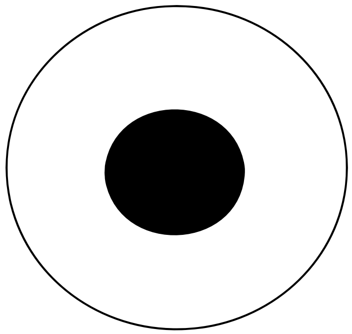
Fundamental Waveguide Theory

Types of Waveguides

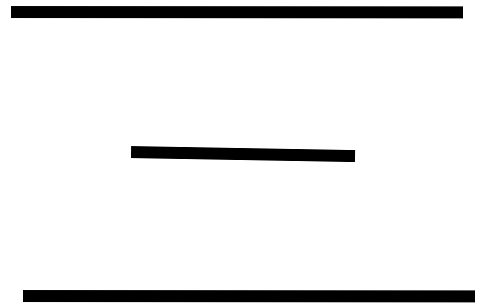
1. TEM and quasi-TEM waveguides
2. Metallic waveguides (TE and TM modes)
3. Dielectric waveguides (TE, TM, TEM or hybrid modes)

TEM and Quasi-TEM Waveguides

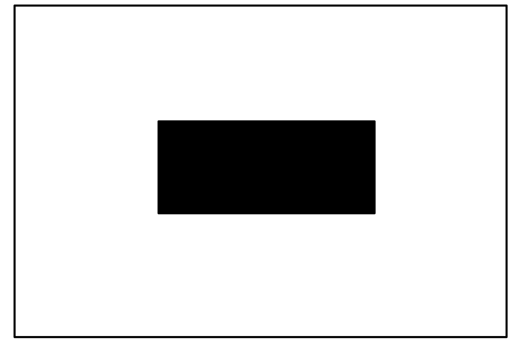
TEM:



Coaxial

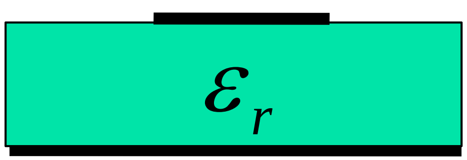


Strip Line

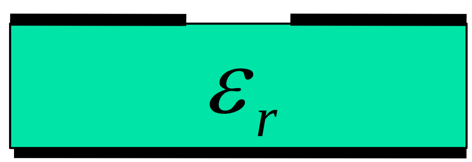


Rectangular Coaxial

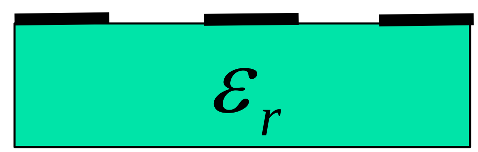
Quasi TEM:



