



## Coaxial Line

## Coaxial Line – TEM Mode (1)

TEM mode,  $\frac{\partial}{\partial \phi} = 0$

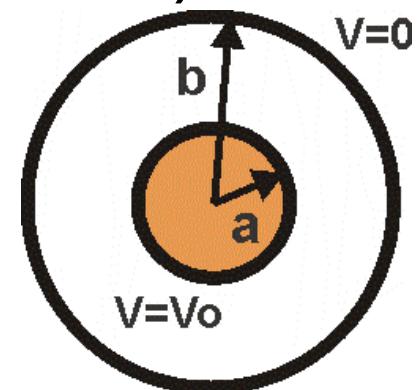
(symmetry in  $\phi$  direction)

$$\nabla_t^2 V = 0$$

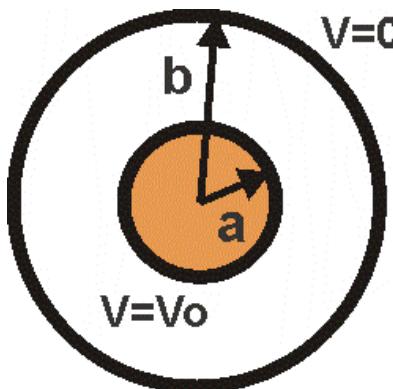
$$\Rightarrow \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial V}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 V}{\partial \phi^2} = 0 \quad \frac{1}{\rho^2} \frac{\partial^2 V}{\partial \phi^2} = 0$$

$$\Rightarrow \frac{1}{\rho} \frac{d}{d \rho} \left( \rho \frac{dV}{d \rho} \right) = 0$$

$$\Rightarrow \rho \frac{dV}{d \rho} = C \quad V = C \ln \rho + D$$



## Coaxial Line – TEM Mode (2)



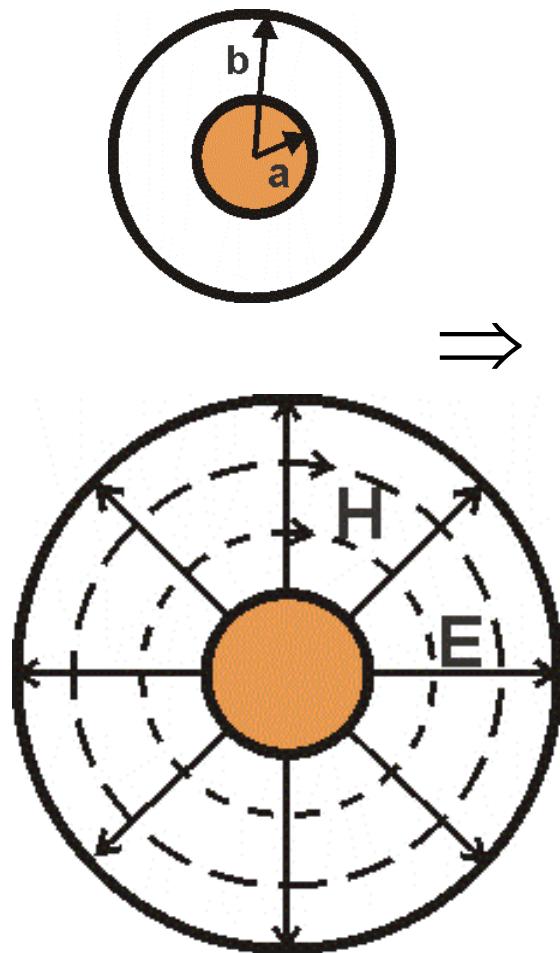
$$V = C \ln \rho + D$$

$$\left. \begin{array}{l} V(\rho = a) = V_0 \quad C \ln a + D = V_0 \\ V(\rho = b) = 0 \quad C \ln b + D = 0 \end{array} \right\}$$

$$\Rightarrow C = \frac{V_0}{\ln a - \ln b} = -\frac{V_0}{\ln \frac{b}{a}} \quad D = \frac{V_0 \ln b}{\ln \frac{b}{a}}$$

$$\Rightarrow V(\rho) = -\frac{V_0}{\ln \frac{b}{a}} \ln \rho + \frac{V_0 \ln b}{\ln \frac{b}{a}} = \frac{V_0 \ln \frac{b}{\rho}}{\ln \frac{b}{a}}$$

