1. **5503 +2**

Let

$$A = \begin{bmatrix} I & B \\ B^* & I \end{bmatrix}$$

with $\|B\|_2 < 1$. Show that

$$\|A\|_2 \, |A^{-1}\,|_2 = \frac{1 + \|B\|_2}{1 - \|B\|_2}.$$

2. **5504 +2**

Let

$$A = \begin{bmatrix} a_1 & b_1 & & & 0 \\ c_1 & a_2 & b_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & b_{n-1} \\ 0 & & & c_{n-1} & a_n \end{bmatrix},$$

where $b_i c_i > 0$. Then there exists a diagonal D such that $D^{-1}AD$ is a symmetric tridiagonal matrix.

3. **5012 +2**

Let

$$B_k = B_{k-1} + B_{k-1}(I - AB_{k-1}), \quad k = 1, 2, \cdots.$$

Show that if $\|I - AB_0\| = c < 1$, then

$$\lim_{k \to \infty} B_k = A^{-1}$$

and

$$\|A^{-1} - B_k\| \le \frac{c^{2^k}}{1 - c} \|B_0\|.$$

4. **5001**

Let $A \in \mathbb{C}^{n \times n}$, $x \in \mathbb{C}^n$ and $X = [x, Ax, \cdots, A^{n-1}x]$. Show that if X is nonsingular, then $X^{-1}AX$ is an upper Hessenberg matrix.

5. 5508 +2

Consider the polynomial recurrence

$$p_{k+1}(x) = (x - \alpha_{k+1})p_k(x) - \beta_{k+1}^2 p_{k-1}(x), \quad k = 0, 1, 2, \ldots$$

where $p_0 = 1$, $p_{-1} = 0$, and α_k and β_k are scalars.

Show that the roots of $p_k(x)$ are the eigenvalues of the tridiagonal matrix

$$J_k = \begin{pmatrix} \alpha_1 & \beta_2 & & & \\ \beta_2 & \alpha_2 & \beta_3 & & \\ & & \ddots & & \\ & & \beta_{k-1} & \alpha_{k-1} & \beta_k \\ & & & \beta_k & \alpha_k \end{pmatrix}.$$

6. 5009

Let A' be a given, $n \times n$, real, positive definite matrix partitioned as follows:

$$A' = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix},$$

where A is an $m \times m$ matrix. First, show:

(a) $C - B^T A^{-1} B$ is positive definite.

7. 5502

Let A, B be Hermitian square matrices and

$$H = \begin{bmatrix} A & C \\ C^H & B \end{bmatrix}.$$

Show: For every eigenvalue $\lambda(B)$ of B there is an eigenvalue $\lambda(H)$ of H such that

$$\left|\lambda(H) - \lambda(B)\right| \leq \sqrt{\mathrm{lub}_2(C^H C)}.$$

$$\begin{bmatrix} I & 0 \\ -AH^{-1} & I \end{bmatrix} \begin{bmatrix} H & A^T \\ A & -C \end{bmatrix} \begin{bmatrix} I & -H^{-1}A^T \\ 0 & I \end{bmatrix} = \begin{bmatrix} H & 0 \\ 0 & S \end{bmatrix}, \qquad (2.4.3)$$

where $S = -(C + AH^{-1}A^T)$ is symmetric negative semidefinite. It therefore

Numerical Linear Algebra Exam (Final)
Department of Mathematics, Iran University of Science and Technology
19-June-2010 (1389/3/29)
Time: 180 minutes

1. Prove the following:

   a) $\|x\|_q \le \|x\|_p$ for $p \le q$

   b) $\|A\|_p = \|A^T\|_q$ where $\frac{1}{p} + \frac{1}{q} = 1$ (Hint: Use Hölder inequality).

2. Given the matrix $A \in \mathbb{R}^{m \times n}$ and a vector $b \in \mathbb{R}^m$. Prove

   a) The normal equations $A^T Ax = A^T b$ are consistent.

   b) The vector x minimizes $\|b - Ax\|_2$ if and only if the residual vector $r = b - Ax$ is orthogonal to the range of A, i.e., $A^T(b - Ax) = 0$.

