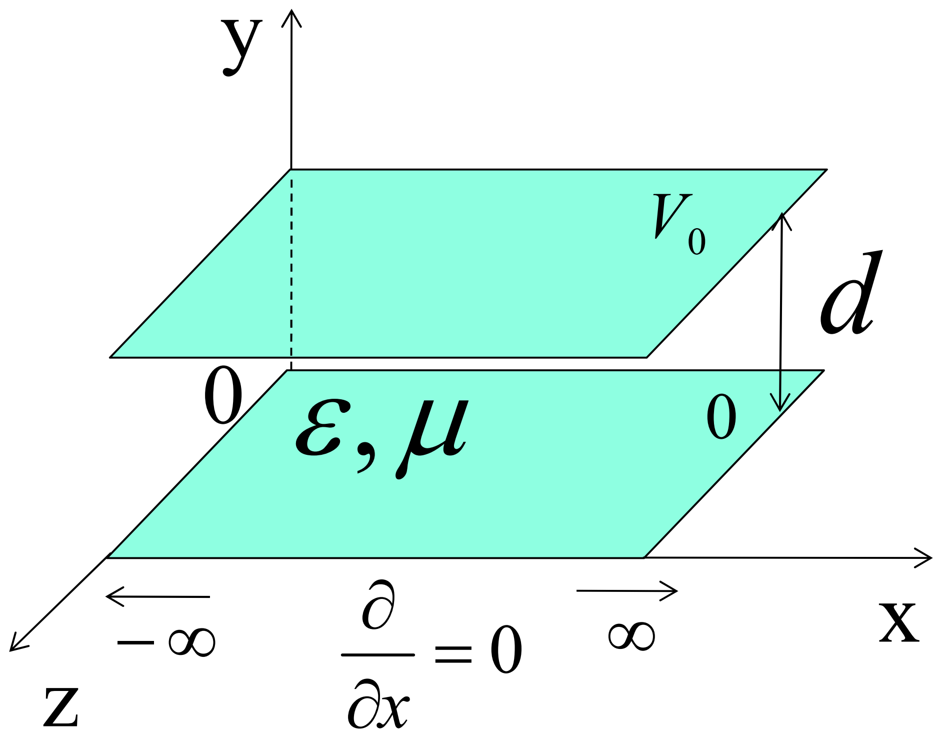
Dr. Vahid Nayyeri

Parallel Plate Waveguides

Parallel Plate Waveguide TEM Mode (1)



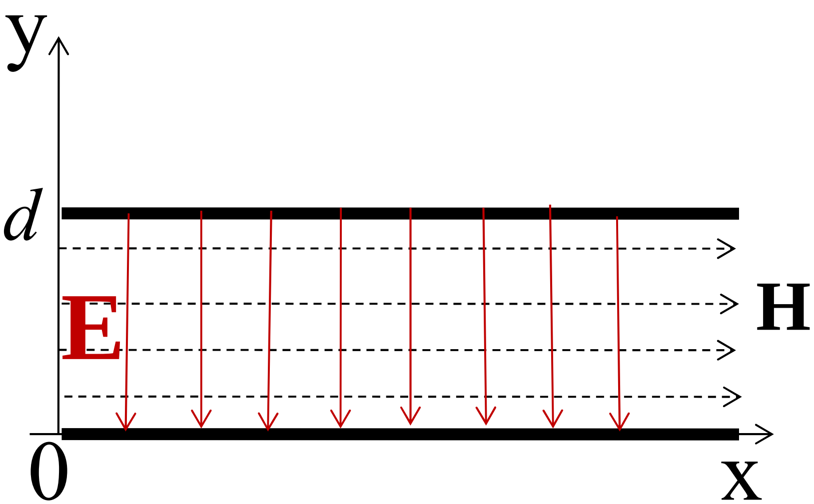
$$\nabla_t^2 V(y) = 0$$

$$\nabla_t^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{d^2}{dy^2}$$

$$\Rightarrow \begin{cases} \dfrac{d^2 V(y)}{dy^2} = 0 \\ V(0) = 0 \\ V(d) = V_0 \end{cases}$$

$$\Rightarrow V(y) = \frac{V_0 y}{d}$$

Parallel Plate Waveguide TEM Mode (2)



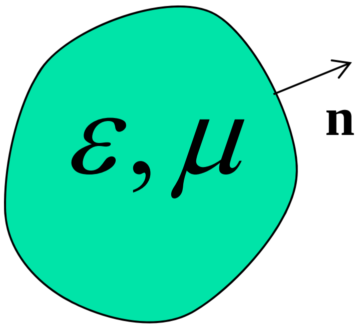
$$\mathbf{e}(y) = -\nabla_t V(y) = -\frac{V_0}{d}\mathbf{a}_y$$

$$\mathbf{h}(y) = \frac{\mathbf{a}_z \times \mathbf{e}(y)}{\eta} = \frac{V_0}{\eta d}\mathbf{a}_x$$

$$\mathbf{E}(y,z) = \mathbf{e}(y)e^{-jkz} = -\frac{V_0}{d}e^{-jkz}\mathbf{a}_y$$

$$\mathbf{H}(y,z) = \mathbf{h}(y)e^{-jkz} = \frac{V_0}{\eta d}e^{-jkz}\mathbf{a}_x$$