Microstrip Line



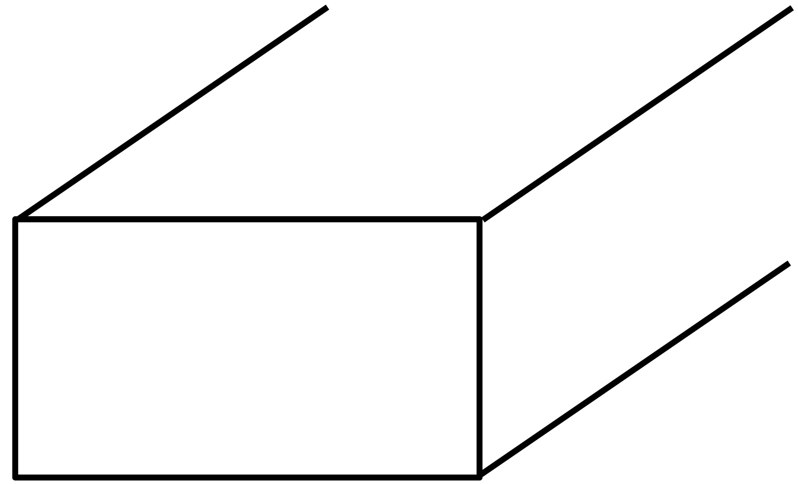
Slot Line



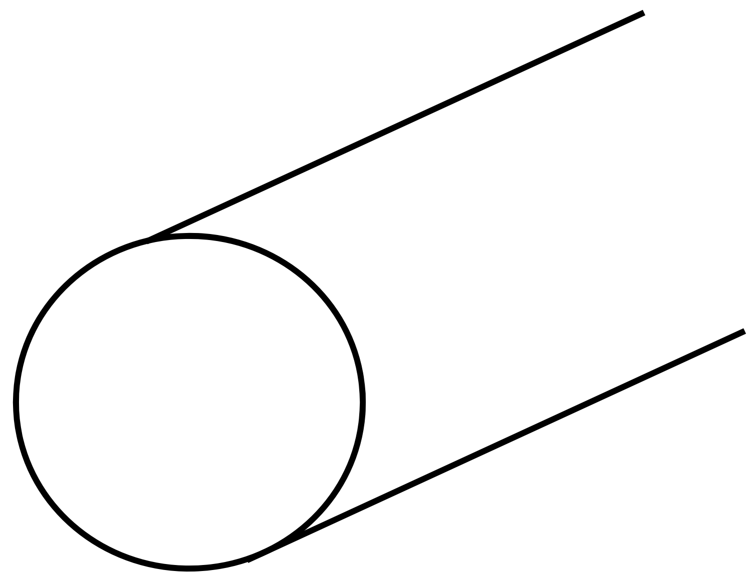
Coplanar Line

Metallic Waveguides

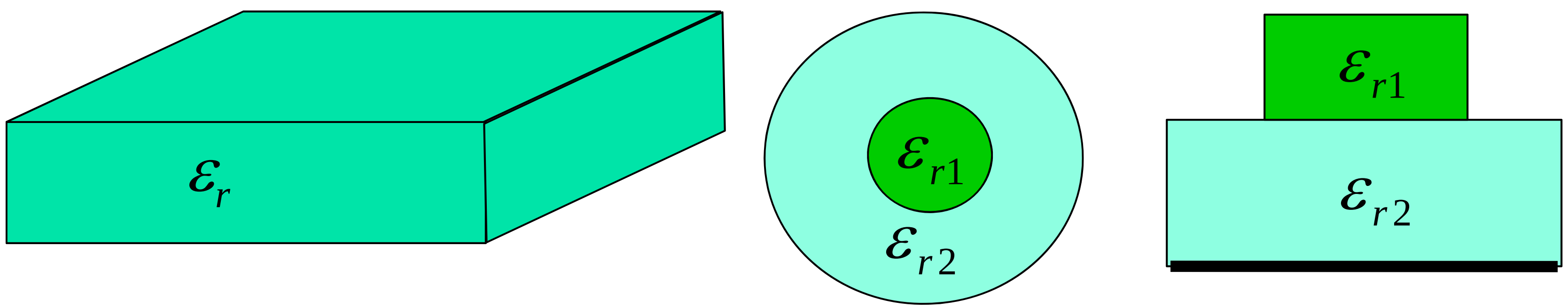
Rectangular waveguide



Circular waveguide



Dielectric Waveguides



Planar dielectric waveguide

Fiber

Ridge waveguide

Modes in Waveguides (1)

Modes: certain field patterns that can propagate independently

TEM mode: Transverse Electromagnetic mode. All the fields are in the cross section or there are no E_z and H_z components

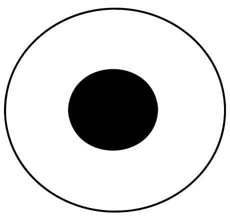
Modes in Waveguides (2)

- TE modes: transverse electric modes
  - Electric fields are in the cross section (or no E_z component).
- Only H_z component in the longitudinal direction. Also called H modes

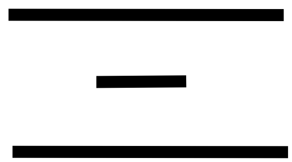
- TM modes: transverse magnetic modes
  - Magnetic fields are in the cross section (or no H_z component).
- Only E_z component in the longitudinal direction. Also called E modes

Conditions for the Existence of TEM Modes

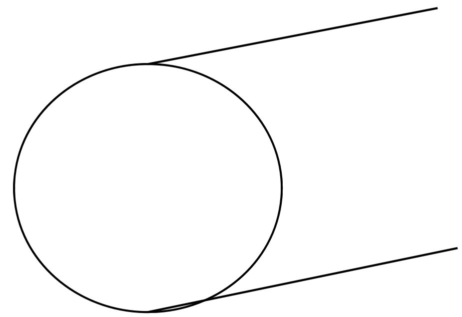
- At least two perfect electric conductors
- Dielectric distribution in the cross section is homogeneous