3. Consider the real system of linear equations

$$Ax = b \tag{1}$$

   where A is a nonsingular matrix and satisfies $\langle v, Av \rangle > 0$ for all real vector v.

   a) Show that $\langle v, Av \rangle = \langle v, Mv \rangle$ for all real vector v where $M = \frac{1}{2}(A + A^T)$ which is the symmetric part of A.

   b) Prove that

$$\frac{\langle v, Av \rangle}{\langle v, v \rangle} \ge \lambda_{min}(M) > 0$$

   where $\lambda_{min}(M)$ is the smallest eigenvalue of M (Hint: Use Principal Axes Theorem).

   c) Now consider the following iteration for computing an approximation solution to (1)

$$x_{k+1} = x_k + \alpha r_k$$

   where $r_k = b - Ax_k$ and α is chosen to minimize $\|r_{k+1}\|_2$ as a function of α.
   Prove

$$\frac{\|r_{k+1}\|_2}{\|r_k\|_2} \le \left(1 - \frac{(\lambda_{min}(M))^2}{\lambda_{max}(A^T A)}\right)^{\frac{1}{2}}.$$

4. Prove that the infinite series

$$I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots + \frac{A^n}{n!} + \cdots$$

converges for any square matrix A.

Denote the sum of the series by e^A.

a) If $A = P^{-1}BP$, show that $e^A = P^{-1}e^B P$.

b) Let$\lambda_1, \lambda_2, \cdots, \lambda_n$ denote the eigenvalues of A, repeated according to their multiplicity, and show that the eigenvalues of e^A are $e^{\lambda_1}, e^{\lambda_2}, \cdots, e^{\lambda_n}$.

5. A symmetric matrix A has dominant eigenvalue λ_1 and corresponding eigenvector x_1. Show that the matrix $B = A - \lambda_1 x_1 x_1^T$ has the same eigenvalues as A except that λ_1 is replaced by zero.

6. Assume A is real, symmetric, positive definite, and of order n. Define

$$f(x) = \frac{1}{2}x^T Ax - b^T x \qquad x,\ b \in \mathbb{R}^n.$$

a) Show that the unique minimum of $f(x)$ is given by solving $Ax = b$.

b) Consider the general iterative method

$$x_{k+1} = x_k + \alpha_k d_k$$

where $x_k, d_k \in \mathbb{R}^n$ and $\alpha_k \in \mathbb{R}$. For given x_k and d_k, show that the value of α_k which minimizes $f(x_k + \alpha d_k)$(as a function of α) is given by

$$\alpha_k = \frac{\langle r_k, d_k\rangle}{\langle d_k, Ad_k\rangle}$$

where $r_k = b - Ax_k$.

`Hope the best`
`Nikazad`

(1-a)

$$\|x\|_q = \left\| \|x\|_p \frac{x}{\|x\|_p} \right\|_q = \|x\|_q \left\| \frac{x}{\|x\|_p} \right\|_q \le C_{p,q} \|x\|_q$$

where $C_{p,q} = \max_{\|y\|_p=1} \|y\|_q$ $\quad y = (y_1, \cdots, y_n)^T$

From $p \le q$, we have

$$\|y\|_q^q = \sum_{i=1}^n |y_i|^q \le \sum_{i=1}^n |y_i|^p = 1 \quad (\text{note } |y_i| \le 1)$$

Therefore $C_{p,q} \le 1$.

(1-b) Using Hölder ineq.

$$|y^T x| \le \|x\|_p \|y\|_q$$

where $\frac{1}{p} + \frac{1}{q} = 1$ or

$$\max\{|x^T y| : \|y\|_q = 1\} = \|x\|_p .$$

We have

$$\|A\|_p = \max_{\|x\|_p=1} \|Ax\|_p = \max_{\|x\|_p=1} \max_{\|y\|_q=1} |(Ax)^T y|$$

$$= \max_{\|y\|_q=1} \max_{\|x\|_p=1} |x^T (A^T y)| = \max_{\|y\|_q=1} \|A^T y\|_q$$

$$= \|A^T\|_q$$

(2-a) $A^T b \in \mathcal{R}(A^T) = \mathcal{R}(A^T A)$

why?

