

P K JAIN • V P GUPTA

**LEBESGUE  
MEASURE  
AND  
INTEGRATION**

**A Halsted Press Book**

# **LEBESGUE MEASURE and INTEGRATION**

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# Preface

One of the basic concepts of analysis is that of integration. The classical theory of integration, perfected in the middle of the nineteenth century by Cauchy and Riemann, was adequate for solving many mathematical problems, both in pure and applied mathematics. However, at the end of the nineteenth century, mathematicians found it inadequate from a more general point of view. The deficiencies of this theory can be roughly summed up in two brief statements. Firstly, the class of functions integrable in sense of Riemann is relatively small being limited by the requirements of continuity, piece-wise continuity or other stringent conditions. Secondly and more seriously, limiting operations often lead to insurmountable difficulties. Given a sequence of Riemann integrable functions converging to some function in a domain, the limit of the sequence of integrated functions may not be the Riemann integral of the limit function. In fact, the Riemann integral of the limit-function may not even exist. This is a major drawback of the classical theory of integration, apart from the fact that even relatively simple functions are not integrable in the Riemann sense. These deficiencies have been removed in the *Modern Theory of Measure and Integration*, developed by some of the leading mathematicians. The origin of this theory lies in the work of Henri Lebesgue, a French mathematician, who in the early part of the present century formulated a more meaningful theory of integration than the classical one by generalizing the concept of the *length* of intervals to the *measure* of sets of real numbers. The modern theory meets the need of a number of important branches of mathematics and allied subjects. It helps in the solution of problems in probability theory, partial differential equations, hydrodynamics and quantum mechanics. The theory of integration has undergone a continuous process of evolution and innovations since the publications of the pioneer work of H. Lebesgue, *Integral, Longueur aire* in 1902.

Today, the theory of 'Lebesgue measure and integration' which has been of tremendous mathematical importance forms the background of modern mathematical analysis. No mathematician can afford today to be ignorant of the modern theories of integration. It is also in the interest of the students of mathematics that they become acquainted with these ideas early in their studies. With this idea in mind, a one-semester course of this subject has been prescribed in various universities at the honours and post-graduate levels.

There are, of course, a number of books, available on the subject, which are too concise to be intelligible to the students. There is hardly any single book which covers the entire syllabus as prescribed in various universities.

The book is written as a text for the standard one-semester course that is usually pursued by the honours and post-graduate students of the various universities in India and abroad with the hope that it will open a path to the Lebesgue theory to the students. The aim of this book is to cover a basic course of Lebesgue measure and integration which flows in a natural, motivated and simple way. The authors have taken great pains to give detailed explanations of reasons of working and of the method, used together with numerous examples and counter-examples at different places in the book. The details are explicitly presented keeping the interest of the students in view. Each topic in the book has been treated in an easy and lucid style. We believe that the students will find the book 'smooth going' and easy to understand. It would also serve as a reference book for persons studying analysis independently.

The material has been arranged by sections, spread out in seven chapters. The text opens with a chapter on preliminaries discussing basic concepts and results which would be taken for granted later in the book; the reader is assumed to be already familiar with most of the material. This chapter is followed by chapters on Infinite Sets, Measurable Sets, Measurable Functions, Lebesgue Integral, Differentiation and Integration and The Lebesgue  $L^p$  Spaces. The last chapter, i.e., The Lebesgue  $L^p$  Spaces, is presented in order to sharpen the student's appetite for functional analysis. The book contains many solved and unsolved problems, remarks and notes at places which would help the students to increase their knowledge by applying previous results or by presenting new material. Some of the problems require extra

effort on the part of students. Hints are also provided for the solution of several problems. A set of problems graded in a proper way has been given at the end of each chapter.

We do not claim any originality of the results but we do claim simplicity and lucidity of presentation coupled with comprehensiveness of the material. The various sources that have inspired the authors are listed in the bibliography. Yet, the works of Natanson (1965), Royden (1968) and Chac (1980) have made significant contribution in making the book useful for the student-readers and research-scholars. In fact, the genesis of the present text lies in the classroom notes prepared by the authors for students in Lebesgue measure and integration at the University of Delhi and Meerut, University over a period of some ten years—such notes were developed, revised, written and expanded more times than one can recall.

We thank in general all our colleagues in the University of Delhi and Meerut University who have inspired us directly or indirectly, for taking up this project. Some of them have enlightened us on some topics by holding discussions. One of the authors (P.K. Jain) conveys his heartfelt thanks to his associates in the Department of Mathematics of the University of Khartoum (Sudan) and Kuwait University (Kuwait). They tempered the ideas and results in the book by valuable discussions from time to time. Above all, we are thankful to the generations of students who have made valuable contributions in injecting simplicity in presentation of the material so as to be intelligible to the student community in general.