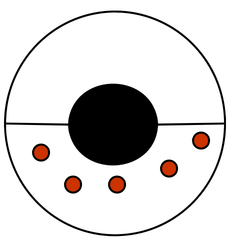
Yes



Yes



No



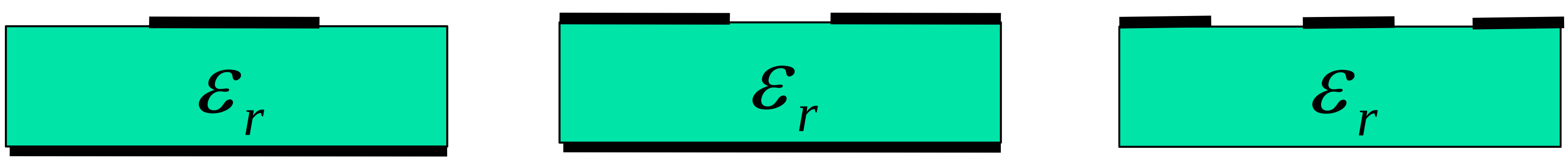
No

Note: TEM line can have higher order TE and TM modes

Quasi-TEM Line

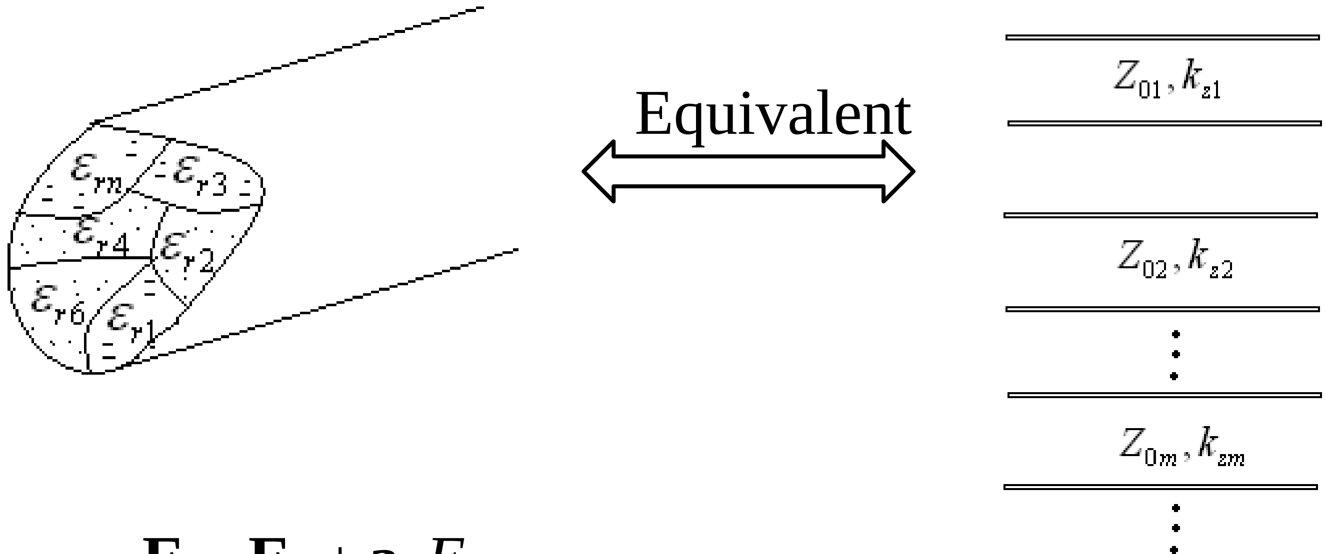
- Some planar waveguide structure with inhomogeneous dielectric distribution.

$E_z \neq 0$, $H_z \neq 0$ But $|E_z| << |\mathbf{E}_t|$, $|H_z| << |\mathbf{H}_t|$



Microstrip Line | Slot Line | Coplanar Line

Basic Waveguide Theory (1)



$$\mathbf{E} = \mathbf{E}_t + \mathbf{a}_z E_z$$

$$\mathbf{H} = \mathbf{H}_t + \mathbf{a}_z H_z$$

with $\mathbf{E}_t = \sum_m \mathbf{e}_{tm}(x,y) V_m(z)$

$$\mathbf{H}_t = \sum_m \mathbf{h}_{tm}(x,y) I_m(z)$$

Generalized
Fourier Transform

Basic Waveguide Theory (2)

$$\frac{dV_m(z)}{dz} = -jk_{zm}Z_{0m}I_m(z)$$

$$\frac{dI_m(z)}{dz} = -jk_{zm}Y_{0m}V_m(z)$$

$Z_{0m} = \frac{1}{Y_{0m}}$ is the characteristic impedance of the m^th^ mode

k_{zm} is the propagation constant of the m^th^ mode

Basic Waveguide Theory (3)

$$V_m(z) = A_m e^{-jk_{zm}z} + B_m e^{jk_{zm}z}$$

A_m ↑ Forward Wave

B_m ↑ Backward Wave

Effective Relative Dielectric Constant of the m^th mode

$$\varepsilon_{eff} = \left(\frac{k_{zm}}{k_0}\right)^2 \rightarrow k_{zm}^2 = k_0^2 \varepsilon_{eff}$$

The mode with the largest ε_{eff} is called the dominant mode.

TEM Mode (1)

Maxwell's Equations

$$\begin{cases} \nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} \\ \nabla \times \mathbf{H} = j\omega\varepsilon\mathbf{E} \\ \nabla \cdot \mathbf{E} = 0 \\ \nabla \cdot \mathbf{H} = 0 \end{cases} \xrightarrow[E_z=0,\; H_z=0]{\text{TEM mode}} \begin{cases} \nabla \times \mathbf{E}_t = -j\omega\mu\mathbf{H}_t \\ \nabla \times \mathbf{H}_t = j\omega\varepsilon\mathbf{E}_t \\ \nabla \cdot \mathbf{E}_t = 0 \\ \nabla \cdot \mathbf{H}_t = 0 \end{cases}$$

$$\text{Let} \quad \nabla = \underbrace{\mathbf{a}_x \frac{\partial}{\partial x} + \mathbf{a}_y \frac{\partial}{\partial y}}_{\nabla_t} + \underbrace{\mathbf{a}_z \frac{\partial}{\partial z}}_{\nabla_z}$$

$$= \nabla_t + \nabla_z \qquad \Rightarrow$$

TEM Mode (2)

$$\Rightarrow \quad (\nabla_t + \nabla_z) \times \mathbf{E}_t = -j\omega\mu \mathbf{H}_t \rightarrow \nabla_t \times \mathbf{E}_t = 0$$

$$(\mathbf{a}_z \frac{\partial}{\partial z}) \times \mathbf{E}_t = -j\omega\mu \mathbf{H}_t \rightarrow \quad \mathbf{a}_z \times (\frac{\partial \mathbf{E}_t}{\partial z}) = -j\omega\mu \mathbf{H}_t$$

$$(\nabla_t + \nabla_z) \times \mathbf{H}_t = -j\omega\varepsilon \mathbf{E}_t \rightarrow \nabla_t \times \mathbf{H}_t = 0$$

$$(\mathbf{a}_z \frac{\partial}{\partial z}) \times \mathbf{H}_t = j\omega\varepsilon \mathbf{E}_t \rightarrow \quad \mathbf{a}_z \times (\frac{\partial \mathbf{H}_t}{\partial z}) = j\omega\varepsilon \mathbf{E}_t$$

$$\nabla_t \cdot \mathbf{E}_t = 0$$

$$\nabla_t \cdot \mathbf{H}_t = 0$$

TEM Mode (3)