(2-b) Let x be a vector for which $A^T(b - Ax) = 0$. Then for any $y \in \mathbb{R}^n$ $b - Ay = (b - Ax) + A(x-y)$. squaring this and using $A^T A x = A^T b$ we obtain

$$\|b - Ay\|_2^2 = \|b - Ax\|_2^2 + \|A(x-y)\|_2^2 \geq \|b - Ax\|_2^2.$$

on the other hand assume that $A^T(b - Ax) = z \neq 0$. Then if $x - y = -\varepsilon z$ we have for sufficiently small $\varepsilon \neq 0$

$$\|b - Ay\|_2^2 = \|b - Ax\|_2^2 - 2\varepsilon\|z\|_2^2 + \varepsilon^2\|Az\|_2^2 < \|b - Ax\|_2^2$$

so x does not minimize $\|b - Ax\|_2$.

(3-a) $\langle v, Mv \rangle = \langle v, \frac{1}{2}(A + A^T) v \rangle$

$$= \frac{1}{2}\langle v, Av \rangle + \frac{1}{2}\langle v, A^T v \rangle$$

$$= \frac{1}{2}\langle v, Av \rangle + \frac{1}{2}\langle v, Av \rangle$$

$$= \langle v, Av \rangle$$

(3-b) $$\frac{\langle v, Av\rangle}{\langle v, v\rangle} = \frac{\langle v, Mv\rangle}{\langle v, v\rangle} = \frac{v^T M v}{v^T v}$$

Using P.A.T we have $M = \underset{\text{orthogonal}}{Q^T} \underset{\text{diagonal}}{\Lambda} Q$

Then $$\frac{\langle v, Av\rangle}{\langle v, v\rangle} = \frac{v^T Q^T \Lambda Q v}{v^T v}$$

Put $Qv =: x$ therefore

$$v = Q^T x, \quad v^T Q^T = x^T, \quad v^T = x^T Q$$

and $$\frac{v^T Q^T \Lambda Q v}{v^T v} = \frac{x^T \Lambda x}{x^T Q\, Q^T x} = \frac{x^T \Lambda x}{x^T x}.$$

But

$$x^T \Lambda x = \lambda_1 x_1^2 + \lambda_2 x_2^2 + \cdots + \lambda_n x_n^2 = \sum_{i=1}^{n} \lambda_i x_i^2$$

then $$\frac{x^T \Lambda x}{x^T x} = \frac{\sum_{i=1}^{n} \lambda_i x_i^2}{\sum_{i=1}^{n} x_i^2} \geq \lambda_{min}(M)$$

From part (a) $\lambda_{min}(M) > 0$

3-C Put $f(\alpha) = \| r_{k+1} \|_2^2$

$$\| r_{k+1} \|_2^2 = \| b - Ax_{k+1} \|_2^2 = b - Ax_k - \alpha A r_k \|_2^2$$

$$= \| r_k - \alpha A r_k \|_2^2$$

$$= \langle r_k, r_k \rangle - 2\alpha \langle Ar_k, r_k \rangle + \alpha^2 \langle Ar_k, Ar_k \rangle$$

$$\Rightarrow f'(\alpha) = -2 \langle Ar_k, r_k \rangle + 2\alpha \langle Ar_k, Ar_k \rangle =: 0$$

$$\alpha = \frac{\langle Ar_k, r_k \rangle}{\langle Ar_k, Ar_k \rangle} \Rightarrow$$

$$f(\alpha) = \| r_{k+1} \|_2^2 = \langle r_k, r_k \rangle - \frac{\langle Ar_k, r_k \rangle^2}{\langle Ar_k, Ar_k \rangle} \Rightarrow$$

$$\frac{\| r_{k+1} \|_2^2}{\| r_k \|_2^2} = 1 - \frac{\langle Ar_k, r_k \rangle^2}{\langle r_k, r_k \rangle \langle Ar_k, Ar_k \rangle}$$

$$= 1 - \frac{\langle Ar_k, r_r \rangle^2 \langle r_k, r_k \rangle}{\langle r_k, r_k \rangle^2 \langle r_k, A^T A r_k \rangle}$$