We owe special gratitude to Marcel Dekker, Inc., New York, for permitting us to reproduce *ad verbum* the English translation of Henri Lebesgue's lecture in the book (Appendix III).

Finally, we thank the publisher for his cooperation in bringing out the book.

P K JAIN  
V P GUPTA



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# I

## Preliminaries

This chapter is to help the reader in reviewing the preliminaries needed subsequently in this book. It is presumed that the reader has pursued an elementary course in real analysis. The approach adopted in this chapter is somewhat different from that used in other chapters. It is descriptive and the arguments given are directly toward plausibility and understanding rather than rigorous proof. The preliminaries are divided into seven sections.

### 1 SET AND SET INCLUSION

A set is any well-defined collection or system of objects. Other words such as collection, class, and aggregate are used synonymously for the term set. 'Well-defined' means that it is possible to determine readily whether an object is a member of a given set or not. The objects that belong to a set are called its **elements** (or **points** or **members**). If  $A$  is a set, then  $a \in A$  denotes that  $a$  is an element of  $A$  and the notion  $a \notin A$  denotes the negation of  $a \in A$ . For any element  $a$  and a set  $A$ , either  $a \in A$  or  $a \notin A$ .

Two methods used frequently to describe sets are the 'tabulation method' and the 'defining-property method'. The first, the tabulation method, enumerates or lists the individual elements separated by commas and enclosed in braces. By this method, the set of vowels of English alphabets is written as  $\{a, e, i, o, u\}$ . Sets which are difficult to describe by an enumeration are described by the second method—the defining-property method. In fact, this method is often more compact and convenient. A defining property of a set is a property which is satisfied by each element of that set and by nothing else. The standard notation for a set so described is  $\{x \mid \}$  or  $\{x : \}$ . Here  $x$  is a dummy symbol and the space between  $:$  and  $\}$  is filled by a defining property. The above set, by this method is described as  $\{x : x \text{ is a vowel of English alphabets}\}$ .

## 2 Lebesgue Measure and Integration

Given two sets  $A$  and  $B$ , if the relation  $a \in A$  implies  $a \in B$  for all  $a$ , we say that  $A$  is a subset of  $B$  (or  $B$  is a superset of  $A$ , or  $A$  is contained in  $B$  or  $B$  contains  $A$ ). In symbol, it is written as  $A \subset B$ . Two sets  $A$  and  $B$  are equal if  $A \subset B$  and  $B \subset A$ . Generally, a set is completely determined by its elements but there is a set which has no element, and we call it as the **empty** (or **void** or **null**) set and denote it by  $\phi$  (phi). If  $A$  is any set, each element of  $\phi$  (there are none) is an element of  $A$ , and so  $\phi \subset A$ . Thus, the empty set is a subset of every set. Further, if  $A$  is a subset of  $B$  with  $A \neq \phi$  and  $A \neq B$ , then  $A$  is a **proper subset** of  $B$  (or  $B$  properly contains  $A$ ). In other words, a set  $A$  is a proper subset of  $B$  if and only if  $a \in A$  implies  $a \in B$ , and there exists at least one  $b \in B$  such that  $b \notin A$ .

Let  $A$  be a set. Then the collection of all the subsets of  $A$  is called the **power set** of  $A$  and is denoted by  $\mathcal{P}(A)$ . For instance, if  $A$  is a set containing  $a$ ,  $b$  and  $c$  as its elements, there are eight subsets of  $A$ . Hence, the power set  $\mathcal{P}(A)$  would contain eight elements, each being a subset of  $A$ . It is obvious that the sets  $\phi$  and  $A$  are always members of  $\mathcal{P}(A)$ . In particular,  $\mathcal{P}(A)$  is always a nonempty set. If  $A$  is a finite set containing  $n$  (distinct) elements,  $\mathcal{P}(A)$  has  $2^n$  elements and this is the reason for the name 'power set'.

