_o \tan(\beta x)$$



Open Circuit $Z_L = \infty\,\Omega$**:**

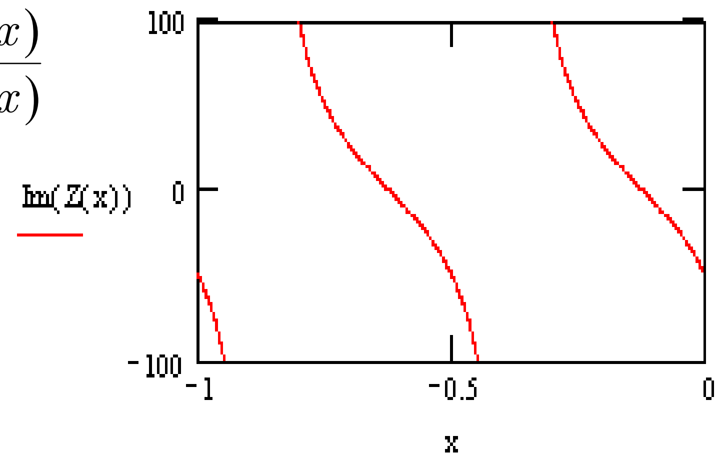
$$Z_{sc}(x) = Z_o \frac{1 - jZ_o \tan(\beta x)/Z_L}{Z_o/Z_L - j1\tan(\beta x)} = jZ_o \cot(\beta x)$$



Purely Reactive Load: $Z_L = jX\,\Omega$**:**

$$Z_{sc}(x) = Z_o \frac{jX - jZ_o \tan(\beta x)}{Z_o - j(jX)\tan(\beta x)} = jZ_o \frac{X - Z_o \tan(\beta x)}{Z_o + X\tan(\beta x)}$$

$$Z_L = -j50\ \Omega$$



Due to the periodicity of the generalized line impedance, some fundamental theorems hold true.

Theorem 1. $Z\left(x+\frac{\lambda}{2}\right)=Z(x)$

Proof:

$$Z\left(x+\frac{\lambda}{2}\right)=Z_o\left(\frac{1+\Gamma_L e^{2j\beta(x+\lambda/2)}}{1-\Gamma_L e^{2j\beta(x+\lambda/2)}}\right)=Z_o\left(\frac{1+\Gamma_L e^{2j\beta x}e^{2j\beta\lambda/2}}{1-\Gamma_L e^{2j\beta x}e^{2j\beta\lambda/2}}\right)$$

$$=Z_o\left(\frac{1+\Gamma_L e^{2j\beta x}e^{2\pi}}{1-\Gamma_L e^{2j\beta x}e^{2\pi}}\right)=Z(x)$$

Theorem 2. $Z\left(x+\frac{\lambda}{4}\right)Z(x)=Z_o^2$

Proof:

$$Z\left(x+\frac{\lambda}{4}\right)Z(x)=Z_o\left(\frac{1+\Gamma_L e^{2j\beta(x+\lambda/4)}}{1-\Gamma_L e^{2j\beta(x+\lambda/4)}}\right)Z_o\left(\frac{1+\Gamma_L e^{2j\beta x}}{1-\Gamma_L e^{2j\beta x}}\right)$$

$$=Z_o^2\left(\frac{1+\Gamma_L e^{2j\beta x}e^{\pi}}{1-\Gamma_L e^{2j\beta x}e^{\pi}}\right)\left(\frac{1+\Gamma_L e^{2j\beta x}}{1-\Gamma_L e^{2j\beta x}}\right)$$

