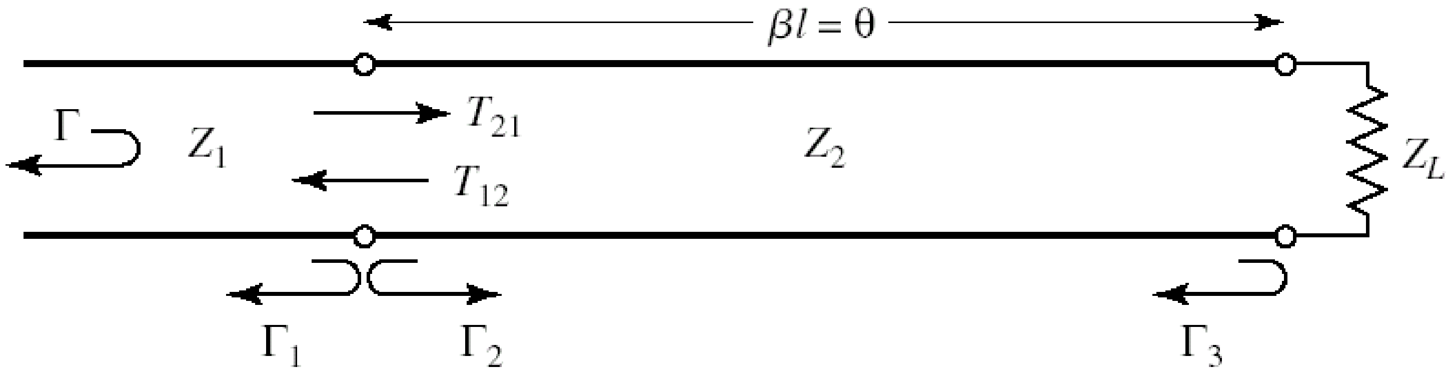
The Theory of Small Reflections

Γ ?



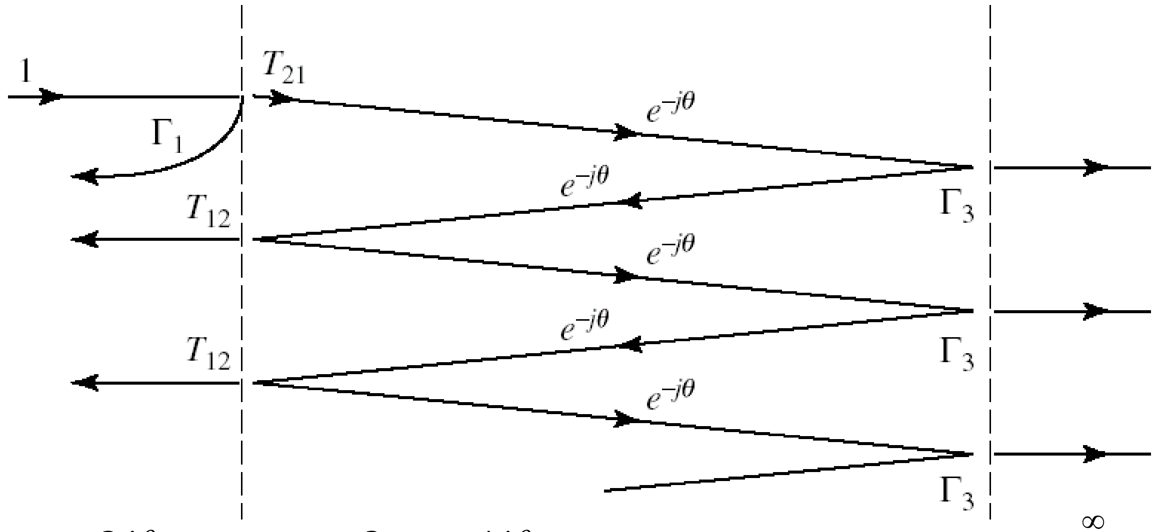
$$\Gamma_1 = \frac{Z_2 - Z_1}{Z_2 + Z_1}$$

$$\Gamma_3 = \frac{Z_L - Z_2}{Z_L + Z_2}$$

$$\Gamma_2 = -\Gamma_1$$

$$T_{21} = 1 + \Gamma_1 = \frac{2Z_2}{Z_1 + Z_2}$$

$$T_{12} = 1 + \Gamma_2 = \frac{2Z_1}{Z_1 + Z_2}$$



$$\Gamma = \Gamma_1 + T_{12}T_{21}\Gamma_3 e^{-2j\theta} + T_{12}T_{21}\Gamma_3^2\Gamma_2 e^{-4j\theta} + \dots = \Gamma_1 + T_{12}T_{21}\Gamma_3 e^{-2j\theta}\sum_{n=0}^{\infty}\Gamma_2^n\Gamma_3^n e^{-2jn\theta}$$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \text{for} \quad |x|<1$$

$$\Gamma = \Gamma_1 + \frac{T_{12}T_{21}\Gamma_3 e^{-2j\theta}}{1-\Gamma_2\Gamma_3 e^{-2j\theta}}$$