Using part (b) we get

$$\frac{\| r_{k+1} \|_2}{\| r_k \|_2} \leqslant \left(1 - \frac{\lambda_{min}(M)^2}{\lambda_{max}(A^T A)} \right)^{\frac{1}{2}}$$

4-0 ~~Pages~~

$$S_n = I + A + \cdots + \frac{A^n}{n!} \Rightarrow S_{n+1} - S_n = \frac{A^{n+1}}{(n+1)!}$$

$$\Rightarrow \| S_m - S_n \| \leqslant \frac{\|A\|^{n+p}}{(n+p)!} + \cdots + \frac{\|A\|^{n+1}}{(n+1)!}$$

$m = n+p$

$p > 0$

Since $\sum_{k=0}^{\infty} \frac{\|A\|^k}{k!}$ converges to a point, we conclude

$\{S_n\}$ is a cauchy sequence, then $S_n \rightarrow S =: e^A$

4-a

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} = \sum_{n=0}^{\infty} \frac{(P^{-1}BP)^n}{n!} = \sum_{n=0}^{\infty} \frac{P^{-1}B^n P}{n!}$$

$$= P^{-1}\left(\sum_{n=0}^{\infty} \frac{B^n}{n!}\right)P = P^{-1} e^B P$$

4-b

Let $Ax = \lambda x$ then

$$e^A x = \left(I + A + A^2/2! + \cdots\right)x = \left(x + \lambda x + \frac{\lambda^2}{2!}x + \cdots\right)$$

$$= \left(1 + \lambda + \frac{\lambda^2}{2!} + \cdots\right)x = e^\lambda x$$

(5)

$$A = QDQ^T \qquad Q = (x_1, x_2, \dots, x_n)$$

$D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$, x_i: column vector

$$B = \underbrace{QDQ^T}_{= \lambda_1 x_1 x_1^T + \cdots + \lambda_n x_n x_n^T} - x_1 \lambda_1 x_1^T$$

$$= \lambda_1 x_1 x_1^T + \underbrace{(e_1\ x_2 \cdots x_n)}_{\bar{Q}} \bar{D} (e_1\ x_2 \cdots x_n)^T - x_1 \lambda_1 x_1^T$$

$\bar{D} = \begin{pmatrix} 0 & & & \\ & \lambda_2 & & 0 \\ & & \ddots & \\ & 0 & & \lambda_n \end{pmatrix}$

$$= \bar{Q} \bar{D} \bar{Q}^T$$

(6)

$$f(x) - f(A^{-1}b) = \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle - \frac{1}{2}\langle AA^{-1}b, A^{-1}b\rangle + \langle A^{-1}b, b\rangle$$

$$= \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle + \frac{1}{2}\langle b, A^{-1}b\rangle = I$$

Let $\{x_i\}_{i=1}^n$ be $\overbrace{\text{orthogonal}}$ eigenvectors of A, $\{\lambda_i\}_{i=1}^n > 0$ its eigenvalues. Then $\exists \alpha_i$: $x = \sum_{i=1}^n \alpha_i x_i$

$\exists \beta_i$: $b = \sum_{i=1}^n \beta_i x_i$

But $I = \sum_{i=1}^n \frac{(\alpha_i \lambda_i - \beta_i)^2}{2\lambda_i} \geq 0$

$$f(x_k+\alpha d_k) = \frac{1}{2}\langle x_k+\alpha d_k, Ax_k+\alpha A d_k\rangle - \langle b, x_k+\alpha d_k\rangle$$

$$= \frac{1}{2}\alpha^2\langle d_k, Ad_k\rangle + \alpha\langle d_k, Ax_k\rangle - \alpha\langle b, d_k\rangle$$

$$+ \frac{1}{2}\langle x_k, Ax_k\rangle - \langle b, x_k\rangle$$

$$\frac{d}{d\alpha} f(x_k+\alpha d_k) = \alpha\langle d_k, Ad_k\rangle + \langle d_k, Ax_k\rangle - \langle b, d_k\rangle = 0$$

$$\alpha\langle d_k, Ad_k\rangle = \langle d_k, b - Ax_k\rangle = \langle d_k, r_k\rangle$$

$$\alpha = \frac{\langle d_k, r_k\rangle}{\langle d_k, Ad_k\rangle}$$

Numerical Linear Algebra (final exam)
Iran University of Science and Technology,
School of Mathematics, Applied Mathematics Department

1.