Let  $A$  and  $B$  be two sets. Using certain operations on  $A$  and  $B$ , we can obtain four other sets. One of these sets is called the **union** of the two sets; written  $A \cup B$  (sometimes, called the **sum** and written as  $A + B$ ); it consists of all elements that are in  $A$  or in  $B$  (or in both, an element that is in both is counted only once). The second is called the **intersection** of two sets, written  $A \cap B$  (sometimes, called the **product** and written as  $A \cdot B$ ); it consists of all elements in  $A$  as well as in  $B$ . The third one is called the **difference** of the two sets, written  $A - B$ ; it consists of all those elements of  $A$  which are not elements of  $B$ . The fourth one is called the **Cartesian product** of the two sets; written as  $A \times B$ , it consists of all ordered pairs  $(a, b)$  where  $a \in A$  and  $b \in B$ . Two sets  $A$  and  $B$  are said to be **disjoint** if  $A \cap B = \phi$ , otherwise  $A$  intersects  $B$ . If  $B \subset A$ ,  $A - B$  is called the **complement** of  $B$  with respect to  $A$ . In case  $A$  is taken as a universal set\*,  $A - B$  is written as  $B^c$  (or  $\sim B$ ) and simply read as complement of  $B$ . If  $A = \mathbf{R}$ , the set of all real numbers, then  $\mathbf{Q}^c$  (complement of  $\mathbf{Q}$ , the set of all rational numbers) is the set of all irrational numbers. We trust that the reader is familiar with the basic properties—commutative laws, associative laws, idem-

\*All sets under consideration are subsets of a single fixed set known as the universal set, and in this book it is taken as  $\mathbf{R}$ , the set of all real numbers, unless specified otherwise.

potent laws, distributive laws, DeMorgan's laws, etc., of complement, union and intersection. It may be noted that the laws given above do not hold good for the operation of Cartesian products.

We now extend the definition of the union and intersection of two sets to collection  $\mathcal{C}$  of sets. Let  $\mathcal{C}$  be a collection of sets in  $\mathcal{P}(X)$ . Then its union and intersection are defined as follows:

$$\bigcup_{A \in \mathcal{C}} A = \{x \in X : x \in A \text{ for some } A \in \mathcal{C}\}$$

and

$$\bigcap_{A \in \mathcal{C}} A = \{x \in X : x \in A \text{ for all } A \in \mathcal{C}\}.$$

If  $\mathcal{C}$  is empty,  $\bigcup_{A \in \mathcal{C}} A = \phi$  and  $\bigcap_{A \in \mathcal{C}} A = X$ . A collection  $\mathcal{C}$  of sets is said to

be **pairwise disjoint** if two distinct members of it are mutually disjoint.

A set whose elements are used as names is called an **index set**. An index set may be finite or infinite. Suppose for each member  $\alpha$  of a fixed set  $A$ , we have a set  $A_\alpha$ . Then  $A$  is the index set and the sets  $A_\alpha$  are called the **indexed sets**, and the subscript  $\alpha$  of  $A_\alpha$ , i.e. each  $\alpha \in A$ , is called an **index**. The collection of  $A_\alpha$  is called an **indexed family of sets** and is denoted by  $\{A_\alpha\}_{\alpha \in A}$ . An index set is usually denoted by  $\Lambda$ . We shall be using this symbol for index set throughout the book. Let  $A$  be a set and  $\Lambda$  an index set. Let  $\{A_\alpha\}_{\alpha \in \Lambda}$  be an indexed family of the subsets of  $A$ . Then the union of all the sets  $A_\alpha$  is the set  $\{x \in A : x \in A_\alpha \text{ for some } \alpha \in \Lambda\}$ . We denote it by  $\bigcup_{\alpha \in \Lambda} A_\alpha$ . We may define  $\bigcap_{\alpha \in \Lambda} A_\alpha$  similarly. It is easy to verify that De Morgan's laws hold good in an indexed family of sets.

## 2 FUNCTIONS

Let  $A$  and  $B$  denote arbitrary given sets. By a **function**  $f : A \rightarrow B$ , we mean a rule which assigns to each element  $a$  of  $A$ , a unique element  $b$  of  $B$ . If  $a \in A$ , the corresponding element  $b$  in  $B$  is called the  **$f$ -image** of  $a$  and is denoted by  $f(a)$ , i.e.  $b = f(a)$ . In this case,  $a$  is called the **pre-image** of  $b$ . The set  $A$  is called the **domain** of the function  $f$ , and  $B$  the **codomain** of  $f$ . The set  $B_1 \subset B$  consisting of all  $f$ -images of elements of  $A$  is called the **range** of  $f$ , denoted by  $f(A)$ . A function  $f$  whose codomain is  $\mathbb{R}$  is called a **real-valued function**.

If  $f$  and  $g$  are two functions defined on the same domain  $A$  and if  $f(a) = g(a)$  for every  $a \in A$ , the functions  $f$  and  $g$  are equal and we write  $f = g$ . Let  $f$  be a function of  $A$  into  $B$ . Then  $f(A) \subset B$ . If  $f(A) = B$ ,  $f$  is a function of  $A$  **onto**  $B$ , or  $f : A \rightarrow B$  is an **onto function**. The function  $f : A \rightarrow B$  is **one-one** if for any two elements  $a_1$  and  $a_2$  of  $A$ ,  $a_1 \neq a_2$

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implies  $f(a_1) \neq f(a_2)$ . A function which is both one-one and onto is called **one-to-one**.