$$\Gamma_2 = -\Gamma_1$$

$$T_{21} = 1+\Gamma_1 = \frac{2Z_2}{Z_1+Z_2}$$

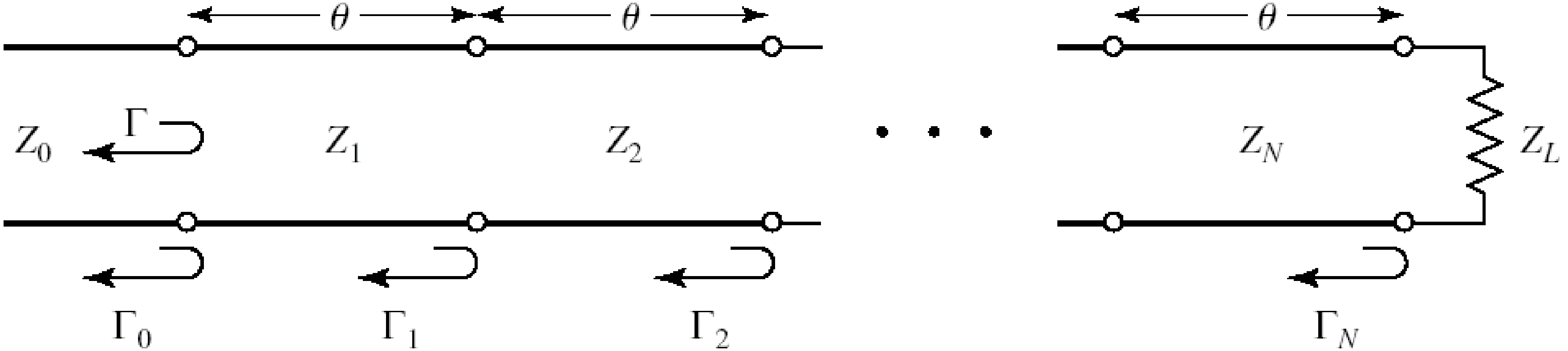
$$T_{12} = 1+\Gamma_2 = \frac{2Z_1}{Z_1+Z_2}$$

$$\Gamma = \Gamma_1 + \frac{(1+\Gamma_1)(1-\Gamma_1)\Gamma_3 e^{-2j\theta}}{1+\Gamma_1\Gamma_3 e^{-2j\theta}} = \Gamma_1 + \frac{(1-\Gamma_1^2)\Gamma_3 e^{-2j\theta}}{1+\Gamma_1\Gamma_3 e^{-2j\theta}} = \frac{\Gamma_1 + \Gamma_1^2\Gamma_3 e^{-2j\theta} + (1-\Gamma_1^2)\Gamma_3 e^{-2j\theta}}{1+\Gamma_1\Gamma_3 e^{-2j\theta}}$$

$$\Gamma = \frac{\Gamma_1 + \Gamma_3 e^{-2j\theta}}{1 + \Gamma_1 \Gamma_3 e^{-2j\theta}}$$

If the discontinuities between the impedances Z_1, Z_2, and Z_2, Z_L are small, then $|\Gamma_1 \Gamma_3| << 1$.

$$\Gamma \cong \Gamma_1 + \Gamma_3 e^{-2j\theta}$$



$$\Gamma_0 = \frac{Z_1 - Z_o}{Z_1 + Z_o} \qquad \Gamma_n = \frac{Z_{n+1} - Z_n}{Z_{n+1} + Z_o} \qquad \Gamma_N = \frac{Z_L - Z_N}{Z_L + Z_N}$$

$$\Gamma \cong \Gamma_1 + \Gamma_3 e^{-2j\theta}$$

$$\Gamma = \Gamma(\theta) \cong \Gamma_0 + \Gamma_1 e^{-2j\theta} + \Gamma_2 e^{-4j\theta} + \cdots + \Gamma_{N-2} e^{-2j(N-2)\theta} + \Gamma_{N-1} e^{-2j(N-1)\theta} + \Gamma_N e^{-2jN\theta}$$

Assume that the transformer is symmetrical

$\Gamma_0 = \Gamma_N \qquad \Gamma_1 = \Gamma_{N-1} \qquad \Gamma_2 = \Gamma_{N-2}$ **etc.**

$$\Gamma(\theta) = \Gamma_0 + \Gamma_1 e^{-2j\theta} + \Gamma_2 e^{-4j\theta} + \cdots + \Gamma_2 e^{-2j(N-2)\theta} + \Gamma_1 e^{-2j(N-1)\theta} + \Gamma_0 e^{-2jN\theta}$$

$$\Gamma(\theta) = \Gamma_0\left(1 + e^{-2jN\theta}\right) + \Gamma_1\left(e^{-2j\theta} + e^{-2j(N-1)\theta}\right) + \Gamma_2\left(e^{-4j\theta} + e^{-2j(N-2)\theta}\right) + \cdots$$

$$\Gamma(\theta) = e^{-jN\theta}\left[\Gamma_0\left(e^{jN\theta} + e^{-jN\theta}\right) + \Gamma_1\left(e^{j(N-2)\theta} + e^{-j(N-2)\theta}\right) + \Gamma_2\left(e^{j(N-4)\theta} + e^{-j(N-4)\theta}\right) + \cdots\right]$$

$$\Gamma(\theta) = 2e^{-jN\theta}\left[\Gamma_0 \cos N\theta + \Gamma_1 \cos(N-2)\theta + \cdots + \Gamma_n \cos(N-2n)\theta + \cdots + \frac{1}{2}\Gamma_{N/2}\right]$$

For N even

$$\Gamma(\theta) = 2e^{-jN\theta}\left[\Gamma_0 \cos N\theta + \Gamma_1 \cos(N-2)\theta + \cdots + \Gamma_n \cos(N-2n)\theta + \cdots + \frac{1}{2}\Gamma_{(N-1)/2} \cos\theta\right]$$

For N odd

Finite Fourier Cosine Series

By choosing the Γ_{ns} and enough sections (N) we can achieve the required response.