## Coaxial Line – TEM Mode (3)



$$V(\rho) = \frac{V_0 \ln \frac{b}{\rho}}{\ln \frac{b}{a}}$$

$$\Rightarrow \mathbf{E} = (-\nabla_t V) e^{-jkz} \quad \mathbf{H} = \frac{1}{\eta} \mathbf{a}_z \times \mathbf{E}$$

$$= \frac{V_0}{\ln \frac{b}{a}} \frac{\mathbf{a}_\rho}{\rho} e^{-jkz}$$

$$= \frac{V_0}{\eta \ln \frac{b}{a}} \frac{\mathbf{a}_\phi}{\rho} e^{-jkz}$$

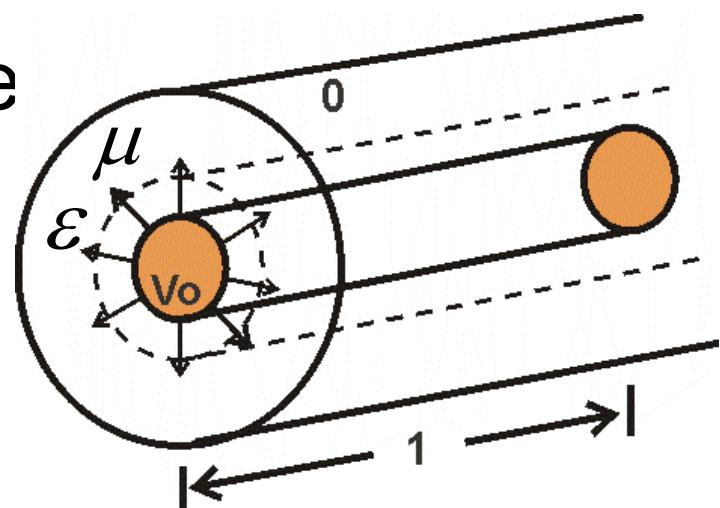
## Coaxial Line – TEM Mode (4)

Per-Unit-Length Capacitance

$$Q = (2\pi\rho \cdot l)\epsilon E$$

$$= 2\pi\rho \cdot \epsilon \left( \frac{V_0}{\ln \frac{b}{a}} \frac{1}{\rho} \right) = 2\pi\rho \cdot \epsilon \frac{V_0}{\ln \frac{b}{a}}$$

$$\Rightarrow C = \frac{Q}{V_0} = \frac{2\pi\epsilon}{\ln \frac{b}{a}}$$



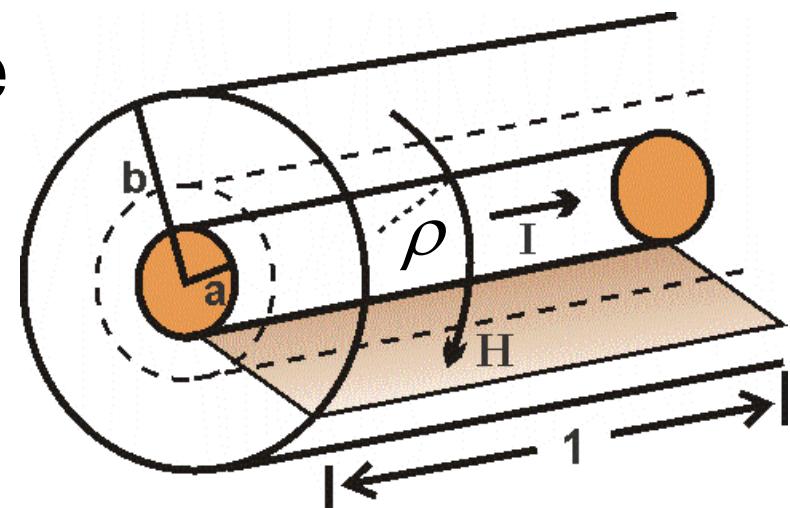
## Coaxial Line – TEM Mode (5)