Boundary Condition for TE Modes in Metallic Waveguide



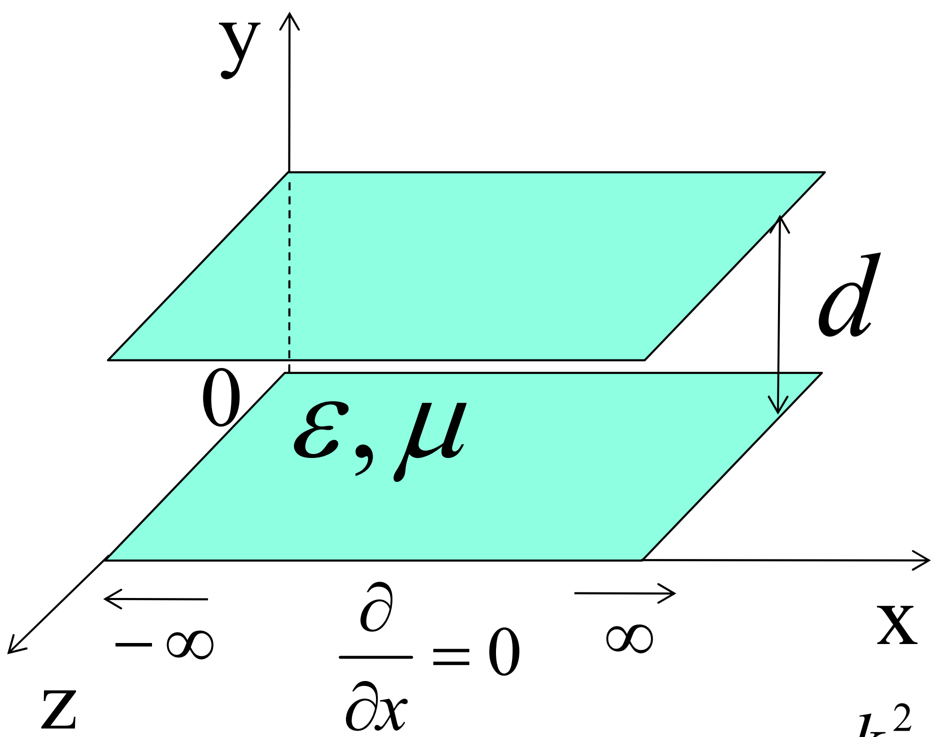
$$\mathbf{h}_{\mathbf{t}} = \frac{-jk_z}{k_c^2} \nabla_t h_z$$

$$\mathbf{n} \cdot (\mu \mathbf{h}_{\mathbf{t}}) = 0$$

$$\Rightarrow \mathbf{n} \cdot (\nabla_t h_z) = 0$$

$$\Rightarrow \frac{\partial h_z}{\partial n} = 0$$

Parallel Plate Waveguide TE Modes (1)



$$\nabla_t^2 h_z(y) + k_c^2 h_z(y) = 0$$

$$\Rightarrow \quad \frac{\partial h_z}{\partial y}\Big|_{y=0} = 0$$

$$\frac{\partial h_z}{\partial y}\Big|_{y=d} = 0$$

$$\nabla_t^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{d^2}{dy^2}$$

$$k_c^2 = k_x^2 + k_y^2 = k_y^2 \quad \text{since } k_x = 0$$

$$\Rightarrow \frac{d^2 h_z(y)}{dy^2} + k_y^2 h_z(y) = 0$$

Parallel Plate Waveguide TE Modes (2)

General solution: $h_z(y) = A\cos(k_y y) + B\sin(k_y y)$

$$\frac{\partial h_z}{\partial y} = -Ak_y \sin(k_y y) + Bk_y \cos(k_y y)$$

$$\left.\frac{\partial h_z}{\partial y}\right|_{y=0} = 0 \;\Rightarrow\; B = 0$$

$$\left.\frac{\partial h_z}{\partial y}\right|_{y=d} = 0 \;\Rightarrow\; \sin(k_y d) = 0$$

$$\Rightarrow k_y = \frac{n\pi}{d}, \quad n = 1,2,\ldots$$

$$k_z = \sqrt{k^2 - \left(\frac{n\pi}{d}\right)^2}$$

$$h_{zn}(y) = A_n \cos\left(\frac{n\pi}{d} y\right) \qquad \text{for TE}_n \text{ mode}$$

Parallel Plate Waveguide TE Modes (3)

Other Field Components:

$$e_{xn} = \frac{j\omega\mu}{k_{yn}} A_n \sin(\frac{n\pi}{d} y)$$

$$e_{yn} = 0$$

$$h_{xn} = 0$$

$$h_{yn} = \frac{jk_{zn}}{k_{yn}} A_n \sin(\frac{n\pi}{d} y)$$

$$k_{yn} = \frac{n\pi}{d}, \quad n = 1,2,\ldots \qquad k_{zn} = \sqrt{k^2 - \left(\frac{n\pi}{d}\right)^2}$$