Binomial Multisection Matching Transformers

Chebyshev Multisection Matching Transformers

The Binomial MultiSection Transformer

We first consider the **Binomial Function:**

$$\Gamma(\theta) = A\left(1 + e^{-j2\theta}\right)^N$$

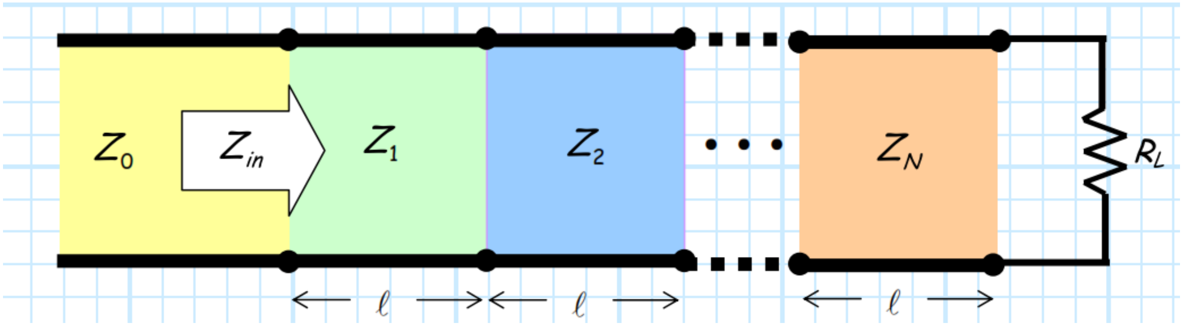
This function has the desirable **properties that:**

$$\begin{aligned}\Gamma(\theta = \pi/2) &= A\left(1 + e^{-j\pi}\right)^N \\ &= A(1-1)^N \\ &= 0\end{aligned}$$

and that:

$$\left.\frac{d^n\,\Gamma(\theta)}{d\theta^n}\right|_{\theta=\pi/2} = 0 \text{ for } n = 1,2,3,\cdots,N-1$$

In other words, this Binomial Function is **maximally flat** at the Point θ=π/2 , where it has a value of Γ(θ=π/2) = 0



Set section lengths ℓ so that they are a **quarter-wavelength** ($\lambda_0/4$) at the design frequency ω_0.

By letting f=0 (or θ=0) he electrical length (βl) of each section will likewise **approach zero. Thus, the input impedance** will simply be equal to R_L, thus

$$\Gamma(0) = A\,2^N = \frac{R_L - Z_0}{R_L + Z_0}$$

So

$$A = 2^{-N}\frac{R_L - Z_0}{R_L + Z_0}$$

Let's **expand** of the Binomial Function:

$$\Gamma(\theta) = A\left(1+e^{-j2\theta}\right)^N$$
$$= A\left(C_0^N + C_1^N e^{-j2\theta} + C_2^N e^{-j4\theta} + C_3^N e^{-j6\theta} + \cdots + C_N^N e^{-j2N\theta}\right)$$

Where

$$C_n^N \doteq \frac{N!}{(N-n)!\,n!}$$

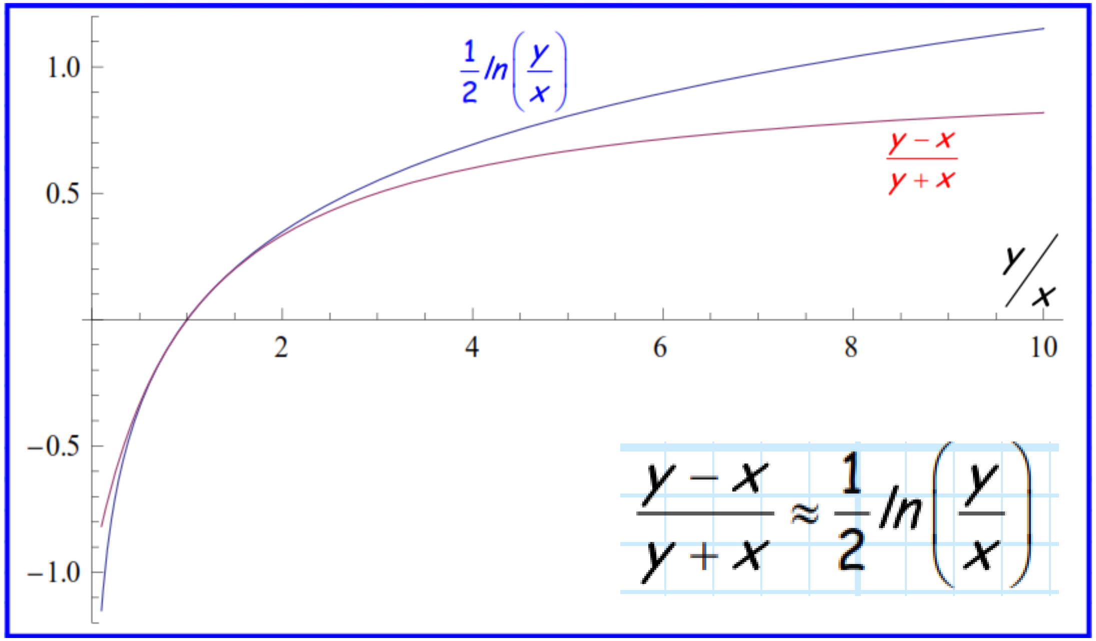
Compare this to an **N-section transformer function:**

$$\Gamma_n = A\, C_n^N$$

Of course, we **also know that these marginal reflection** coefficients are physically related to the characteristic impedances of each section as:

$$\Gamma_n = \frac{Z_{n+1}-Z_n}{Z_{n+1}+Z_n} \quad \Rightarrow \quad Z_{n+1} = Z_n\frac{1+\Gamma_n}{1-\Gamma_n} = Z_n\frac{1+A\,C_n^N}{1-A\,C_n^N}$$