Let  $A$  be any set. Then  $f: A \rightarrow A$  defined by  $f(x) = x$  for all  $x \in A$  is called the **identity function**, denoted by  $I_A$ . An identity function is one-to-one. A function  $f$  is called a **constant function** if its range consists of only one element. Let  $f: A \rightarrow B$  and  $g: B \rightarrow C$  be two functions such that  $f(a) = b$ ,  $a \in A$  and  $b \in B$ ; and  $g(b) = c$ , where  $c \in C$ . Then the function  $h: A \rightarrow C$  defined by

$$h(a) = c = g(b) = g(f(a)), \quad a \in A$$

is called the **composite function** of two functions  $f$  and  $g$ , denoted by  $g \circ f$ . If  $f: A \rightarrow B$ , then  $I_B \circ f = f$  and  $f \circ I_A = f$ . Let  $f: A \rightarrow B$  be a function and  $E \subset A$ . The function  $f \circ I_E: E \rightarrow B$  is called the **restriction** of  $f$  to the set  $E$ , denoted by  $f|_E$ ; dually, the function  $f$  is referred to as the **extension** of  $f|_E$  to the set  $A$ .

Let  $f: A \rightarrow B$  and  $b \in B$ . Then the  $f$ -inverse of  $b$ , denoted by  $f^{-1}(b)$ , consists of those elements of  $A$  which are mapped onto  $b$  by  $f$ , i.e. those elements of  $A$  which have  $b$  as their image. More briefly, if  $f: A \rightarrow B$ , then

$$f^{-1}(b) = \{x \in A : f(x) = b\}.$$

It is obvious that  $f^{-1}(b)$  is a subset of  $A$ . We read  $f^{-1}$  as ' $f$  inverse'. It is easy to verify that a function  $f: A \rightarrow B$  is **one-one** if and only if for each  $b \in B$ ,  $f^{-1}(b)$  is either empty or singleton (set consisting of only one element). Let  $f: A \rightarrow B$  and  $B_1$  be a subset of  $B$ . Then the inverse of  $B_1$  under the function  $f$ , denoted by  $f^{-1}(B_1)$ , consists of those elements of  $A$  which are mapped by  $f$  onto an element in  $B_1$ . More briefly,

$$f^{-1}(B_1) = \{x \in A : f(x) \in B_1\}.$$

It is easy to prove that a function  $f: A \rightarrow B$  is onto if and only if for every non-empty subset  $B_1$  of  $B$ ,  $f^{-1}(B_1)$  is a nonempty set. For a function  $f: A \rightarrow B$  which is one-to-one, we note that

$$f^{-1} \circ f = I_A \quad \text{and} \quad f \circ f^{-1} = I_B.$$

It may also be seen that  $I_A^{-1} = I_A$ .

An indexed family of sets  $\{A_\alpha\}_{\alpha \in \Lambda}$  is a function  $f: \Lambda \rightarrow \mathcal{P}(A)$ , where the domain of  $f$  is the index set  $\Lambda$ . Let  $\{A_\alpha\}_{\alpha \in \Lambda}$  be a nonempty family of nonempty subsets of a set  $A$ . A function  $f: \{A_\alpha\}_{\alpha \in \Lambda} \rightarrow A$  is called a **choice function** if  $f(A_\alpha) = a_\alpha \in A_\alpha$ , for every  $\alpha \in \Lambda$ . The set of all choice functions defined on  $\{A_\alpha\}_{\alpha \in \Lambda}$  is called the **Cartesian product**

of  $\{A_\alpha\}_{\alpha \in A}$  and is denoted by  $\prod_{\alpha \in A} \{A_\alpha\}$ . If  $\{A_\alpha\}_{\alpha \in A}$  be any non-empty family of nonempty sets, the axiom of choice states that *it is possible to choose simultaneously an element from each set*. In other words, the axiom of choice states that the *Cartesian product of a non-empty family of non-empty sets is nonempty*. The axiom of choice may also be stated as 'there exists a choice function for any nonempty family of nonempty sets'.