$$e_x = \frac{-j\omega\mu}{k_c^2} \frac{\partial h_z}{\partial y}$$

$$e_y = \frac{j\omega\mu}{k_c^2} \frac{\partial h_z}{\partial x}$$

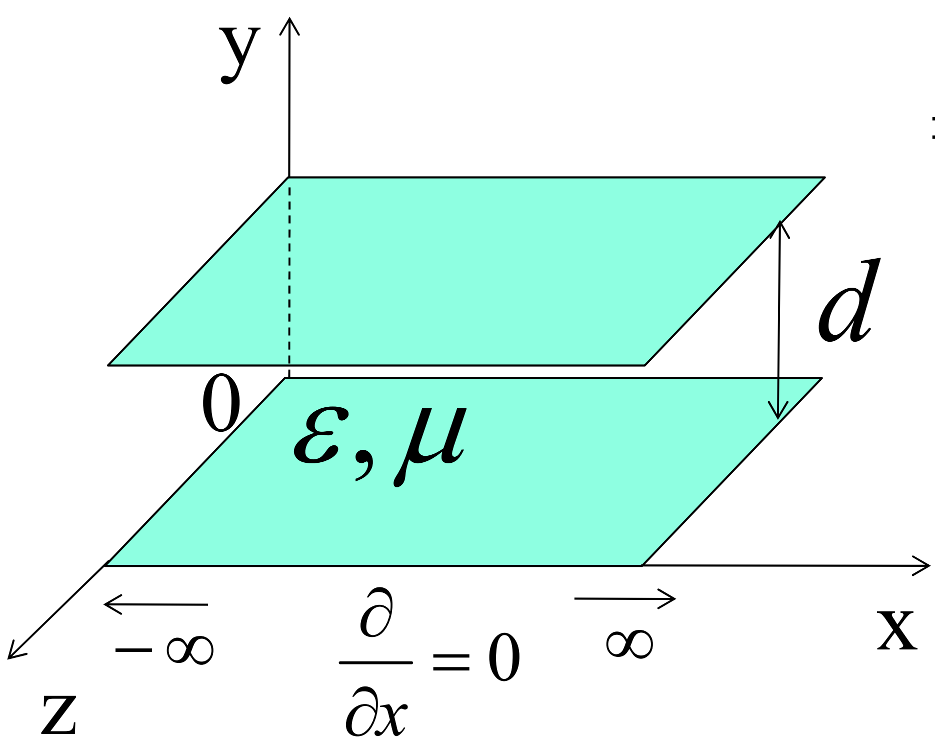
$$h_x = \frac{-jk_z}{k_c^2} \frac{\partial h_z}{\partial x}$$

$$h_y = \frac{-jk_z}{k_c^2} \frac{\partial h_z}{\partial y}$$

Cut off Frequency

$$f_c = \frac{k_c}{2\pi\sqrt{\mu\varepsilon}} = \frac{n}{2d\sqrt{\mu\varepsilon}}$$

Parallel Plate Waveguide TM Modes (1)



$$\Rightarrow \begin{cases} \nabla_t^2 e_z(y) + k_c^2 e_z(y) = 0 \\ e_z(0) = 0 \\ e_z(d) = 0 \end{cases}$$

$$\nabla_t^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$= \frac{d^2}{dy^2} \quad \text{since } \frac{\partial}{\partial x} = 0$$

$$k_c^2 = k_x^2 + k_y^2 = k_y^2 \quad \text{since } k_x = 0$$

$$\Rightarrow \frac{d^2 e_z(y)}{dy^2} + k_y^2 e_z(y) = 0$$

Parallel Plate Waveguide
TM Modes (2)

General solution: $e_z(y) = A\cos(k_y y) + B\sin(k_y y)$

$$e_z(0) = 0 \;\Rightarrow\; A = 0$$

$$e_z(d) = 0 \;\Rightarrow\; \sin(k_y d) = 0$$

$$\Rightarrow k_y = \frac{n\pi}{d}, \quad n = 1,2,\ldots$$

$$k_z = \sqrt{k^2 - \left(\frac{n\pi}{d}\right)^2}$$

$$e_{zn}(y) = B_n \sin(\frac{n\pi}{d} y)$$ for $\mathrm{TM_n}$ mode

Parallel Plate Waveguide TM Modes (3)

Other Field Components:

$$e_{xn} = 0$$

$$e_{yn} = \frac{-jk_{zn}}{k_{yn}} B_n \cos(\frac{n\pi}{d} y)$$

$$h_{xn} = \frac{j\omega\varepsilon}{k_{yn}} B_n \cos(\frac{n\pi}{d} y)$$

$$h_{yn} = 0$$

$$k_{yn} = \frac{n\pi}{d}, \quad n = 1,2,...$$

$$k_{zn} = \sqrt{k^2 - \left(\frac{n\pi}{d}\right)^2}$$

$$e_x = \frac{-jk_z}{k_c^2} \frac{\partial e_z}{\partial x}$$

$$e_y = \frac{-jk_z}{k_c^2} \frac{\partial e_z}{\partial y}$$

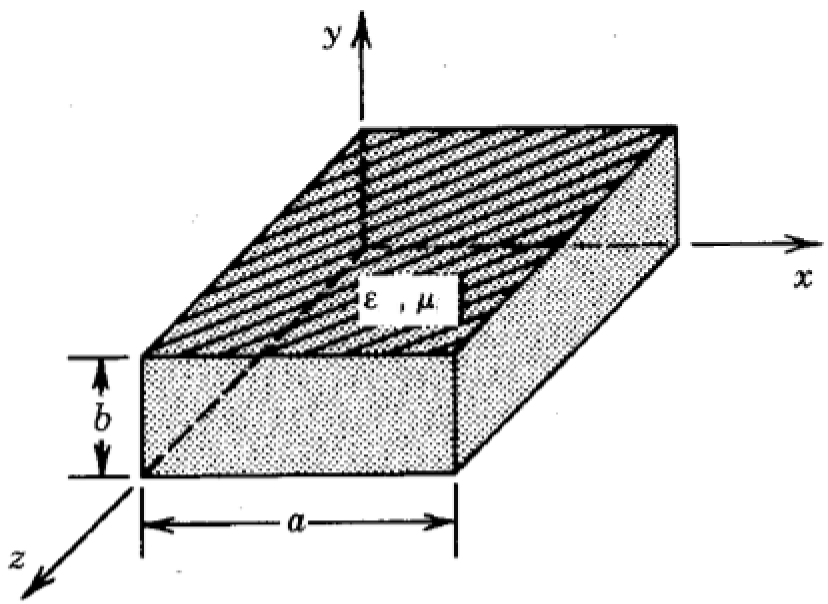
$$h_x = \frac{j\omega\varepsilon}{k_c^2} \frac{\partial e_z}{\partial y}$$

$$h_y = \frac{-j\omega\varepsilon}{k_c^2} \frac{\partial e_z}{\partial x}$$

Cut off Frequency $f_c = \frac{k_c}{2\pi\sqrt{\mu\varepsilon}} = \frac{n}{2d\sqrt{\mu\varepsilon}}$