You undoubtedly have previously used the approximation:



Now, we know that the values of Z_{n+1} and Z_n in a multi-section matching network are typically **very close, such that**

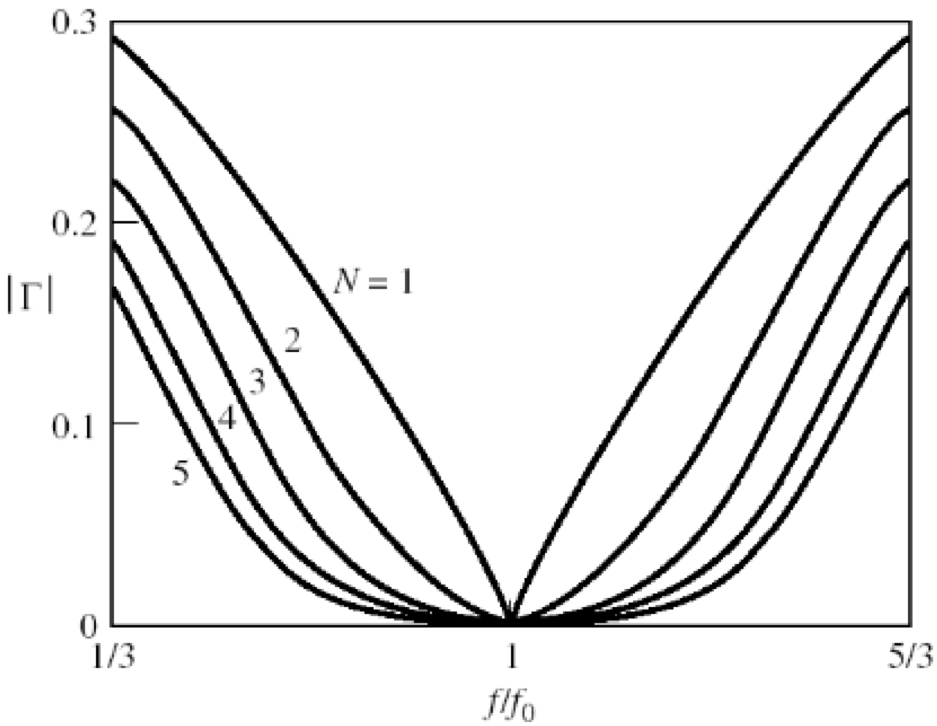
$$\Gamma_n = \frac{Z_{n+1} - Z_n}{Z_{n+1} + Z_n} \approx \frac{1}{2} ln\left(\frac{Z_{n+1}}{Z_n}\right)$$

Then we have

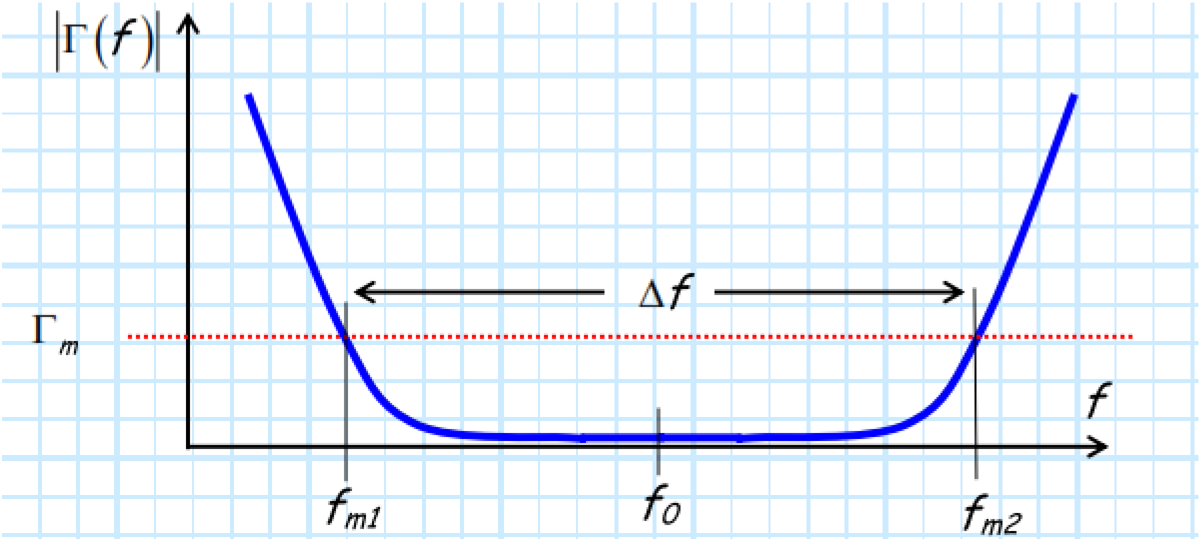
$$Z_{n+1} = Z_n \exp\left[2\Gamma_n\right]$$

Where

$$\Gamma_n = A\, C_n^N$$



Bandwidth



$$\Gamma(\theta) = A\left(1 + e^{-j2\theta}\right)^N$$

$$|\Gamma(\theta)| = 2^N |A| |e^{-jN\theta}| |\cos\theta|^N = 2^N |A| |\cos\theta|^N$$