Let  $S$  be any set. A binary relation  $R$  on a set is defined as a subset of  $S \times S$ . If  $R$  is a relation on a set  $S$ , then for  $x, y \in S$  we write  $xRy$  to mean  $(x, y) \in R$  and read it as 'x is related to y under R'. A relation  $R$  defined on a set  $S$  is said to be **reflexive** if  $xRx$  for every  $x \in S$ ; **symmetric** if  $xRy$  implies  $yRx$ ; and **transitive** if  $xRy$  and  $yRz$  imply  $xRz$ . A relation is said to be an **equivalence relation** if it is reflexive, symmetric and transitive. Closely associated with the concept of an equivalence relation is that of the **partition** of a set. A **partition** of a set  $S$  is a pairwise disjoint collection of nonempty subsets of  $S$  whose union is  $S$ . *An equivalence relation in  $S$  defines a partition of  $S$  and, conversely, a partition of  $S$  yields an equivalence relation in  $S$* . Let  $R$  be an equivalence relation in  $S$ . Then for each  $s \in S$ , let

$$R(s) = \{x \in S : xRs\}.$$

The collection  $\{R(s) : s \in S\}$  is a partition of  $S$ . The members of this collection are called the **equivalence classes** and the collection is called the **quotient set** of  $S$  with respect to  $R$ .

### 3 SUPREMUM AND INFIMUM

A set  $A \subset \mathbb{R}$  is said to be **bounded below** if there is a real number  $m$  such that  $x \geq m$ , for all  $x \in A$ . The number  $m$ , in this case, is called a **lower bound** of  $A$ . It is easily seen that if  $m$  is a lower bound of  $A$ , then any number  $m' \leq m$  is also a lower bound of  $A$ . The set  $A$  is said to be **bounded above** if there is a real number  $M$  such that  $x \leq M$ , for all  $x \in A$ . The number  $M$ , in this case, is called an **upper bound** of  $A$ . Again, it is easily seen that if  $M$  is an upper bound of  $A$ , then any number  $M' \geq M$  is also an upper bound of  $A$ . The set  $A$  is **bounded** if it is both bounded above and bounded below, i.e., the set  $A$  is bounded if  $\exists$  a real number  $k > 0$  such that  $|x| \leq k$  for all  $x \in A$ . Otherwise  $A$  is said to be unbounded. (Note that an unbounded set may be unbounded above, unbounded below or both.) A real number  $M$  is called the **least upper bound** (or **supremum**) of a nonempty set  $A$ , if (i)  $x \leq M$  for all  $x \in A$  and (ii) given any  $\epsilon > 0$ , however small, there is a number  $x_0 \in A$  such that  $M - \epsilon < x_0$ . In other words,  $M$  is an

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upper bound of  $A$  and no other real number less than  $M$  is an upper bound of  $A$ , i.e., the least upper bound of a set is the smallest of all the upper bounds of  $A$ . We denote the least upper bound of  $A$  by  $\text{lub}(A)$  or  $\text{sup}(A)$  or  $\text{sup } x$ . A real number  $m$  is called the **greatest lower**

**bound** (or **infimum**) of a nonempty set  $A$ , if (i)  $x \geq m$  for all  $x \in A$  and (ii) given any  $\epsilon > 0$ , there is a number  $x_0 \in A$  such that  $x_0 < m + \epsilon$ . We denote the greatest lower bound of  $A$  by  $\text{glb}(A)$  or  $\text{inf}(A)$  or  $\text{inf } x$ .

*Every non-empty set of real numbers bounded above possesses the supremum while the one bounded below possesses the infimum.* It is obvious that the supremum and infimum, provided they exist, are determined uniquely. The supremum and the infimum of a set may or may not belong to the set. For finite sets, the supremum coincides with the greatest real number of the set and the infimum with the smallest real number of the set. For an unbounded set  $A$  having no upper bound, we write  $\text{sup}(A) = +\infty$ , and for a set  $A$  having no lower bound,  $\text{inf}(A) = -\infty$ . The following equalities of supremum and infimum are obvious

$$\text{inf}(A) = \text{inf}_{x \in A} x = -\text{sup}_{x \in A} (-x) = -\text{sup}(-A).$$

Since the least upper bound of a set  $A$  is a special upper bound, it is clear that only sets bounded above can have the least upper bound. However, the empty set  $\phi$  has no least upper bound even though it is bounded above by any real number.

## 4 INTERVALS

Let  $a, b \in \mathbb{R}$  and  $a < b$ . Then the sets  $\{x \in \mathbb{R} : a < x < b\}$ , denoted by  $]a, b[$  and  $\{x \in \mathbb{R} : a \leq x \leq b\}$ , denoted by  $[a, b]$  are, respectively, called the **open interval** and the **closed interval** from  $a$  to  $b$ . The sets  $\{x \in \mathbb{R} : a \leq x < b\}$ , denoted by  $[a, b[$  and  $\{x \in \mathbb{R} : a < x \leq b\}$ , denoted by  $]a, b]$  are called **semi-open intervals**; the former is **closed at left** (or at  $a$ ) and **open at right** (or at  $b$ ) while the latter is described analogously. Obviously, for  $b < a$ , all the four intervals defined above are empty. All these intervals are also bounded.