Per-Unit-Length Inductance

$$2\pi\rho H_\phi = I$$

$$\Rightarrow H_\phi = \frac{I}{2\pi\rho}$$

$$\psi = \int_a^b \frac{\mu I}{2\pi\rho} d\rho = \frac{\mu I}{2\pi\rho} \ln \frac{b}{a} \quad \Rightarrow L = \frac{\psi}{I} = \frac{\mu}{2\pi} \ln \frac{b}{a}$$



Note:

$$\frac{I}{2\pi\rho} = \frac{V_0}{\eta \ln \frac{b}{a}} \frac{1}{\rho} = H_\phi \quad \Rightarrow \quad I = \frac{2\pi V_0}{\eta \ln \frac{b}{a}}$$

## Coaxial Line – TEM Mode (6)

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$$C = \frac{2\pi\epsilon}{\ln \frac{b}{a}} \quad L = \frac{\mu}{2\pi} \ln \frac{b}{a}$$

$$\Rightarrow \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\mu\epsilon}} \quad Z_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{\frac{\mu}{2\pi} \ln \frac{b}{a}}{\frac{2\pi\epsilon}{\ln \frac{b}{a}}}}$$

$$= \frac{1}{2\pi} \sqrt{\frac{\mu}{\epsilon}} \ln \frac{b}{a} = 60 \sqrt{\frac{\mu_r}{\epsilon_r}} \ln \frac{b}{a}$$

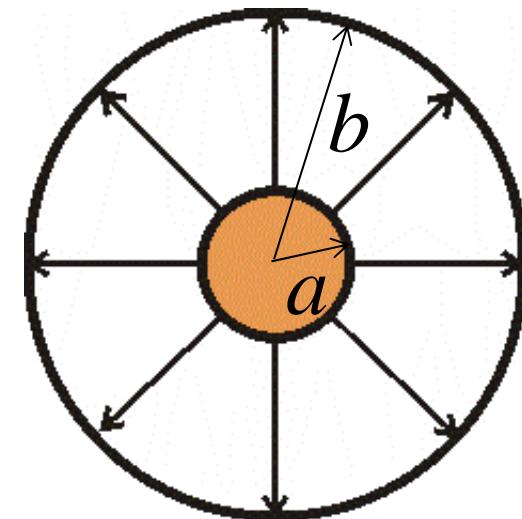
$Z_0$  depends on the relative size of  $a$  &  $b$

## Coaxial Line – TEM Mode (7)

From  $\mathbf{E} = \frac{V_0}{\ln \frac{b}{a}} \frac{\mathbf{a}_\rho}{\rho} e^{-jkz}$

$$|\mathbf{E}|_{\max} = |\mathbf{E}|_{\rho=a} = \frac{V_0}{\ln \frac{b}{a}} \frac{1}{a}$$

$$\Rightarrow V_0 = |\mathbf{E}|_{\max} \ln \frac{b}{a} \cdot a$$



# Coaxial Line – TEM Mode (8)

$$\begin{aligned}
P_{av} &= \frac{1}{2} \operatorname{Re} \left[ \int_a^b \int_0^{2\pi} (\mathbf{E} \times \mathbf{H}^*) \cdot \mathbf{a}_z \rho d\rho d\phi \right] \\
&= \frac{\pi V_0^2}{\eta \ln \frac{b}{a}} \quad V_0 = |\mathbf{E}|_{\max} \ln \frac{b}{a} \cdot a \\
&= \frac{\pi}{\eta} |\mathbf{E}|_{\max} a^2 \ln \frac{b}{a}
\end{aligned}$$