In the figure above Γ_m is maximum allowed reflection

So we have

$$\Gamma_m = \left|\Gamma\left(\theta = \theta_m\right)\right| = 2^N |A| \left|\cos\theta_m\right|^N$$

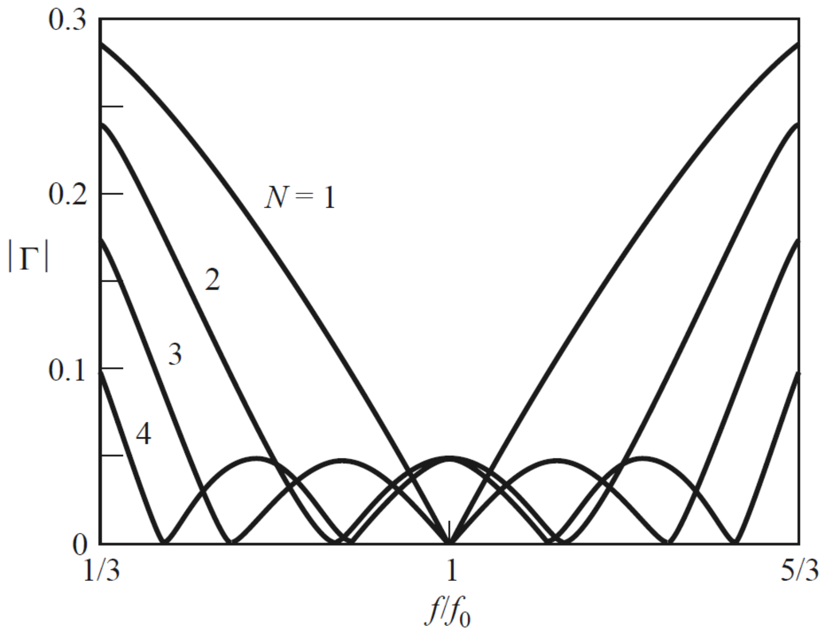
Doing some math we will have

$$\theta_{m1} = \cos^{-1}\left[\frac{1}{2}\left(\frac{\Gamma_m}{|A|}\right)^{1/N}\right] \qquad \theta_{m2} = \cos^{-1}\left[-\frac{1}{2}\left(\frac{\Gamma_m}{|A|}\right)^{1/N}\right]$$

On the other word

$$\Delta f = 2\left(f_0 - f_{m1}\right) = 2f_0 - \frac{4f_0}{\pi}\cos^{-1}\left[+\frac{1}{2}\left(\frac{\Gamma_m}{|A|}\right)^{1/N}\right]$$

Chenushev MultiSection Transformer



Reading Assignment: section 5.7

Chebyshev Multisection Matching Transformers

- We can also build a multisection matching network such that the function Γ(f) is a **Chebyshev function.**
- Chebyshev functions **maximize bandwidth, although at the cost** of **pass-band ripple.**
- Chebyshev solutions can provide functions Γ(ω) with **wider**
- **bandwidth than the Binomial case—although at the "expense" of passband ripple.**
- Chebyshev transformers are **symmetric**.
- The reflection coefficient of a **Chebyshev matching** network has the form:

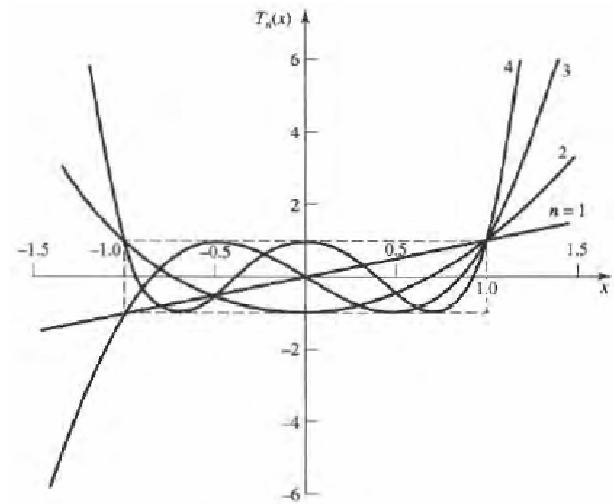
$$\Gamma(\theta) = A\, e^{-jN\theta}\, T_N\left(\frac{\cos\theta}{\cos\theta_m}\right) = A\, e^{-jN\theta}\, T_N(\cos\theta \sec\theta_m)$$

where θm =ωmT

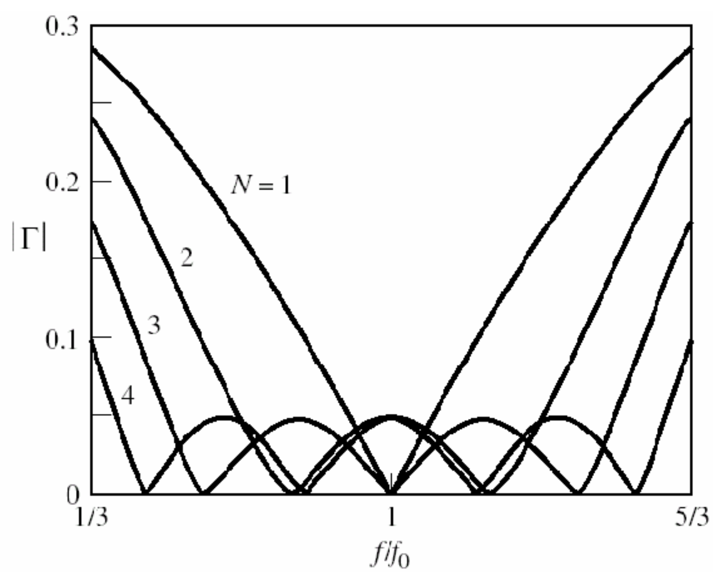
- The function $T_N(\cos\theta \sec\theta)$ is a **Chebyshev polynomial of** order N.