Rectangular Waveguides

Rectangular Waveguide
TE Modes (1)



$$\nabla_t^2 h_z + k_c^2 h_z = 0$$

Or $$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_c^2\right) h_z(x, y) = 0$$

$$\left.\frac{\partial h_z(x, y)}{\partial y}\right|_{y=0,b} = 0$$

$$\left.\frac{\partial h_z(x, y)}{\partial x}\right|_{x=0,a} = 0$$

Rectangular Waveguide TE Modes (2)

Let $h_z(x,y) = X(x)Y(y)$ (separation of variables)

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_c^{\ 2}\right) X(x)Y(y) = 0$$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + k_c^{\ 2} = 0$$

Define $\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -k_x^{\ 2}$ $\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -k_y^{\ 2}$

Rectangular Waveguide TE Modes (3)

$$\frac{d^2 X}{dx^2} + k_x{}^2 X = 0$$

$$\frac{d^2 Y}{dy^2} + k_y{}^2 Y = 0$$

$$k_x{}^2 + k_y{}^2 = k_c{}^2$$

General Solution

$$h_z(x, y) = [A\cos(k_x x) + B\sin(k_x x)][C\cos(k_y y) + D\sin(k_y y)]$$

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Rectangular Waveguide
TE Modes (4)

$$h_z(x,y) = [A\cos(k_x x) + B\sin(k_x x)][C\cos(k_y y) + D\sin(k_y y)]$$

$$\left.\frac{\partial h_z}{\partial x}\right|_{x=0} = 0 \quad \Rightarrow B = 0$$

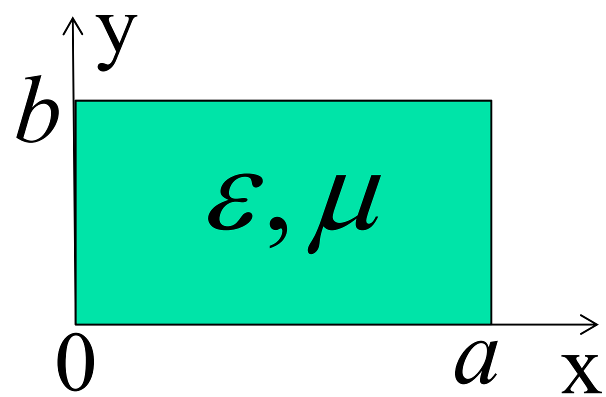
$$\left.\frac{\partial h_z}{\partial x}\right|_{x=a} = 0 \quad \Rightarrow \sin(k_x a) = 0$$

$$\Rightarrow k_x = \frac{m\pi}{a} \qquad m = 0,1,2,.....$$

$$\left.\frac{\partial h_z}{\partial y}\right|_{y=0} = 0 \quad \Rightarrow D = 0$$

$$\left.\frac{\partial h_z}{\partial y}\right|_{y=b} = 0 \quad \Rightarrow \sin(k_y b) = 0$$

$$\Rightarrow k_y = \frac{n\pi}{b} \qquad n = 0,1,2,.....$$



m and *n* cannot be equal to 0 at the same at the same time

Rectangular Waveguide TE Modes (5)

$$h_{zmn}(x,y) = A_{mn}\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right) \qquad (\text{TE}_{mn}\ \text{Mode})$$

$$\Rightarrow e_{xmn}(x,y) = \frac{j\omega\mu}{k_c^{\,2}}\frac{n\pi}{b}A_{mn}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

$$e_{ymn}(x,y) = \frac{-j\omega\mu}{k_c^{\,2}}\frac{m\pi}{a}A_{mn}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

$$h_{xmn}(x,y) = \frac{jk_z}{k_c^{\,2}}\frac{m\pi}{a}A_{mn}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

$$h_{ymn}(x,y) = \frac{jk_z}{k_c^{\,2}}\frac{n\pi}{b}A_{mn}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{-jk_z z}$$

$$k_z = \sqrt{k^2 - k_c^{\,2}} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

Rectangular Wave Guide TE modes (6)

Characteristic Impedance $Z_{TE} = \frac{E_x}{H_y} = -\frac{E_y}{H_x} = \frac{\omega\mu}{k_z}$

$$k_z = \sqrt{k^2 - k_c^2} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

EDC $\varepsilon_{eff} = \left(\frac{k_z}{k_0}\right)^2 = \frac{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}{k_0^2}$

Cutoff frequency $f_{c_{mn}} = \frac{k_c}{2\pi\sqrt{\mu\varepsilon}} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$

$$m, n = 0, 1, 2, \ldots..$$

m and *n* cannot be equal to 0 at the same at the same time

Dominant mode: The mode with the largest ε_{eff} or the lowest cutoff frequency

If $a > b \Rightarrow \mathrm{TE}_{10}$ (or H_{10}) mode

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Rectangular Waveguide TE Modes (7)

- Guide Wavelength

$$\lambda_g = \frac{2\pi}{k_z} = \frac{2\pi}{\sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}} \geq \frac{2\pi}{k} = \lambda$$

- Phase Velocity

$$v_p = \frac{\omega}{k_z} = \frac{\omega}{\sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}} \geq \frac{\omega}{k} = \frac{1}{\sqrt{\mu\varepsilon}} = v_c$$