From

$$\nabla_t \times \mathbf{E}_t = 0$$
$$\nabla_t \cdot \mathbf{E}_t = 0$$
$$\nabla_t \times \mathbf{H}_t = 0$$
$$\nabla_t \cdot \mathbf{H}_t = 0$$

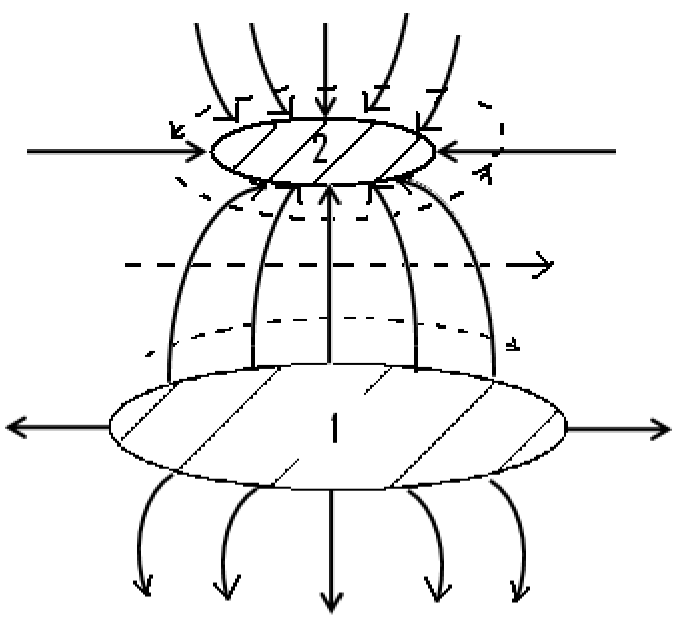
⇒

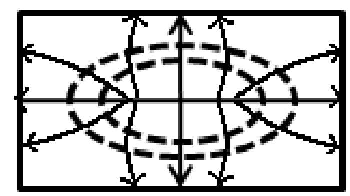
The fields in the cross-section are similar to 2-D electrostatic & 2-D magnetostatic fields for TEM mode even if actual operating frequency can be very high.

TEM Mode (4)

Let $\mathbf{E}_t = -\nabla_t \Phi$

$$\Rightarrow \nabla_t^2 \Phi = 0$$ —— Laplace Equation





$V_{12} = \int_1^2 \mathbf{E} \cdot d\mathbf{l}$ For voltage

$I = \oint \mathbf{H} \cdot d\mathbf{l}$ For current

TEM Mode (5)

$$\mathbf{a}_z \times (\frac{\partial \mathbf{E}_t}{\partial z}) = -j\omega\mu \mathbf{H}_t \quad \Rightarrow \quad \mathbf{H}_t = -\frac{1}{j\omega\mu}\mathbf{a}_z \times (\frac{\partial \mathbf{E}_t}{\partial z})$$

$$\mathbf{a}_z \times (\frac{\partial \mathbf{H}_t}{\partial z}) = j\omega\varepsilon \mathbf{E}_t \quad \Rightarrow \quad \mathbf{a}_z \times \left[-\frac{1}{j\omega\mu}\frac{\partial}{\partial z}(\mathbf{a}_z \times \frac{\partial \mathbf{E}_t}{\partial z}) \right] = j\omega\varepsilon \mathbf{E}_t$$

$$\mathbf{a}_z \times (\mathbf{a}_z \times \frac{\partial^2 \mathbf{E}_t}{\partial z^2}) = \omega^2 \mu\varepsilon \mathbf{E}_t$$

$$\mathbf{a}_z (\mathbf{a}_z \cdot \frac{\partial^2 \mathbf{E}_t}{\partial z^2}) - \frac{\partial^2 \mathbf{E}_t}{\partial z^2}(\mathbf{a}_z \cdot \mathbf{a}_z) = \omega^2 \mu\varepsilon \mathbf{E}_t \quad \text{where} \quad \mathbf{a}_z (\mathbf{a}_z \cdot \frac{\partial^2 \mathbf{E}_t}{\partial z^2}) = 0$$

$$\Rightarrow \frac{\partial^2 \mathbf{E}_t}{\partial z^2} + k^2 \mathbf{E}_t = 0 \quad \text{where} \quad k^2 = \omega^2 \mu\varepsilon$$

Likewise $$\frac{\partial^2 \mathbf{H}_t}{\partial z^2} + k^2 \mathbf{H}_t = 0$$

TEM Mode (6)

For +z direction propagation mode, we have

$$\mathbf{E}_t = \mathbf{e}_t(x, y)e^{-jk_z z}$$

$$\mathbf{H}_t = \mathbf{h}_t(x, y)e^{-jk_z z}$$

$$\rightarrow \frac{\partial}{\partial z} = -jk_z$$

From $$\frac{\partial^2 \mathbf{E}_t}{\partial z^2} + k^2 \mathbf{E}_t = 0 \qquad \Rightarrow (k^2 - k_z^2)\mathbf{e}_t = 0$$

Since $\mathbf{e}_t \neq 0 \quad \Rightarrow \quad k^2 = k_z^2$ For TEM mode

Or: $$k_c^2 = k^2 - k_z^2 = 0$$

where $$k_c^2 = k_x^2 + k_y^2$$

TEM Mode (7)