The set  $\mathbb{R} \cup \{-\infty, \infty\}$ , denoted by  $\mathbb{R}^*$  is called the **extended real number system**. For  $a \in \mathbb{R}$ , the sets  $\{x \in \mathbb{R} : x > a\}$ , denoted by  $]a, \infty[$ , and  $\{x \in \mathbb{R} : x \geq a\}$ , denoted by  $[a, \infty[$ , are called, respectively, the **open interval** and the **closed interval** from  $a$  to  $\infty$ . It may be noted

'sup' and 'inf' are, respectively, the abbreviations of the Latin words *supremum* 'the greatest' and *infimum* 'the least'.

that  $\infty$  and  $-\infty$  are not real numbers. By convention,  $] -\infty, \infty[$  is the entire set  $\mathbb{R}$ . An interval which has at least one endpoint as  $\infty$  or  $-\infty$  is called an **unbounded interval**.

## 5 OPEN, CLOSED AND PERFECT SETS

A set  $N \subset \mathbb{R}$  is said to be a **neighbourhood\*** of the point  $x_0 \in \mathbb{R}$  if  $\exists$  an open interval  $I$  containing  $x_0$  such that  $x_0 \in I \subset N$ . It is obvious that an open interval is a neighbourhood of each of its points.

A point  $x_0$  is called an **interior point** of a set  $G$  if there exists a nbd of  $x_0$  contained in  $G$ . The set of all interior points of the set  $G$  is called the **interior** of  $G$  which is generally denoted by  $\text{Int}(G)$ . The interior of  $[a, b]$  is  $]a, b[$ . The interior of a finite set is empty. In fact, the interior of a set  $G$  is the largest open subset of  $G$ . A set  $G$  is said to be **open** if it is a nbd of each of its points. Thus, every point of an open set is its interior point, i.e. for an open set  $G$ ,  $\text{Int}(G) = G$ . Evidently,  $G$  is open if and only if  $\text{Int}(G) = G$ . It is easy to verify that *every open interval is an open set*. Also, *every open set of real numbers can be expressed as a union of a countable\*\* collection of mutually disjoint open intervals*.

A point  $x_0$  is a **point of closure (adherent point)** of  $F$  if every nbd of  $x_0$  contains at least one point of  $F$ . The set of all points of closure of  $F$  is called the **closure** of  $F$  and is generally denoted by  $\bar{F}$  or  $C(F)$ . The closure of  $]a, b[$  is  $[a, b]$ . A set is **closed** if its complement is an open set. In fact, the closure of a set  $F$  is the smallest closed superset of  $F$ . Evidently, a set  $F$  is closed if and only if  $F = \bar{F}$ . If  $F$  is a non-empty, closed and bounded set, then the lub and glb of  $F$  always belong to  $F$ . Note that every finite set in  $\mathbb{R}$  is closed.

It must be remarked here that the terms 'open' and 'closed' are neither inclusive nor mutually exclusive. For example, each of the void set and the set  $\mathbb{R}$  is both open and closed, on the other hand, a semi-open interval is neither open nor closed. There is a relationship, however, between open and closed sets. The complement of an open (closed) set is closed (open). An arbitrary union (intersection) of any number of open (closed) sets is open (closed). The union (intersection) of a finite number of closed (open) sets is closed (open). It may be noted that the intersection of an arbitrary number of open sets may not be an open set; for example

$$\bigcap_{n=1}^{\infty} \left] -\frac{1}{n}, \frac{1}{n} \right[ = \{0\}$$

\*The neighbourhood is abbreviated as 'nbd' throughout the book.

\*\*For definition of countable collection, see Chapter II.

which is not an open set. Moreover, the union of any number of closed sets may not be a closed set; for example

$$\bigcup_{n=1}^{\infty} \left[ \frac{n}{n+1}, \frac{n+1}{n+2} \right] = \left[ \frac{1}{2}, 1 \right]$$

which is not a closed set.

A family  $\mathcal{C}$  of sets is said to be a *cover* of a set  $A$  if  $A$  is contained in the union of sets forming  $\mathcal{C}$ . If each element of  $\mathcal{C}$  is an open set then the cover  $\mathcal{C}$  is said to be an *open cover* of  $A$ . It is called a *finite cover* if  $\mathcal{C}$  contains only a finite number of sets. A family  $\mathcal{D} \subset \mathcal{C}$  is said to be a *subcover* of  $\mathcal{C}$  if it itself is a cover of  $A$ . We state and prove below an important result concerning the cover of a bounded closed set of real numbers which we shall be using in our subsequent chapters.