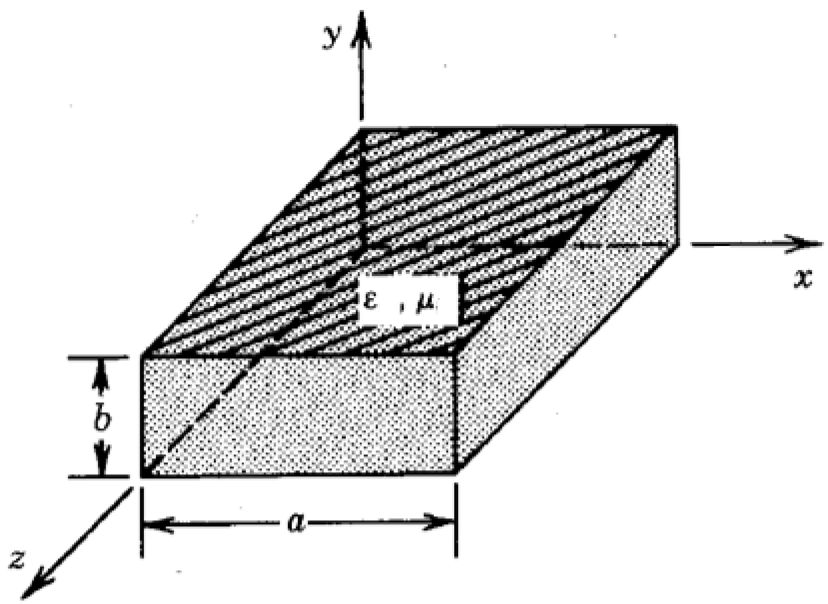
fast wave

v_c is the velocity of light in (ε,μ)

- Group Velocity

$$v_g = \left(\frac{dk_z}{d\omega}\right)^{-1} = \frac{v_c k_z}{k} \leq v_c$$

Rectangular Waveguide
TM Modes (1)



$$\nabla_t^{\ 2} e_z + k_c^{\ 2} e_z = 0$$

Or $$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_c^{\ 2} \right) e_z(x,y) = 0$$

$$e_z(x,y)\Big|_{x=0,a} = 0$$

$$e_z(x,y)\Big|_{y=0,b} = 0$$

General Solution

$$e_z(x,y) = [A\cos(k_x x) + B\sin(k_x x)][C\cos(k_y y) + D\sin(k_y y)]$$

Rectangular Waveguide TM modes (2)

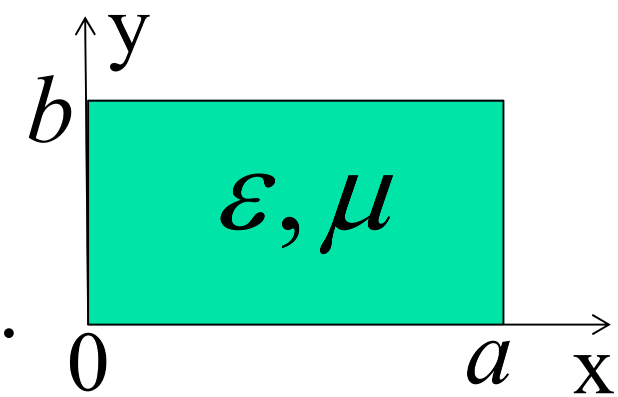
$$e_z(x,y)=[A\cos(k_x x)+B\sin(k_x x)][C\cos(k_y y)+D\sin(k_y y)]$$

$$e_z\big|_{x=0}=0\Rightarrow A=0$$

$$e_z\big|_{x=a}=0\Rightarrow k_x=\frac{m\pi}{a},\ m=1,2,\ldots$$

$$e_z\big|_{y=0}=0\Rightarrow C=0$$

$$e_z\big|_{y=b}=0\Rightarrow k_y=\frac{n\pi}{b},\ n=1,2,\ldots$$



Rectangular Waveguide TM modes (3)

$$e_{zmn}(x,y) = B_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \qquad (\text{TM}_{mn}\ \text{Mode})$$

$$\Rightarrow e_{xmn}(x,y) = \frac{-jk_z}{k_c^2}\frac{m\pi}{a}B_{mn}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

$$e_{ymn}(x,y) = \frac{-jk_z}{k_c^2}\frac{n\pi}{b}B_{mn}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

$$h_{xmn}(x,y) = \frac{j\omega\varepsilon}{k_c^2}\frac{n\pi}{b}B_{mn}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

$$h_{ymn}(x,y) = \frac{-j\omega\varepsilon}{k_c^2}\frac{m\pi}{a}B_{mn}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{-jk_z z}$$

$$m, n = 1, 2, 3, \ldots$$

Rectangular Waveguide TM modes (4)

$$k_z = \sqrt{k^2 - k_c^2} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

Characteristic Impedance

$$Z_{TM} = \frac{E_x}{H_y} = \frac{-E_y}{H_x} = \frac{k_z}{\omega\varepsilon}$$

$$\text{Note : } \mathrm{TE}_{mn} \text{ mode is degenerate to } \mathrm{TM}_{mn} \text{ mode!}$$

Rectangular Waveguide TE_{10} (Dominant) Mode (1)

$$k_x = \frac{m\pi}{a}, \quad m = 0, 1, 2\ldots \text{ for TE modes}$$

$$m = 1, 2, \ldots \text{ for TM modes}$$

$$k_y = \frac{n\pi}{b}, n = 0, 1, 2\ldots \text{ for TE modes}$$

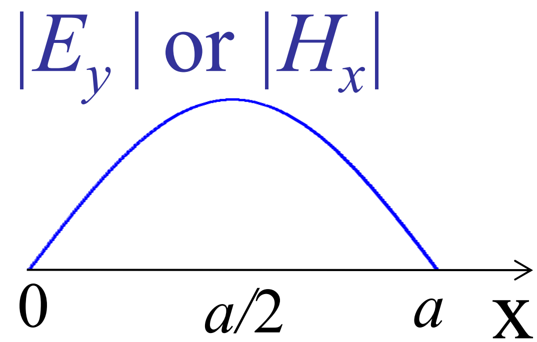
$$(\text{No } m = 0 \;\&\; n = 0)$$

$$k_c^2 = k_x^2 + k_y^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

When $a>b$, the dominant mode is TE_{10} mode.