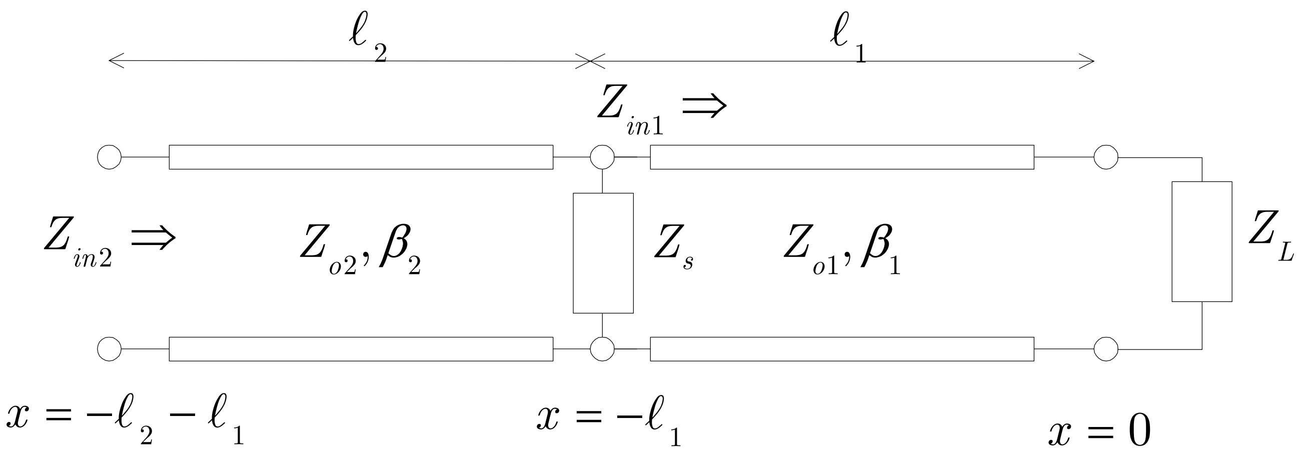
$$=Z_o^2\left(\frac{1-\Gamma_L e^{2j\beta x}}{1+\Gamma_L e^{2j\beta x}}\right)\left(\frac{1+\Gamma_L e^{2j\beta x}}{1-\Gamma_L e^{2j\beta x}}\right)=Z_o^2$$

Theorem 3. $$\frac{Z\left(x \pm \frac{\lambda}{4}\right)}{Z_o} = \frac{Y(x)}{Y_o}, \text{ where } Y_o = \frac{1}{Z_o}, \; Y(x) = \frac{1}{Z(x)}$$

Proof: Homework!

Shunt Loads

A. Parallel Loads



Determine $Z_{in2} = Z_2(-\ell_1 - \ell_2)$:

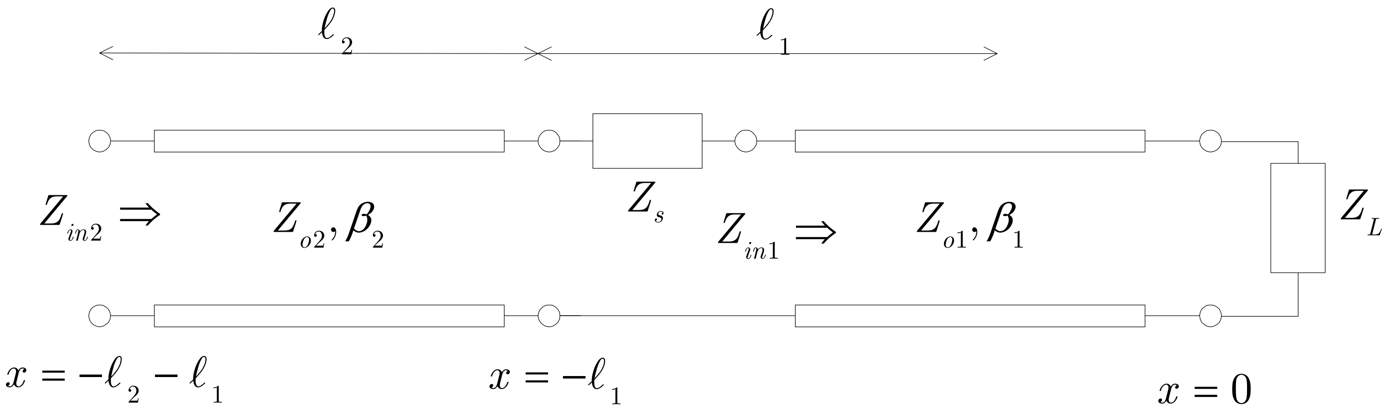
Solution Procedure:

1) Apply impedance match at *x=0*
2) Determine Z_{in1}
3) combine Z_{in1} with Z_s (How do we do this?)
4) Determine Z_{in2}

Solution:

$$Z_{in1} = Z_o \frac{Z_L + jZ_o \tan(\beta \ell_1)}{Z_o + jZ_L \tan(\beta \ell_1)}, \quad Z_{\parallel} = \frac{Z_s Z_{in1}}{Z_s + Z_{in1}}, \quad Z_{in2} = Z_o \frac{Z_{\parallel} + jZ_o \tan(\beta \ell_2)}{Z_o + jZ_{\parallel} \tan(\beta \ell_2)}$$