Show that if $X \in \mathbb{R}^{n\times r}$ with $r \leq n$, and $\|X^T X - I\|_2 = \tau < 1$, then

$$\sigma_{\min}(X) \geq 1 - \tau,$$

where $\sigma_{\min}$ denotes the smallest singular value.

2.

Given an $m \times n$ matrix $\mathbf{\Lambda}$ with $m > n$ and a positive number λ, a *regularized* least squares solution $\mathbf{x}_\lambda$ may be computed by solving

$$\min \| \begin{pmatrix} \mathbf{A} \\ \mu I \end{pmatrix} \mathbf{x} - \begin{pmatrix} \mathbf{b} \\ 0 \end{pmatrix} \|_2,$$

where $\mu = \sqrt{\lambda}$.

a. Derive the normal equations for the *regularized* least squares problem given above.

b. Show $\mathbf{\Lambda}^T\mathbf{\Lambda} + \lambda\mathbf{I}$ is symmetric and positive definite for every positive value of λ. Prove that the regularized least squares solution x_λ is unique for every postive value of λ.

c. Use the Singular Value Decomposition of A to express the solution x_λ to the problem

$$min\| \begin{pmatrix} \mathbf{A} \\ \mu\mathbf{I} \end{pmatrix} \mathbf{x} - \begin{pmatrix} \mathbf{b} \\ 0 \end{pmatrix} \|_2$$

where $\mathbf{b} \in \mathbf{R}^m$ and $\mu^2 = \lambda$.

d. Prove that $\lim_{\lambda\to 0^+} \mathbf{x}_\lambda = \mathbf{x}_{LS}$ the minimum norm least squares solution (Regardless of the rank of $\mathbf{\Lambda}$).

3.

Use the singular value decomposition to show that if $A \in \mathbb{R}^{m\times n}$ with $m \geq n$, then there exist a matrix $Q \in \mathbb{R}^{m\times n}$ with $Q^T Q = I$ and a positive semi-definite matrix $P \in \mathbb{R}^{n\times n}$ such that $A = QP$.

4.

Let

$$A = \begin{bmatrix} I & B \\ B^* & I \end{bmatrix}$$

with $\|B\|_2 < 1$. Show that

$$\|A|_2 \|A^{-1}\|_2 = \frac{1 + \|B\|_2}{1 - \|B\|_2}.$$

5.

Let

$$A = \begin{bmatrix} a_1 & b_1 & & & 0 \\ c_1 & a_2 & b_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & b_{n-1} \\ 0 & & & c_{n-1} & a_n \end{bmatrix},$$

where $b_i c_i > 0$. Then there exists a diagonal D such that $D^{-1}AD$ is a symmetric tridiagonal matrix.

6.

Let $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ be symmetric and satisfy

(1) $a_{ii} > 0, \quad i = 1, 2, \cdots, n,$

(2) $a_{ij} \leq 0, \quad i \neq j,$

(3) $\sum\limits_{i=1}^{n} a_{i1} > 0,$

(4) $\sum\limits_{i=1}^{n} a_{ij} = 0, \quad j = 2, 3, \cdots, n.$

Prove that the eigenvalues of A are nonnegative.

7.

Consider $Ax = b$ where

$$A = \begin{bmatrix} 1 & 0 & \alpha \\ 0 & 1 & 0 \\ \alpha & 0 & 1 \end{bmatrix}.$$

(1) For which α, is A positive definite?

(2) For which α, does the Jacobi method converge?

(3) For which α, does the Gauss-Seidel method converge?

8.

Let

$$B_k - B_{k-1} + B_{k-1}(I - AB_{k-1}), \quad k - 1, 2, \cdots.$$

Show that if $\|I - AB_0\| - c < 1$, then

$$\lim_{k\to\infty} B_k = A^{\ 1}$$

and

$$\|A^{-1} - B_k\| \leq \frac{c^{2^k}}{1-c}\|B_0\|.$$

9.

Let $A \in \mathbb{C}^{n\times n}$, $x \in \mathbb{C}^n$ and $X = [x, Ax, \cdots, A^{n-1}x]$. Show that if X is nonsingular, then $X^{-1}AX$ is an upper Hessenberg matrix.