Rectangular Waveguide TE_{10} (Dominant) Mode (2)

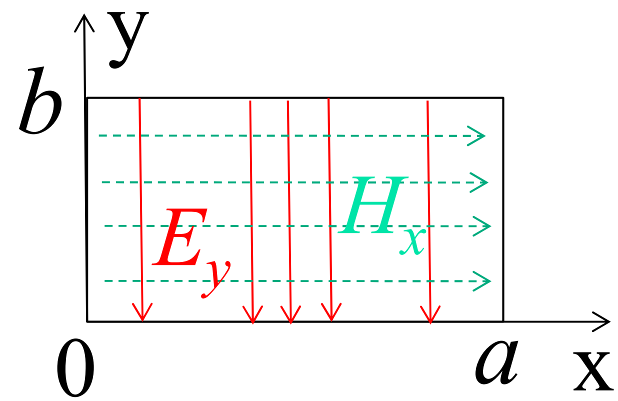
- For TE_{10} mode:



$$H_z = A_{10}\cos\left(\frac{\pi x}{a}\right)e^{-jk_z z}$$

$$E_y = \frac{-j\omega\mu a}{\pi}A_{10}\sin\left(\frac{\pi x}{a}\right)e^{-jk_z z}$$

$$H_x = \frac{jk_z a}{\pi}A_{10}\sin\left(\frac{\pi x}{a}\right)e^{-jk_z z}$$



$$E_x = E_z = H_y = 0\ ,\ k_x = \frac{\pi}{a},\ k_y = 0$$

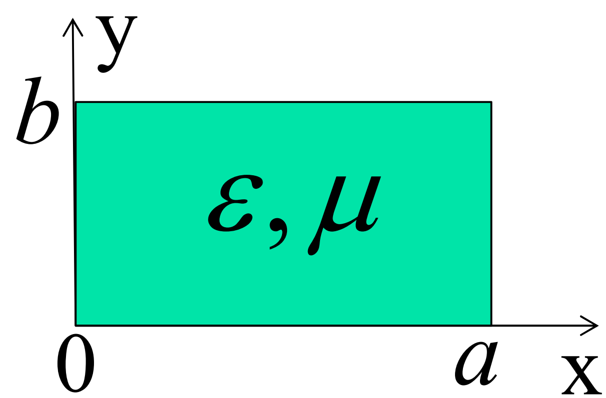
Rectangular Waveguide TE_{10} (Dominant) Mode (3)

Power flow down the guide

$$P_{10} = \frac{1}{2}\mathrm{Re}\left[\int_0^a\int_0^b (\mathbf{E}\times\mathbf{H}^*)\cdot\mathbf{a}_z\,dydx\right]$$

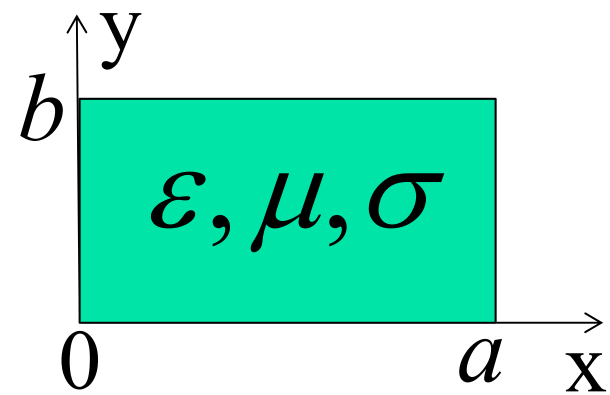
$$= \frac{\omega\mu a^2}{2\pi^2}\mathrm{Re}(k_z)|A_{10}|^2\int_0^a\int_0^b \sin^2\left(\frac{\pi x}{a}\right)dx$$

$$= \frac{\omega\mu a^3\beta}{4\pi^2}|A_{10}|^2 b$$



Rectangular Waveguide
TE_{10} (Dominant) Mode (4)

Power Loss Per Unit Length



$$P_l = \frac{R_s}{2} \oint |H|^2 dl$$

$$= \frac{R_s}{2}\left[2\int_0^b |A_{10}|^2 dy\right] + \frac{R_s}{2}\left\{2\int_0^a |A_{10}|^2 \left[\cos^2\left(\frac{\pi x}{a}\right) + \left(\frac{\beta a}{\pi}\right)^2 \sin^2\left(\frac{\pi x}{a}\right)\right] dx\right\}$$

$$= R_s |A_{10}|^2 \left(b + \frac{a}{2} + \frac{\beta^2 a^3}{2\pi^2}\right)$$

Rectangular Waveguide TE_{10} (Dominant) Mode (5)

Use $P(z) = P(0)e^{-2\alpha z}$

$\mathbf{E} \propto e^{-\alpha z}$

$\mathbf{H} \propto e^{-\alpha z}$

$k_z = \beta - j\alpha$

$$\Rightarrow P_l = \frac{-dP(z)}{dz} = +2\alpha P(z)$$

$$\Rightarrow \alpha_c = \frac{P_l}{2P(z)} = \frac{P_l}{2P_{10}}$$

$$= \frac{R_s}{a^3 b\beta\omega\mu}\left(2\pi^2 b + \pi^2 a + \beta^2 a^3\right)\text{Nep/m}$$

Rectangular Waveguide TE_{10} (Dominant) Mode (6)

The first higher order mode:
The mode who has the second largest ε_{eff}
(or the second cutoff frequency).