B. Series Load



Determine $Z_{in2} = Z_2\left(-\ell_1 - \ell_2\right)$:

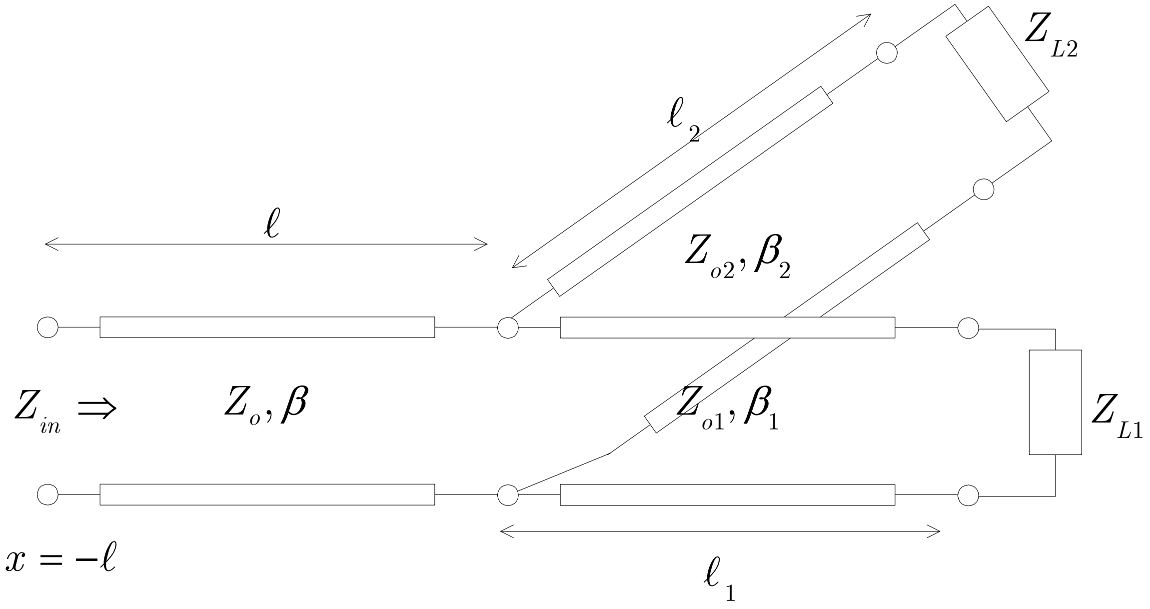
Solution Procedure:

5) Apply impedance match at *x=0*
6) Determine Z_{in1}
7) combine Z_{in1} with Z_s (How do we do this?)
8) Determine Z_{in2}

Solution:

$$Z_{in1} = Z_{o1}\frac{Z_L + jZ_{o1}\tan\left(\beta_1\ell_1\right)}{Z_{o1} + jZ_L\tan\left(\beta_1\ell_1\right)}, \quad Z_{ser} = Z_s + Z_{in1}, \quad Z_{in2} = Z_{o2}\frac{Z_{ser} + jZ_{o2}\tan\left(\beta_2\ell_2\right)}{Z_{o2} + jZ_{ser}\tan\left(\beta_2\ell_2\right)}$$

Parallel Lines:



Procedure for Determination of Input Impedance:

1) Determine Z_{in} of lines 1 and 2
2) Determine effective load (how do they combine?)
3) Determine Z_{in}

Solution:

$$Z_{in1} = Z_{o1} \frac{Z_{L1} + jZ_{o1} \tan(\beta_1 \ell_1)}{Z_{o1} + jZ_{L1} \tan(\beta_1 \ell_1)}, \quad Z_{in2} = Z_{o2} \frac{Z_{L2} + jZ_{o2} \tan(\beta_2 \ell_2)}{Z_{o2} + jZ_{L2} \tan(\beta_2 \ell_2)}$$

$$Z_{\parallel} = \frac{Z_{in1} Z_{in2}}{Z_{in1} + Z_{in2}}, \quad Z_{in} = Z_o \frac{Z_{\parallel} + jZ_o \tan(\beta \ell)}{Z_o + jZ_{\parallel} \tan(\beta \ell)}$$