For +z direction propagating TEM mode:

$$\mathbf{E} = \mathbf{e}(x,y)e^{-jkz}$$
$$\mathbf{H} = \mathbf{h}(x,y)e^{-jkz}$$

$$\nabla_t^{\,2} V = 0$$

$$\mathbf{e} = -\nabla_t V$$

From: $\mathbf{H}_t = -\dfrac{1}{j\omega\mu}\mathbf{a}_z \times \left(\dfrac{\partial \mathbf{E}_t}{\partial z}\right) \qquad \dfrac{\partial}{\partial z} = -jk$

$$\Rightarrow \mathbf{H} = \frac{jk}{j\omega\mu}\mathbf{a}_z \times \mathbf{E} \qquad \Rightarrow \mathbf{h} = \frac{\mathbf{a}_z \times \mathbf{e}}{\eta}$$

$\eta = \dfrac{\omega\mu}{k} = \sqrt{\dfrac{\mu}{\varepsilon}}$ wave impedance of the media inside the TEM waveguide

TE, TM and Hybrid Modes (1)

From

$$\begin{cases} \nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} \\ \nabla \times \mathbf{H} = j\omega\varepsilon\mathbf{E} \\ \nabla \cdot \mathbf{E} = 0 \\ \nabla \cdot \mathbf{H} = 0 \end{cases} \Longrightarrow \begin{cases} \nabla^2\mathbf{E} + k^2\mathbf{E} = 0 \\ \nabla^2\mathbf{H} + k^2\mathbf{H} = 0 \end{cases}$$

For z component:

$$\begin{cases} \nabla^2 E_z + k^2 E_z = 0 \\ \nabla^2 H_z + k^2 H_z = 0 \end{cases}$$

or

$$\begin{cases} \nabla_t^2 E_z + k_c^2 E_z = 0 \\ \nabla_t^2 H_z + k_c^2 H_z = 0 \end{cases}$$

since $\dfrac{\partial^2}{\partial z^2} = -k_z^2, \quad k_c^2 = k^2 - k_z^2$

TE, TM and Hybrid Modes (2)

For a mode propagating in +z direction: $\frac{\partial}{\partial z} = -jk_z$

From $\nabla \times \mathbf{E} = -j\omega\mu \mathbf{H}$ $\qquad \nabla \times \mathbf{H} = j\omega\varepsilon \mathbf{E}$

$$\Rightarrow \begin{cases} \frac{\partial E_z}{\partial y} + jk_z E_y = j\omega\mu H_x \\ -jk_z E_x - \frac{\partial E_z}{\partial x} = -j\omega\mu H_y \\ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\mu H_z \end{cases} \qquad \begin{cases} \frac{\partial H_z}{\partial y} + jk_z H_y = -j\omega\varepsilon E_x \\ -jk_z H_x - \frac{\partial H_z}{\partial x} = j\omega\varepsilon E_y \\ \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\omega\varepsilon E_z \end{cases}$$

TE, TM and Hybrid Modes (3)

⇨

$$E_x = \frac{-jk_z}{k_c^2}\frac{\partial E_z}{\partial x} + \frac{-j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial y}$$

$$E_y = \frac{-jk_z}{k_c^2}\frac{\partial E_z}{\partial y} + \frac{j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial x}$$

$$H_x = \frac{j\omega\varepsilon}{k_c^2}\frac{\partial E_z}{\partial y} + \frac{-jk_z}{k_c^2}\frac{\partial H_z}{\partial x}$$

$$H_y = \frac{-j\omega\varepsilon}{k_c^2}\frac{\partial E_z}{\partial x} + \frac{-jk_z}{k_c^2}\frac{\partial H_z}{\partial y}$$

Or

$$\mathbf{E_t} = \frac{-jk_z}{k_c^2}\nabla_t E_z + \frac{j\omega\mu}{k_c^2}\mathbf{a}_z \times \nabla_t H_z$$

$$\mathbf{H_t} = \frac{-j\omega\varepsilon}{k_c^2}\mathbf{a}_z \times \nabla_t E_z + \frac{-jk_z}{k_c^2}\nabla_t H_z$$

TE Modes (1)

$$E_x = \frac{-j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial y}$$

$$E_y = \frac{j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial x}$$

$$H_x = \frac{-jk_z}{k_c^2}\frac{\partial H_z}{\partial x}$$

$$H_y = \frac{-jk_z}{k_c^2}\frac{\partial H_z}{\partial y}$$

$$E_z = 0$$

$$\nabla_t^2 H_z + k_c^2 H_z = 0$$

$$\mathbf{E_t} = \frac{j\omega\mu}{k_c^2}\mathbf{a}_z \times \nabla_t H_z$$

$$= -\frac{\omega\mu}{k_z}\mathbf{a}_z \times \mathbf{H_t}$$

$$\mathbf{H_t} = \frac{-jk_z}{k_c^2}\nabla_t H_z$$

TE Modes (2)