Example: a = 2.286cm, b = 1.016cm
Rectangular guide, air filled.

Mode	m	n	f_c (GHz)	
TE	1	0	6.562	
TE	2	0	13.123	$\rightarrow TE_{20}$
TE	0	1	14.764	
TE,TM	1	1	16.156	

Single Mode Operation

$$f_{c\ \text{dominant}} < f < f_{c\ \text{first higher order mode}}$$

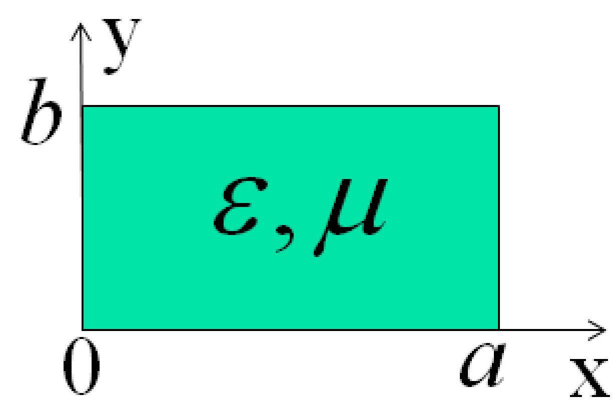
For the previous example:

$$6.562\,\text{GHz} < f < 13.123\,\text{GHz}$$

so that only the TE_{10} mode can propagate

Dielectric Loss

$$k_z^2 = k^2 - k_c^2 \qquad \tilde{\varepsilon} = \varepsilon - j\frac{\sigma}{\omega}$$



$$k_z = \sqrt{\omega^2\mu\left(\varepsilon - j\frac{\sigma}{\omega}\right) - k_c^2}$$

$$= \sqrt{(\omega^2\mu\varepsilon - k_c^2) - j\omega\sigma}$$

$$= \sqrt{(\omega^2\mu\varepsilon - k_c^2)\left(1 - \frac{j\omega\mu\sigma}{\omega^2\mu\varepsilon - k_c^2}\right)}$$

$$\approx \sqrt{\omega^2\mu\varepsilon - k_c^2}\left[1 - \frac{j\omega\mu\sigma}{2(\omega^2\mu\varepsilon - k_c^2)}\right]$$

$$= \beta - j\alpha_d$$

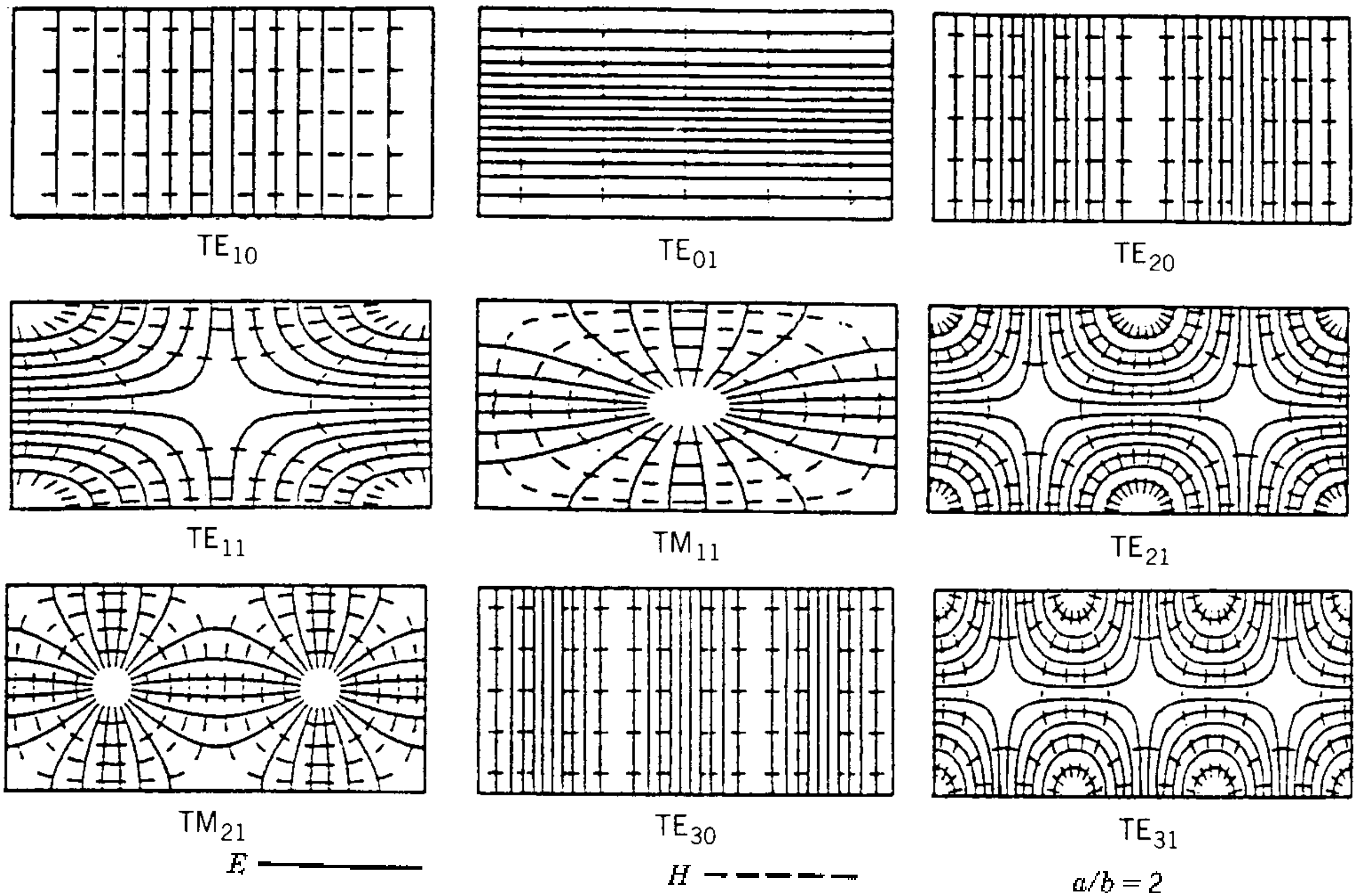
$$\Rightarrow \beta = \sqrt{\omega^2\mu\varepsilon - k_c^2}$$