10.

Let $H = \{x \in \mathbb{R}^n \mid < a, x >= b\}$ (hyperplane). Compute orthogonal projection of $z \in \mathbb{R}^n$ onto H.

11.

Consider the polynomial recurrence

$$p_{k+1}(x) = (x - \alpha_{k+1})p_k(x) - \beta_{k+1}^2 p_{k-1}(x), \quad k = 0, 1, 2, \ldots$$

where $p_0 = 1$, $p_{-1} = 0$, and α_k and β_k are scalars.

Show that the roots of $p_k(x)$ are the eigenvalues of the tridiagonal matrix

$$J_k = \begin{pmatrix} \alpha_1 & \beta_2 & & & \\ \beta_2 & \alpha_2 & \beta_3 & & \\ & & \ddots & & \\ & & \beta_{k-1} & \alpha_{k-1} & \beta_k \\ & & & \beta_k & \alpha_k \end{pmatrix}.$$

12.

A symmetric matrix A has dominant eigenvalue λ_1 and corresponding eigenvector x_1. Show that the matrix $B = A - \lambda_1 x_1 x_1^T$ has the same eigenvalues as A except that λ_1 is replaced by zero.

- ***Please send your answers to tnikazad@iust.ac.ir***
- ***The deadline is 1/12/91***

Numerical Linear Algebra (final exam)
Iran University of Science and Technology,
School of Mathematics, Applied Mathematics Department

1.

Show that if $X \in \mathbb{R}^{n\times r}$ with $r \le n$, and $\|X^T X - I\|_2 = \tau < 1$, then

$$\sigma_{\min}(X) \ge 1 - \tau,$$

where $\sigma_{\min}$ denotes the smallest singular value.

2.

Given an $m \times n$ matrix $\mathbf{A}$ with $m > n$ and a positive number λ, a *regularized* least squares solution $\mathbf{x}_\lambda$ may be computed by solving

$$\min \| \begin{pmatrix} \mathbf{A} \\ \mu I \end{pmatrix} \mathbf{x} - \begin{pmatrix} \mathbf{b} \\ 0 \end{pmatrix} \|_2,$$

where $\mu = \sqrt{\lambda}$.

a. Derive the normal equations for the *regularized* least squares problem given above.

b. Show $\mathbf{A}^T\mathbf{A} + \lambda\mathbf{I}$ is symmetric and positive definite for every positive value of λ. Prove that the regularized least squares solution x_λ is unique for every postive value of λ.

c. Use the Singular Value Decomposition of A to express the solution x_λ to the problem

$$min\| \begin{pmatrix} \mathbf{A} \\ \mu \mathbf{I} \end{pmatrix} \mathbf{x} - \begin{pmatrix} \mathbf{b} \\ 0 \end{pmatrix} \|_2$$

where $\mathbf{b} \in \mathbf{R}^m$ and $\mu^2 = \lambda$.

d. Prove that $\lim_{\lambda\to 0+} \mathbf{x}_\lambda = \mathbf{x}_{LS}$ the minimum norm least squares solution (Regardless of the rank of $\mathbf{A}$).

3.

Use the singular value decomposition to show that if $A \in \mathbb{R}^{m\times n}$ with $m \ge n$, then there exist a matrix $Q \in \mathbb{R}^{m\times n}$ with $Q^T Q = I$ and a positive semi-definite matrix $P \in \mathbb{R}^{n\times n}$ such that $A = QP$.

4.

Let

$$A = \begin{bmatrix} I & B \\ B^* & I \end{bmatrix}$$

with $\|B\|_2 < 1$. Show that

$$\|A\|_2 \|A^{-1}\|_2 = \frac{1 + \|B\|_2}{1 - \|B\|_2}.$$

5.

Let

$$A = \begin{bmatrix} a_1 & b_1 & & & 0 \\ c_1 & a_2 & b_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & b_{n-1} \\ 0 & & & c_{n-1} & a_n \end{bmatrix},$$

where $b_i c_i > 0$. Then there exists a diagonal D such that $D^{-1}AD$ is a symmetric tridiagonal matrix.