For a mode propagating in +z direction:

$$\mathbf{E} = \mathbf{e}(x, y)e^{-jk_z z}$$

$$\mathbf{H} = \mathbf{h}(x, y)e^{-jk_z z}$$

$$\nabla_t^2 h_z + k_c^2 h_z = 0$$ + Boundary Conditions

$$e_x = \frac{-j\omega\mu}{k_c^2}\frac{\partial h_z}{\partial y}$$

$$e_y = \frac{j\omega\mu}{k_c^2}\frac{\partial h_z}{\partial x}$$

$$h_x = \frac{-jk_z}{k_c^2}\frac{\partial h_z}{\partial x}$$

$$h_y = \frac{-jk_z}{k_c^2}\frac{\partial h_z}{\partial y}$$

$$\mathbf{h_t} = \frac{-jk_z}{k_c^2}\nabla_t h_z$$

$$\mathbf{e_t} = -\frac{\omega\mu}{k_z}\mathbf{a}_z \times \mathbf{h_t} = -Z_{TE}\,\mathbf{a}_z \times \mathbf{h_t}$$

$$Z_{TE} = \frac{\omega\mu}{k_z}$$ characteristic impedance of TE modes

TM Modes (1)

$$E_x = \frac{-jk_z}{k_c^2}\frac{\partial E_z}{\partial x}$$

$$E_y = \frac{-jk_z}{k_c^2}\frac{\partial E_z}{\partial y}$$

$$H_x = \frac{j\omega\varepsilon}{k_c^2}\frac{\partial E_z}{\partial y}$$

$$H_y = \frac{-j\omega\varepsilon}{k_c^2}\frac{\partial E_z}{\partial x}$$

$$H_z = 0$$

$$\nabla_t^2 E_z + k_c^2 E_z = 0$$

$$\mathbf{E}_\mathbf{t} = \frac{-jk_z}{k_c^2}\nabla_t E_z$$

$$\mathbf{H}_\mathbf{t} = \frac{-j\omega\varepsilon}{k_c^2}\mathbf{a}_z \times \nabla_t E_z$$

$$= \frac{\omega\varepsilon}{k_z}\mathbf{a}_z \times \mathbf{E}_\mathbf{t}$$

TM Modes (2)

For a mode propagating in +z direction:

$$\mathbf{E} = \mathbf{e}(x, y)e^{-jk_z z}$$

$$\mathbf{H} = \mathbf{h}(x, y)e^{-jk_z z}$$

$$\nabla_t^2 e_z + k_c^2 e_z = 0$$ + Boundary Conditions

$$e_x = \frac{-jk_z}{k_c^2}\frac{\partial e_z}{\partial x}$$

$$e_y = \frac{-jk_z}{k_c^2}\frac{\partial e_z}{\partial y}$$

$$h_x = \frac{j\omega\varepsilon}{k_c^2}\frac{\partial e_z}{\partial y}$$

$$h_y = \frac{-j\omega\varepsilon}{k_c^2}\frac{\partial e_z}{\partial x}$$

$$\mathbf{e_t} = \frac{-jk_z}{k_c^2}\nabla_t e_z$$

$$\mathbf{h_t} = \frac{\omega\varepsilon}{k_z}\mathbf{a}_z \times \mathbf{E_t} = \frac{1}{Z_{TM}}\mathbf{a}_z \times \mathbf{E_t}$$

$$Z_{TM} = \frac{k_z}{\omega\varepsilon}$$ characteristic impedance of TM modes

Hybrid Modes

For a mode propagating in +z direction:

$$\mathbf{E} = \mathbf{e}(x,y)e^{-jk_z z}$$

$$\mathbf{H} = \mathbf{h}(x,y)e^{-jk_z z}$$

$$\begin{cases} \nabla_t^2 e_z + k_c^2 e_z = 0 \\ \nabla_t^2 h_z + k_c^2 h_z = 0 \end{cases}$$ + Boundary Conditions

$$e_x = \frac{-jk_z}{k_c^2}\frac{\partial e_z}{\partial x} + \frac{-j\omega\mu}{k_c^2}\frac{\partial h_z}{\partial y}$$

$$e_y = \frac{-jk_z}{k_c^2}\frac{\partial e_z}{\partial y} + \frac{j\omega\mu}{k_c^2}\frac{\partial h_z}{\partial x}$$

$$h_x = \frac{j\omega\varepsilon}{k_c^2}\frac{\partial e_z}{\partial y} + \frac{-jk_z}{k_c^2}\frac{\partial h_z}{\partial x}$$

$$h_y = \frac{-j\omega\varepsilon}{k_c^2}\frac{\partial e_z}{\partial x} + \frac{-jk_z}{k_c^2}\frac{\partial h_z}{\partial y}$$

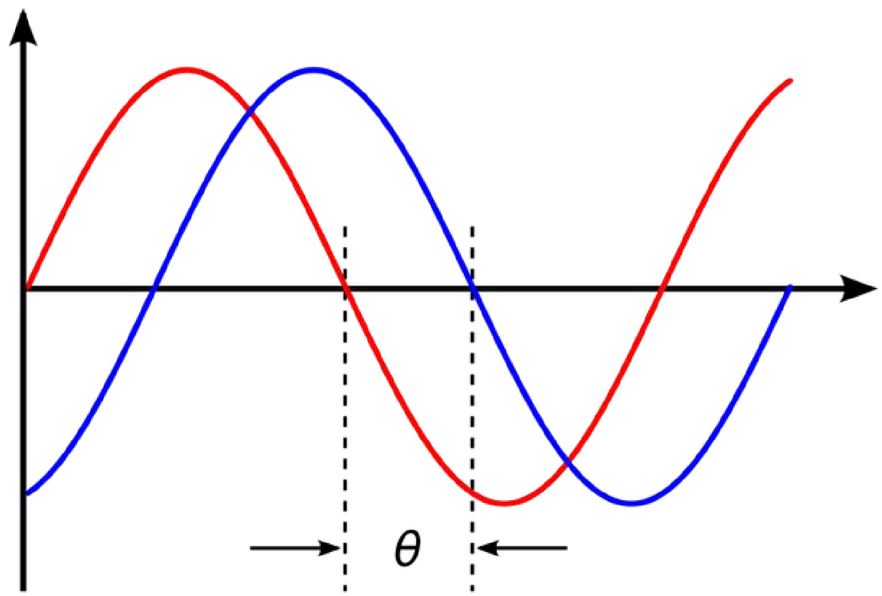
$$\mathbf{e}_\mathbf{t} = \frac{-jk_z}{k_c^2}\nabla_t e_z + \frac{j\omega\mu}{k_c^2}\mathbf{a}_z \times \nabla_t h_z$$

$$\mathbf{h}_\mathbf{t} = \frac{-j\omega\varepsilon}{k_c^2}\mathbf{a}_z \times \nabla_t e_z + \frac{-jk_z}{k_c^2}\nabla_t h_z$$