- The first four Chebyshev polynomials are,

$$T_1(x) = x,$$
$$T_2(x) = 2x^2 - 1,$$
$$T_3(x) = 4x^3 - 3x,$$
$$T_4(x) = 8x^4 - 8x^2 + 1.$$



- Inserting the **substitution:** x = cosθsecθ$_m$ into the Chebyshev polynomials above



$$T_1(\cos\theta\sec\theta_m) = \cos\theta\sec\theta_m$$

$$T_2(\cos\theta\sec\theta_m) = \sec^2\theta_m(1+\cos 2\theta) - 1$$
$$= \sec^2\theta_m \cos 2\theta + (\sec^2\theta_m - 1)$$

$$T_3(\cos\theta\sec\theta_m) = \sec^3\theta_m(\cos 3\theta + 3\cos\theta) - 3\sec\theta_m\cos\theta$$
$$= \sec^3\theta_m\cos 3\theta + (3\sec^2\theta_m - 3)\sec\theta_m\cos\theta$$

$$T_4(\cos\theta\sec\theta_m) = \sec^4\theta_m(\cos 4\theta + 4\cos 2\theta + 3)$$
$$-4\sec^2\theta_m(\cos 2\theta + 1) + 1$$
$$= \sec^4\theta_m\cos 4\theta$$
$$+ 4\sec^2\theta_m(\sec^2\theta_m - 1)\cos 2\theta$$
$$+ (3\sec^4\theta_m - 4\sec^2\theta_m + 1)$$

- We can now synthesize a Chebyshev equal-ripple passband by making Γ(θ) proportional to T_N(secθmcosθ) where N is the number of sections in the transformer.

$$\Gamma(\theta) = 2e^{-jN\theta}[\Gamma_0 \cos N\theta + \Gamma_1 \cos(N-2)\theta + \cdots + \Gamma_n \cos(N-2n)\theta + \cdots]$$
$$= Ae^{-jN\theta} T_N(\sec\theta_m \cos\theta),$$

- As in the binomial transformer case we can find the constant "A" by letting θ = 0 at zero frequency.

$$\Gamma(0) = \frac{Z_L - Z_0}{Z_L + Z_0} = AT_N(\sec\theta_m),$$

$$A = \frac{Z_L - Z_0}{Z_L + Z_0} \frac{1}{T_N(\sec\theta_m)}.$$

- The maximum allowable reflection coefficient magnitude in the passband is Γm.
- But Γm = lAl
- The maximum value of T_N(secθmcosθ) is in the passband is unity.
- We can find θm by using,

$$\sec\theta_m = \cosh\left[\frac{1}{N}\cosh^{-1}\left(\frac{1}{\Gamma_m}\left|\frac{Z_L - Z_0}{Z_L + Z_0}\right|\right)\right]$$
$$\simeq \cosh\left[\frac{1}{N}\cosh^{-1}\left(\left|\frac{\ln Z_L/Z_0}{2\Gamma_m}\right|\right)\right].$$

- The fractional bandwidth can be calculated using the following equation once θm is known.

$$\frac{\Delta f}{f_0} = 2 - \frac{4\theta_m}{\pi}$$

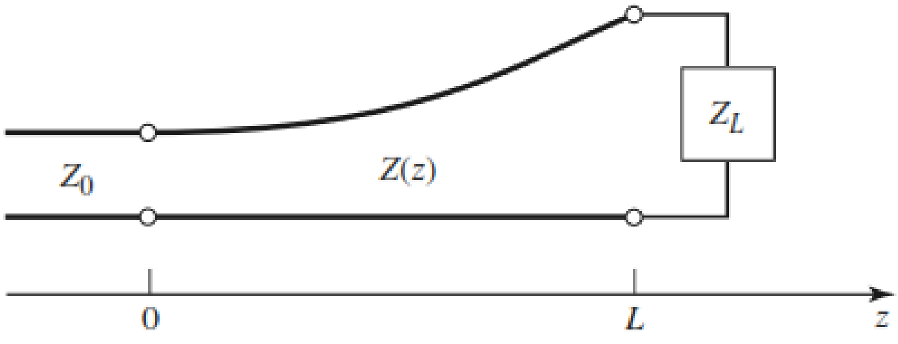
- Summarizing the he Chebyshev matching network design **procedure**

- 1. **Determine the value N required to meet the** bandwidth and ripple m Γ requirements.
- 2. Determine the **Chebychev function.**
- **3.** Determine all Γn by **equating terms with the** symmetric multisection transformer expression:
- 4. **Calculate all Zn using the approximation:**

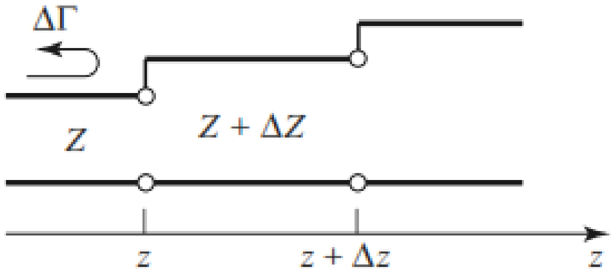
$$\Gamma_n \simeq \frac{1}{2} \ln \frac{Z_{n+1}}{Z_n}.$$