6.

Let $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ be symmetric and satisfy

(1) $a_{ii} > 0, \quad i = 1, 2, \cdots, n,$

(2) $a_{ij} \leq 0, \quad i \neq j,$

(3) $\sum_{i=1}^{n} a_{i1} > 0,$

(4) $\sum_{i=1}^{n} a_{ij} = 0, \quad j = 2, 3, \cdots, n.$

Prove that the eigenvalues of A are nonnegative.

7.

Let $A \in \mathbb{C}^{n \times n}$, $x \in \mathbb{C}^n$ and $X = [x, Ax, \cdots, A^{n-1}x]$. Show that if X is nonsingular, then $X^{-1}AX$ is an upper Hessenberg matrix.

8.

Let $H = \{x \in \mathbb{R}^n \mid < a, x >= b\}$ (hyperplane). Compute orthogonal projection of $z \in \mathbb{R}^n$ onto H.

Numerical Linear Algebra (1/15/2013)
Iran University of Science and Technology
School of Mathematics, Applied Mathematics Department

1. Show that if $X \in \mathbb{R}^{n\times r}$ with $r \le n$, and $\|X^TX - I\|_2 = \tau < 1$, then $\sigma_{min}(X) \ge 1 - \tau$, where $\sigma_{min}(X)$ denotes the smallest singular value of X.

2. Given an $m \times n$ matrix A with $m > n$ and a positive number λ, a *regularized* least squares solution x_λ may be computed by solving

$$min\| \begin{pmatrix} A \\ \mu I \end{pmatrix} x - \begin{pmatrix} b \\ 0 \end{pmatrix} \|_2, \tag{1}$$

where $\mu = \sqrt{\lambda}$ and $b \in \mathbb{R}^m$.

   a. Derive the normal equations for the *regularized* least squares problem given above.

   b. Prove that the regularized least squares solution x_λ is unique for every positive value of λ.

   c. Use the Singular value Decomposition of A to express the solution x_λ to the problem (1).

   d. Prove that $\lim_{x\to 0^+} x_\lambda = x_{LS}$ the minimum norm least squares solution (regardless of the rank of A).

3. Use the singular value decomposition to show that if $A \in \mathbb{R}^{m\times n}$ with $m \ge n$, then there exists a matrix $Q \in \mathbb{R}^{m\times n}$ with $Q^TQ = I$ and a positive semi-definite matrix $P \in \mathbb{R}^{n\times n}$ such that $A = QP$.

4. Let $A = \begin{pmatrix} I & B \\ B^* & I \end{pmatrix}$ with $\|B\|_2 < 1$. Show that $\|A\|_2\|A^{-1}\|_2 = \frac{1+\|B\|_2}{1-\|B\|_2}$ (where $'*'$ shows conjugate transpose operation).

5. Let

$$A = \begin{pmatrix} a_1 & b_1 & & & 0 \\ c_1 & a_2 & b_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & b_{n-1} \\ 0 & & & c_{n-1} & a_n \end{pmatrix}$$

where $b_ic_i > 0$. Then there exists a diagonal D such that $D^{-1}AD$ is a symmetric tridiagonal matrix.

6. Let $A \in \mathbb{C}^{n\times n}$, $x \in \mathbb{C}^n$ and $X = [x, Ax, A^2x, \ldots, A^{n-1}x]$. Show that if X is nonsingular, then $X^{-1}AX$ is an upper Hessenberg matrix.

Numerical Linear Algebra (04/01/2014)
Iran University of Science and Technology
School of Mathematics
11 am–13:30 pm

1. Given an $m \times n$ matrix A with $m > n$ and a positive number λ, a *regularized* least squares solution x_λ may be computed by solving

$$min\| \begin{pmatrix} A \\ \mu I \end{pmatrix} x - \begin{pmatrix} b \\ 0 \end{pmatrix} \|_2, \tag{1}$$

where $\mu = \sqrt{\lambda}$ and $b \in \mathbb{R}^m$.

   a. Derive the normal equations for the *regularized* least squares problem given above.
   b. Prove that the regularized least squares solution x_λ is unique for every positive value of λ.
   c. Use the Singular value Decomposition of A to express the solution x_λ to the problem (1).
   d. Prove that $\lim_{x \to 0^+} x_\lambda = x_{LS}$ the minimum norm least squares solution (regardless of the rank of A).