**1.1 Theorem (Heine-Borel Theorem).** *Let  $F$  be a bounded closed set of real numbers. Then each open cover of  $F$  has a finite subcover.*

*Proof. Case 1* Suppose  $F$  is a closed interval, say,  $[a, b]$ . Also, let  $\mathcal{C}$  be an open cover of  $[a, b]$ . Take  $E$  to be the set of numbers  $x \leq b$  such that the interval  $[a, x]$  is contained in the union of a finite number of sets in  $\mathcal{C}$ . Clearly,  $E$  is a non-empty set as  $a \in E$  and is bounded above by  $b$ . So, it has the least upper bound, say  $c$ . Since  $c \in [a, b]$ , there exists an open set  $G$  in  $\mathcal{C}$  which contains  $c$ . Since  $G$  is open, there is an  $\epsilon > 0$  such that the interval  $]c - \epsilon, c + \epsilon[ \subset G$ . Now  $c - \epsilon$  is not an upper bound of  $E$ , and hence  $x > c - \epsilon$  for some  $x \in E$ . Since  $x \in E$ ,  $[a, x]$  is contained in the union of a finite number of sets in  $\mathcal{C}$ . Consequently, the finite collection obtained by adding one more set  $G$  to the finite number of sets, already required to cover  $[a, x]$ , covers  $[a, c + \epsilon[$ . Thus  $[c, c + \epsilon[ \subset E$  if each point of  $[c, c + \epsilon[$  is less than or equal to  $b$ . Since no point of  $[c, c + \epsilon[$  except  $c$  can belong to  $E$ , we must have  $c = b$  and  $b \in E$ . Thus  $[a, b]$  can be covered by a collection consisting of finite number of sets in  $\mathcal{C}$ . Thus, the result is proved when  $F = [a, b]$ .

*Case 2* Suppose  $F$  is any closed and bounded set of  $\mathbb{R}$ . Let  $\mathcal{C}$  be an open cover of  $F$ .  $F$  being a bounded set, we enclose  $F$  in a closed interval  $[a, b]$ . Let  $\mathcal{D}$  be the collection obtained by adding  $F^c$  to  $\mathcal{C}$ ; i.e.  $\mathcal{D} = \mathcal{C} \cup \{F^c\}$ . Clearly,  $\mathcal{D}$  is an open cover of  $\mathbb{R}$  as

$$\mathbb{R} = F \cup F^c \subseteq F^c \cup \{G : G \in \mathcal{C}\} = \bigcup \{G : G \in \mathcal{D}\}.$$

and hence of  $[a, b]$ . By Case 1, there exists a finite subcover  $\mathcal{E}$  of  $\mathcal{D}$  which covers  $[a, b]$  and hence  $F$ . Since  $F \cap F^c = \phi$ ,  $\mathcal{E} - \{F^c\}$  covers  $F$ . However,  $\mathcal{E} - \{F^c\}$  is a finite subcollection of  $\mathcal{C}$ . ■

A real number  $\zeta$  is a **limit (accumulation) point** of a set  $A \subset \mathbb{R}$  if every nbd of  $\zeta$  contains infinitely many points of  $A$ . It is obvious that a finite set cannot have a limit point. The set of all limit points of the set  $A$  is known as the **derived set** of  $A$ , denoted by  $A'$ . It is easy to verify that: (i)  $\bar{A} = A \cup A'$  and (ii)  $A$  is closed if and only if  $A' \subset A$ . A set  $A \subset \mathbb{R}$  is said to be **dense** (or dense in  $\mathbb{R}$ ) if every point of  $\mathbb{R}$  is a point of  $A$  or a limit point of  $A$  or, equivalently, if  $\bar{A} = \mathbb{R}$ . A set  $A$  is said to be **nowhere dense** (nondense) in  $\mathbb{R}$  if no nbd in  $\mathbb{R}$  is contained in the closure of  $A$ , i.e., if the complement of  $\bar{A}$  is dense in  $\mathbb{R}$ . The set  $A$  is **perfect** if either  $A = A'$  or  $A = B$ , where  $B$  is a closed set which is dense in itself.