- 5. Determine section **length l = λo/4.**

Tapered Lines



(a)



(b)

As we know

$$\Delta\Gamma = \frac{(Z + \Delta Z) - Z}{(Z + \Delta Z) + Z} \simeq \frac{\Delta Z}{2Z}.$$

Remember that

$$\frac{d(\ln f(z))}{dz} = \frac{1}{f}\frac{df(z)}{dz}.$$

we will have:

$$d\Gamma = \frac{dZ}{2Z} = \frac{1}{2}\frac{d(\ln Z/Z_0)}{dz}dz,$$

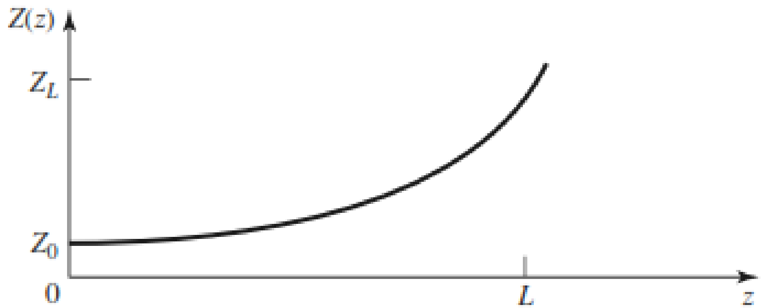
Finally the reflection coefficient will be

$$\Gamma(\theta) = \frac{1}{2}\int_{z=0}^{L} e^{-2j\beta z}\frac{d}{dz}\ln\left(\frac{Z}{Z_0}\right)dz,$$

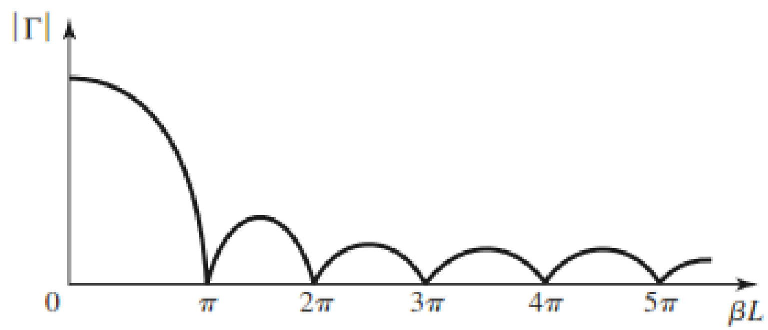
If Z(z) is known, Γ(θ) can be found as a function of frequency. Alternatively, if Γ(θ) is specified, then in principle Z(z) can be found by inversion. This latter procedure is difficult, and is generally avoided in practice

Exponential Taper

$$Z(z) = Z_0 e^{az} \quad \text{for } 0 < z < L,$$



(a)



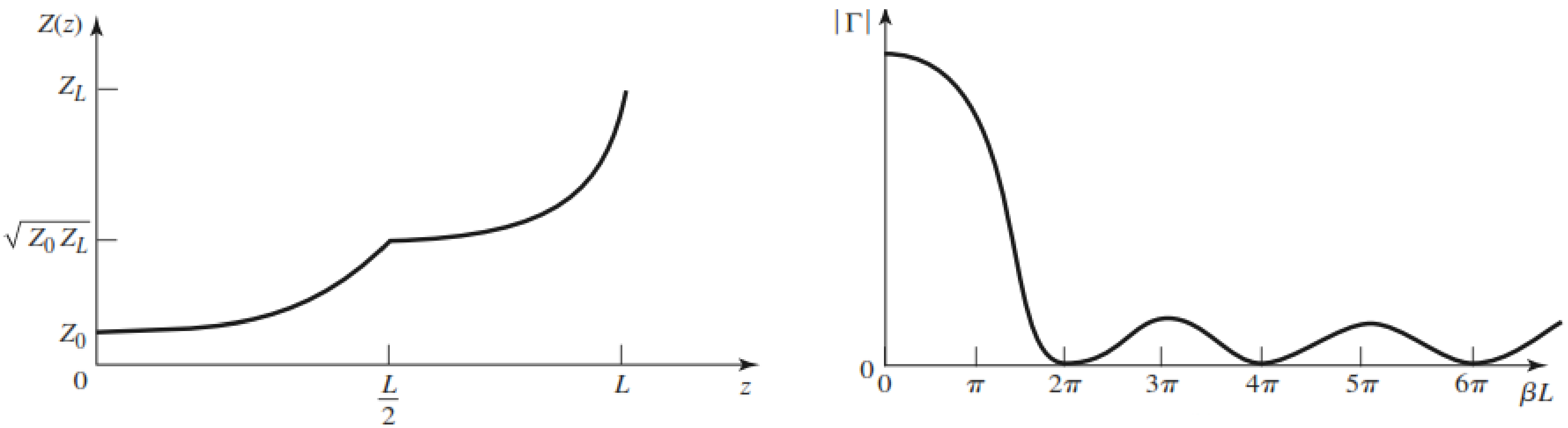
(b)

$$a = \frac{1}{L} \ln\left(\frac{Z_L}{Z_0}\right).$$

$$\begin{aligned}\Gamma &= \frac{1}{2}\int_0^L e^{-2j\beta z}\frac{d}{dz}(\ln e^{az})dz \\ &= \frac{\ln Z_L/Z_0}{2L}\int_0^L e^{-2j\beta z}dz \\ &= \frac{\ln Z_L/Z_0}{2} e^{-j\beta L}\frac{\sin\beta L}{\beta L}.\end{aligned}$$