Phase Velocity and Group Velocity (1)

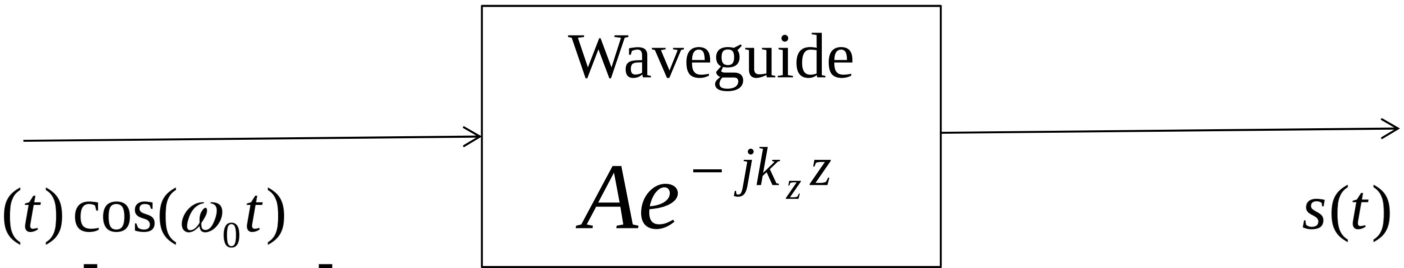
For a single frequency: $e^{-jk_z z} \Longrightarrow \cos(\omega t - k_z z)$

Phase velocity: the velocity of constant phase plane (ωt-$k_z z$=const)



$$v_p = \frac{\omega}{k_z}$$

Phase Velocity and Group Velocity (2)



$$f(t)\cos(\omega_0 t) = \mathrm{Re}\left[f(t)e^{j\omega_0 t}\right]$$

$$s(t)$$

$$\updownarrow$$

$$F(\omega-\omega_0)$$

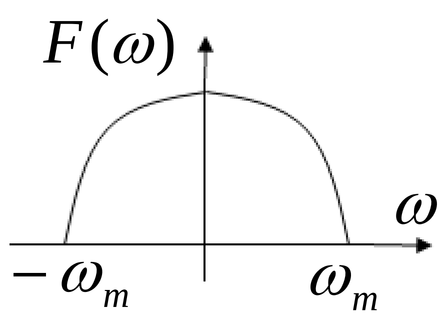
$$\mathfrak{F}\left[f(t)\right] = F(\omega)$$

$$S(\omega) = F(\omega-\omega_0)Ae^{-jk_z z}$$

$$\Rightarrow s(t) = \frac{1}{2\pi}\mathrm{Re}\left[\int_{-\infty}^{+\infty} S(\omega)e^{j\omega t}d\omega\right]$$

$$= \frac{1}{2\pi}\mathrm{Re}[\int_{\omega_0-\omega_m}^{\omega_0+\omega_m} S(\omega)e^{j\omega t}d\omega]$$

$$= \frac{1}{2\pi}\mathrm{Re}[\int_{\omega_0-\omega_m}^{\omega_0+\omega_m} AF(\omega-\omega_0)e^{j(\omega t-k_z z)}d\omega]$$



Phase Velocity and Group Velocity (3)

$$s(t) = \frac{1}{2\pi}\mathrm{Re}[\int_{\omega_0-\omega_m}^{\omega_0+\omega_m} AF(\omega-\omega_0)e^{j(\omega t - k_z z)}d\omega]$$

Expand $$k_z(\omega) = k_z(\omega_0) + \frac{dk_z}{d\omega}\Big|_{\omega=\omega_0}(\omega-\omega_0) + \ldots$$

$$\approx k_z(\omega_0) + \frac{dk_z}{d\omega}\Big|_{\omega=\omega_0}(\omega-\omega_0)$$

Let $$k_{z0} = k_z(\omega_0)$$

$$k'_{z0} = \frac{dk_z}{d\omega}\Big|_{\omega=\omega_0}$$

$$\Rightarrow k_z(\omega) = k_{z0} + k'_{z0}(\omega-\omega_0)$$

Phase Velocity and Group Velocity (4)

$$s(t) = \frac{1}{2\pi}\mathrm{Re}[\int_{\omega_0-\omega_m}^{\omega_0+\omega_m} AF(\omega-\omega_0)e^{j\omega t}e^{-j\left[k_{z0}+k_{z0}^{'}(\omega-\omega_0)\right]z}d\omega]$$

$$\overset{p=\omega-\omega_0}{=} \frac{A}{2\pi}\mathrm{Re}[e^{j(\omega_0 t-k_{z0}z)}\int_{-\omega_m}^{\omega_m} F(p)e^{j\left[t-k_{z0}^{'}z\right]p}dp]$$