Require  $|\mathbf{E}|_{\max} < |\mathbf{E}_{br}|$

⇒ Power capacity:

broken

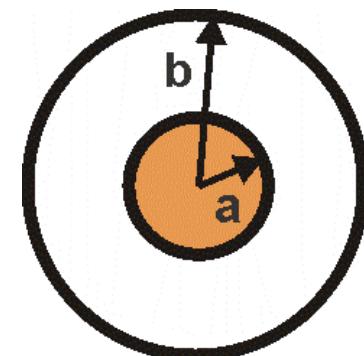
$$P_{br} = \frac{\pi}{\eta} |\mathbf{E}_{br}|^2 a^2 \ln \frac{b}{a}$$

# Coaxial Line – TEM Mode (9)

Maximum Power Capacity:

$$P_{br} = \frac{\pi}{\eta} |E_{br}|^2 a^2 \ln \frac{b}{a}$$

For b fixed, from:  $\frac{\partial P_{br}}{\partial a} = 0$



$$\Rightarrow \frac{b}{a} = 1.649 \quad \text{for maximum power capacity}$$

$$Z_0 = 30\Omega \quad \text{for } \epsilon_r = \mu_r = 1$$

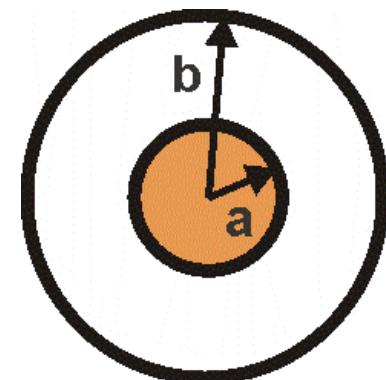
# Coaxial Line – TEM Mode (10)

Conductor Loss:

$$P_l = \frac{R_s}{2} \left[ \int_0^{2\pi} \left| H_\phi \right|_{\rho=b}^2 \cdot bd\phi + \int_0^{2\pi} \left| H_\phi \right|_{\rho=a}^2 \cdot ad\phi \right]$$

$$= \frac{R_s}{2} 2\pi \left| \frac{V_0}{\eta \ln \frac{b}{a}} \right|^2 b + \frac{R_s}{2} 2\pi \left| \frac{V_0}{\eta \ln \frac{b}{a}} \right|^2 a$$

$$= \frac{R_s \pi V_0^2}{\eta^2} \frac{a+b}{ab}$$



$$\alpha_c = \frac{P_l}{2P_{av}} = \frac{R_s}{2\eta b} \frac{1 + \frac{b}{a}}{\ln \frac{b}{a}} \text{ Nep/m}$$

# Coaxial Line – TEM Mode (11)

Minimum Conductor Loss :  $\alpha_c = \frac{R_s}{2\eta b} \frac{1 + b/a}{\ln(b/a)}$

For b fixed, from  $\frac{\partial \alpha_c}{\partial a} = 0$

$$\Rightarrow \frac{b}{a} = 3.591 \quad \rightarrow Z_0 = 76.71\Omega$$

for  $\epsilon_r = \mu_r = 1$

To balance maximum power capacity  $Z_0 \approx 30\Omega$

and minimum conductor loss  $Z_0 \approx 76.71\Omega$

$\Rightarrow$  Standard Coaxial Line  $Z_0 = 50\Omega$

$$\begin{cases} b/a = 2.303 \\ \text{for } \epsilon_r = \mu_r = 1 \end{cases}$$

Sometime  $Z_0 = 75\Omega$  is used for minimum  $\alpha_c$

## Coaxial Line – Higher Order Modes (1)

TE Modes

$$\left\{ \begin{array}{l} \nabla_t^2 h_z + k_c^2 h_z = 0 \\ \left. \frac{\partial h_z}{\partial \rho} \right|_{\rho=a,b} = 0 \end{array} \right.$$

Let  $h_z(\rho, \phi) = \overbrace{[A \sin(m\phi) + B \cos(m\phi)]}^{\text{Polarization degenerate}} [C J_m(k_c \rho) + D Y_m(k_c \rho)]$

Boundary Conditions

$$\Rightarrow \begin{cases} C J_m'(k_c a) + D Y_m'(k_c a) = 0 \\ C J_m'(k_c b) + D Y_m'(k_c b) = 0 \end{cases}$$

$$\begin{bmatrix} J_m'(k_c a) & Y_m'(k_c a) \\ J_m'(k_c b) & Y_m'(k_c b) \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = 0$$