Triangular Taper

$$Z(z) = \begin{cases} Z_0 e^{2(z/L)^2 \ln Z_L/Z_0} & \text{for } 0 \le z \le L/2 \\ Z_0 e^{(4z/L - 2z^2/L^2 - 1) \ln Z_L/Z_0} & \text{for } L/2 \le z \le L, \end{cases}$$



$$\Gamma(\theta) = \frac{1}{2} e^{-j\beta L} \ln\left(\frac{Z_L}{Z_0}\right) \left[\frac{\sin(\beta L/2)}{\beta L/2}\right]^2 .$$

Klopfenstein Taper

$$\ln Z(z) = \frac{1}{2}\ln Z_0 Z_L + \frac{\Gamma_0}{\cosh A} A^2 \phi(2z/L - 1,\ A) \quad \text{for } 0 \le z \le L,$$

where

$$\phi(x,\ A) = -\phi(-x,\ A) = \int_0^x \frac{I_1(A\sqrt{1-y^2})}{A\sqrt{1-y^2}}\, dy \quad \text{for } |x| \le 1,$$

And I_1 is modified Bessel function

It can be shown that

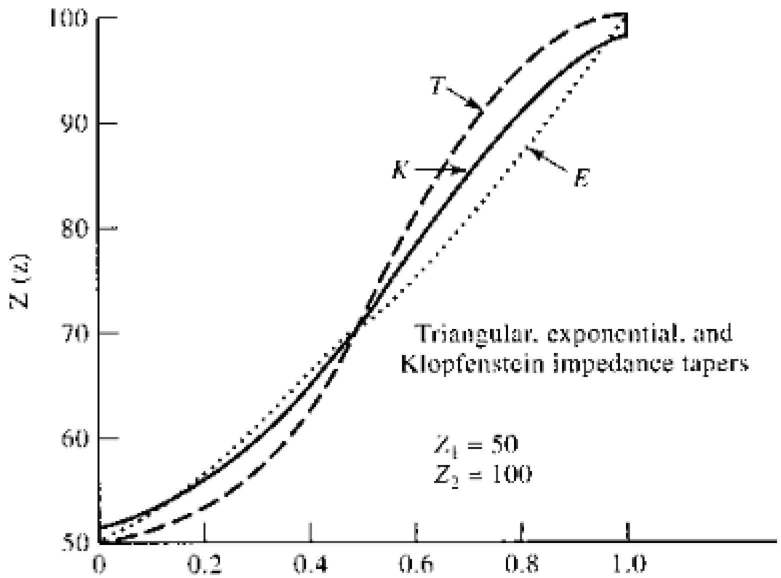
$$\Gamma(\theta) = \Gamma_0 e^{-j\beta L} \frac{\cos\sqrt{(\beta L)^2 - A^2}}{\cosh A} \quad \text{for } \beta L > A.$$

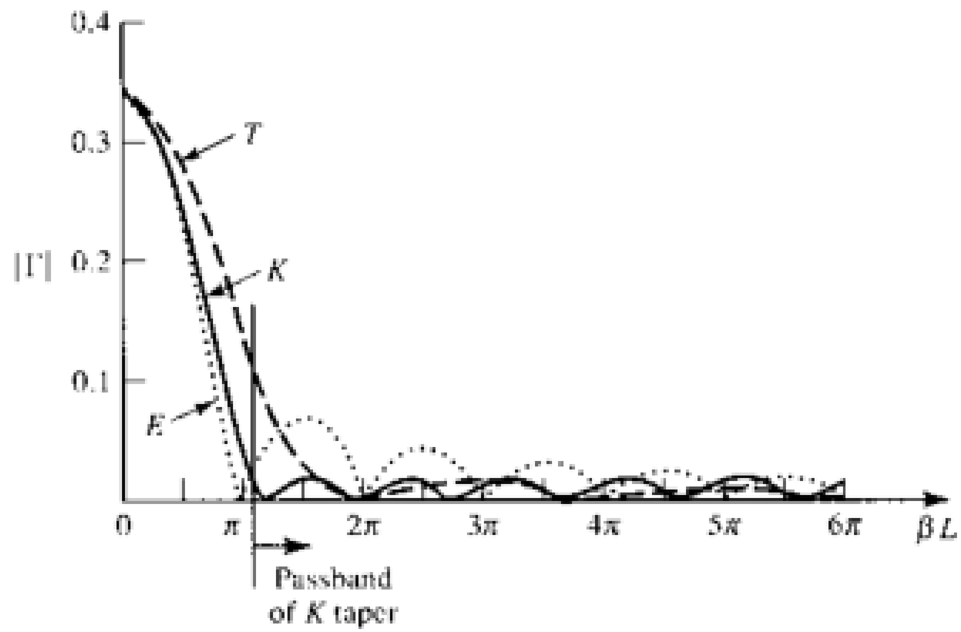
where

$$\Gamma_0 = \frac{Z_L - Z_0}{Z_L + Z_0} \simeq \frac{1}{2}\ln\left(\frac{Z_L}{Z_0}\right)$$

Clearly

$$\Gamma_m = \frac{\Gamma_0}{\cosh A}$$


100
90
80
70
60
50
Z (z)
0
0.2
0.4
0.6
0.8
1.0
T
K
E
Triangular, exponential, and
Klopfenstein impedance tapers
$Z_1 = 50$
$Z_2 = 100$


0.4
0.3
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|Γ|
0
π
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3π
4π
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T
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Passband
of K taper