$$= A\,\mathrm{Re}\left[f(t-k_{z0}^{'}z)e^{j(\omega_0 t-k_{z0}z)}\right]$$

$$= \underline{Af(t-k_{z0}^{'}z)}\,\underline{\cos(\omega_0 t-k_{z0}z)}$$

Information with group velocity

$$k_{z0} = k_z(\omega_0)$$

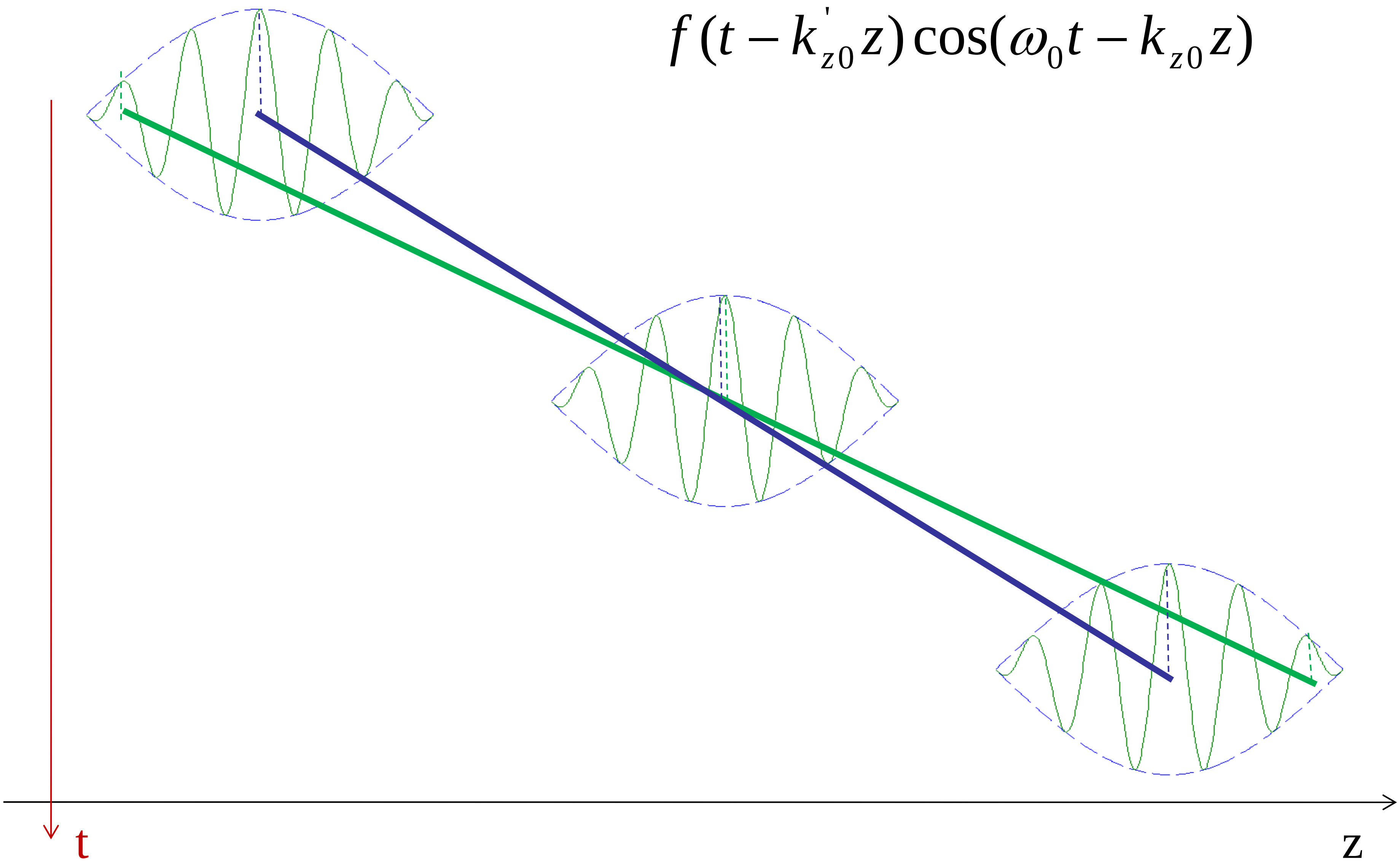
$$v_g = \frac{1}{k'_{z0}} = \frac{1}{\left.\frac{dk_z}{d\omega}\right|_{\omega=\omega_0}}$$

Carrier with phase velocity

$$v_p = \frac{\omega_0}{k_{z0}}$$

Phase Velocity and Group Velocity (5)

$$f(t - k'_{z0} z)\cos(\omega_0 t - k_{z0} z)$$



Cutoff Frequency in Metallic Waveguide

For $e^{-jk_z z}$

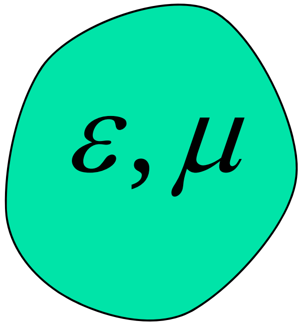
When $k_z^2 < 0,$ the mode is an evanescent mode.

Find cutoff frequency

$$k_z^2 = k^2 - k_c^2 = 0$$

k_c is cutoff wavenumber.

From $k_c = 2\pi f_c \sqrt{\mu\varepsilon}$



cutoff frequency $f_c = \dfrac{k_c}{2\pi\sqrt{\mu\varepsilon}}$

Degenerate Modes

If two or more modes have the same eigenvalue (propagation constant kz) but different eigenvectors (field patterns), they are called degenerate modes.