## Coaxial Line – Higher Order Modes (2)

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The only nontrivial solution occurs when

$$\begin{vmatrix} J_m'(k_c a) & Y_m'(k_c a) \\ J_m'(k_c b) & Y_m'(k_c b) \end{vmatrix} = 0$$

$$\Rightarrow J_m'(k_c a)Y_m'(k_c b) - J_m'(k_c b)Y_m'(k_c a) = 0$$

Characteristic equation for TE<sub>11</sub> (H<sub>11</sub>) mode:  $k_c \approx \frac{2}{a+b}$

Note:  $m$  must be integers because of periodic boundary conditions

$$h_z(\rho, \phi + 2\pi) = h_z(\rho, \phi)$$

$$\begin{aligned} \Rightarrow A \sin[m(\phi + 2\pi)] + B \cos[m(\phi + 2\pi)] &= A \sin(m\phi + 2m\pi) + B \cos(m\phi + 2m\pi) \\ &= A \sin(m\phi) + B \cos(m\phi) \end{aligned}$$

$\Rightarrow m$  must be integers

# Coaxial Line – Higher Order Modes (3)

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TM Modes

$$\begin{cases} \nabla_t^2 e_z + k_c^2 e_z = 0 \\ e_z|_{\rho=a,b} = 0 \end{cases}$$

$$\text{Let } e_z(\rho, \phi) = [A \sin(m\phi) + B \cos(m\phi)][C J_m(k_c \rho) + D Y_m(k_c \rho)]$$

Boundary Conditions

$$\Rightarrow \begin{cases} C J_m(k_c a) + D Y_m(k_c a) = 0 \\ C J_m(k_c b) + D Y_m(k_c b) = 0 \end{cases}$$

$$\begin{bmatrix} J_m(k_c a) & Y_m(k_c a) \\ J_m(k_c b) & Y_m(k_c b) \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = 0$$

## Coaxial Line – Higher Order Modes (4)

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The only nontrivial solution occurs when

$$\begin{vmatrix} J_m(k_c a) & Y_m(k_c a) \\ J_m(k_c b) & Y_m(k_c b) \end{vmatrix} = 0$$

$$\Rightarrow J_m(k_c a)Y_m(k_c b) - J_m(k_c b)Y_m(k_c a) = 0$$

Characteristic equation for TM modes

## Coaxial Line – Higher Order Modes (5)

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$$\text{TE : } J_m'(k_c a)Y_m'(k_c b) - J_m'(k_c b)Y_m'(k_c a) = 0$$

$$\text{TM : } J_m(k_c a)Y_m(k_c b) - J_m(k_c b)Y_m(k_c a) = 0$$

Since  $J_0'(x) = -J_1(x)$  for Bessel functions

$$Y_0'(x) = -Y_1(x)$$

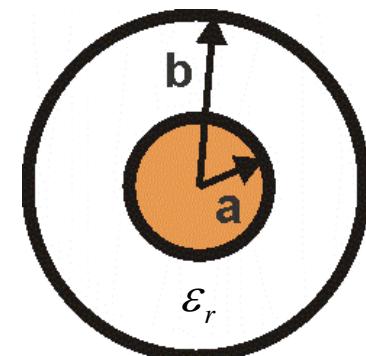
$\Rightarrow \text{TE}_{0n} (\text{H}_{0n})$  mode is degenerate to  $\text{TM}_{1n} (\text{E}_{1n})$  mode

## Coaxial Line – Higher Order Modes (6)

The first higher order mode is  $H_{11}$  ( $TE_{11}$ ) mode with

$$k_c \approx \frac{2}{a+b}$$

$$\Rightarrow f_c = \frac{ck_c}{2\pi\sqrt{\epsilon_r}} = \frac{c}{\pi(a+b)\sqrt{\epsilon_r}}$$



Single mode operation

$$0 < f < \frac{c}{\pi(a+b)\sqrt{\epsilon_r}}$$