2. A symmetric matrix A has dominant eigenvalue λ_1 and corresponding eigenvector x_1. Show that the matrix

$$B = A - \lambda_1 x_1 x_1^T$$

has the same eigenvalues as A except that λ_1 is replaced by zero.

3. Let $v = A^T M(b - Az)$ where $z \in \mathbb{R}^n, A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m$ and $M \in \mathbb{R}^{m\times m}$ is a symmetric positive definite matrix. Prove that: if $\langle v, B^s v\rangle = 0$ then $v = 0$. Here $B = A^T MA$ and $s \in \mathbb{N}$ (any arbitrary natural number).

4. Use the singular value decomposition to show that if $A \in \mathbb{R}^{m\times n}$ with $m \geq n$, then there exists a matrix $Q \in \mathbb{R}^{m\times n}$ with $Q^T Q = I$ and a positive semi-definite matrix $P \in \mathbb{R}^{n\times n}$ such that $A = QP$.

5. A matrix of the form $I - \alpha xy^T$ ($\alpha \in \mathbb{R}$ and $x, y \in \mathbb{R}^n$) is called an *elementary* matrix.

   a. Compute all eigenvalues of such matrix.
   b. Under which condition(s) this kind of matrix is invertible and compute its inverse.
   c. Show that any lower triangular matrix of size n, with "1" on its diagonal, can be written as the product of $n-1$ elementary matrices.

6. Let $T \in \mathbb{C}^{n\times n}$ such that

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}.$$

Define $\Phi : \mathbb{C}^{p\times q} \to \mathbb{C}^{p\times q}, \Phi(X) = T_{11}X - XT_{22}$. Show that Φ is nonsingular if and only if $\lambda(T_{11}) \cap \lambda(T_{22}) = \emptyset$.

Shahid Rajaee Teacher Training University
Numerical Linear Algebra
Final Exam

1. Let W be a subspace of $\mathbb{R}^n$. For $x \in \mathbb{R}^n$, define
$$\rho(x) = \inf_{y \in W} \|x - y\|_2.$$
Let $\{u_1, \cdots, u_m\}$ be an orthogonal basis of W, where m is the dimension of W. Extend this to an orthogonal basis $\{u_1, \cdots, u_m, \cdots, u_n\}$ of all of $\mathbb{R}^n$.

   (a) Show that
   $$\rho(x) = \left[\sum_{j=m+1}^{n} |\langle x, u_j \rangle|^2 \right]^{1/2}$$
   and that it is uniquely attained at
   $$y = Px \text{ where } P = \sum_{j=1}^{m} u_j u_j^T.$$

   (b) Show $P^2 = P$ and $P^T = P$.

   (c) Show $\langle Px, z - Pz \rangle = 0$ for all $x, z \in \mathbb{R}^n$.

   (d) Show $\|x\|_2^2 = \|Px\|_2^2 + \|x - Px\|_2^2$, for all $x \in \mathbb{R}^n$

2. Show that the infinite series
$$I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots + \frac{A^n}{n!} + \cdots$$
converges for any square matrix A, and denote the sum of the series by e^A.

   (a) If $A = P^{-1}BP$, show that $e^A = P^{-1}e^B P$.

   (b) Let $\lambda_1, \cdots, \lambda_n$ denote the eigenvalues of A, repeated according to their multiplicity, and show that the eigenvales of e^A are $e^{\lambda_1}, \cdots, e^{\lambda_n}$.

3. Suppose that x_0 is an approximation to the solution of non-singular equations $Ax = b$. Show that
$$\frac{\|\delta x\|}{\|x\|} \leq K(A) \frac{\|r_0\|}{\|b\|}$$
where $\delta x = x - x_0$ and $r_0 = Ax_0 - b$.

4. A symmetric matrix A has dominant eigenvalue λ_1 and corresponding eigenvector x_1. Show that the matrix
$$B = A - \lambda_1 x_1 x_1^T$$
has the same eigenvalues as A except that λ_1 is replaced by zero.

Hope the best

Touraj Nikazad