$$\alpha_d = \sqrt{\omega^2\mu\varepsilon - k_c^2}\,\frac{\omega\mu\sigma}{2(\omega^2\mu\varepsilon - k_c^2)} = \frac{\omega\mu\sigma}{2\sqrt{\omega^2\mu\varepsilon - k_c^2}}$$

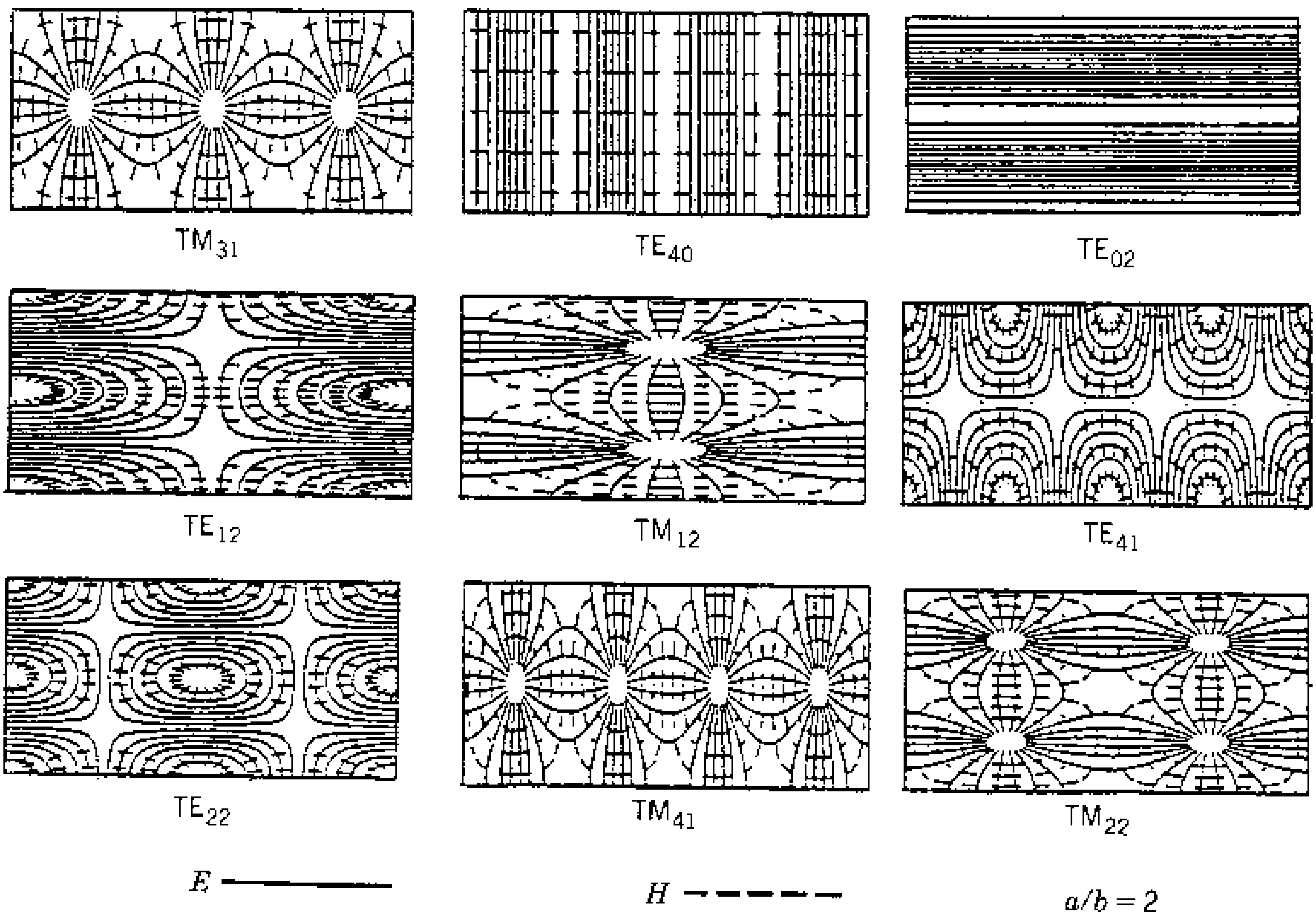
Low loss:

$$\omega\mu\sigma << \omega^2\mu\varepsilon - k_c^2$$

Field Patterns (1)



Field Patterns (2)



Field Patterns (3)

TE modes:

$$\mathbf{H}_t = \frac{-jk_z}{k_c^2}\nabla_t H_z$$

$$\mathbf{E}_t = -\frac{\omega\mu}{k_z}\mathbf{a}_z \times \mathbf{H}_t$$

Constant H_z line // $\mathbf{E}_t$

TM modes:

$$\mathbf{E}_t = \frac{-jk_z}{k_c^2}\nabla_t E_z$$

$$\mathbf{H}_t = \frac{\omega\varepsilon}{k_z}\mathbf{a}_z \times \mathbf{E}_t$$

Constant E_z line // $\mathbf{H}_t$

Coupling