## 6 SEQUENCES AND SERIES

A function  $f: \mathbb{N} \rightarrow X$ , where  $X$  is any set is called a **sequence** in  $X$ . Because a sequence is uniquely and completely determined by the values  $a_n (= f(n))$  for  $n \in \mathbb{N}$ , a sequence is usually denoted by  $\{a_n\}$  without explicit reference to  $f$ . The value  $a_n$  is called the  $n$ th term of the sequence  $\{a_n\}$ . A sequence  $\{a_n\}$  is said to **converge to a limit**  $l$  if for each  $\epsilon > 0$ , there exists a natural number  $N$  (dependent on  $\epsilon$ ) such that  $|a_n - l| < \epsilon$  for  $n > N$ . We write  $\lim_{n \rightarrow \infty} a_n = l$ . A sequence having

a limit is said to be **convergent**. It may be noted that if a sequence has a limit, the limit is unique. Also, every convergent sequence is bounded but the converse is not necessarily true. For example,  $\{(-1)^n\}$  is a bounded but not a convergent sequence. It is easy to prove that if  $\{a_n\}$  and  $\{b_n\}$  are two sequences converging, respectively, to  $l_1$  and  $l_2$ , then:

1. The sequence  $\{a_n \pm b_n\}$  converges to  $l_1 \pm l_2$ .
2. The sequence  $\{a_n b_n\}$  converges to  $l_1 l_2$ .
3. The sequence  $\{k a_n\}$ ,  $k$  is constant converges to  $k l_1$ .
4. The sequence  $\left\{ \frac{a_n}{b_n} \right\}$  converges to  $l_1/l_2$  provided  $l_2 \neq 0$ .

Further, a sequence  $\{a_n\}$  is said to be **increasing** if  $a_{n+1} \geq a_n$  for all  $n$ . By repeated application of this inequality, we may deduce that  $a_{n_1} \geq a_{n_2}$  whenever  $n_1 > n_2$ . In particular,  $a_n \geq a_1$  for all  $n$ , and so every increasing sequence is bounded below. A decreasing sequence is defined in a similar way. Sequences which are either increasing or decreasing are called **monotone (monotonic)** sequences. A monotone sequence either tends to a limit or to  $\pm\infty$ . A monotone sequence bounded below (above) converges to its infimum (respectively, supremum). A sequence  $\{a_n\}$  of real numbers is said to be a **Cauchy**

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sequence if for each  $\epsilon > 0$ , there is a positive integer  $n_0$  such that  $|a_n - a_m| < \epsilon$  for  $n, m > n_0$ . It is trivial, from the definition, that a sequence of real numbers is convergent if and only if it is a Cauchy sequence.

There are many bounded sequences which are not convergent. For such sequences, the concepts of limit superior and limit inferior are introduced. Let  $\{a_n\}$  be a sequence. For any increasing sequence  $\{n_1, n_2, n_3, \dots, n_m, \dots\}$  of positive integers, the sequence  $\{a_{n_k}\}$ , ( $k=1, 2, 3, \dots$ ) is called a **subsequence** of  $\{a_n\}$ . It is clear that a sequence  $\{a_n\}$  converges to  $l$  if and only if each of its subsequences converges to  $l$ . Though a bounded sequence may not be a convergent sequence, it has a convergent subsequence. Let  $\{a_n\}$  be a sequence of real numbers and  $M_k = \sup_{n \geq k} a_n$ . It is evident that  $M_{k+1} \leq M_k$ . Thus, the sequence  $\{M_k\}$  has a limit; namely, the infimum of the  $M_k$ 's. This infimum is defined to be the **limit superior** or the **upper limit** of  $\{a_n\}$ . It is denoted by  $\limsup_{n \rightarrow \infty} a_n$  or  $\overline{\lim}_{n \rightarrow \infty} a_n$  or simply  $\lim a_n$ . In other words

$$\limsup_{n \rightarrow \infty} a_n = \inf_{k \geq 1} \sup_{n \geq k} a_n.$$

The **limit inferior** or the **lower limit** of the sequence  $\{a_n\}$  is defined in a similar way. It is denoted by  $\liminf_{n \rightarrow \infty} a_n$  or  $\underline{\lim}_{n \rightarrow \infty} a_n$  or simply  $\underline{\lim} a_n$ .

It may be noted that

$$\liminf_{n \rightarrow \infty} a_n = \sup_{k \geq 1} \inf_{n \geq k} a_n.$$

From definition, it is obvious that  $\liminf_{n \rightarrow \infty} a_n \leq \limsup_{n \rightarrow \infty} a_n$ , and a sequence  $\{a_n\}$  converges to  $l$  and only if  $\limsup_{n \rightarrow \infty} a_n = \liminf_{n \rightarrow \infty} a_n = l$ .

The idea of upper and lower limits can also be generalised to include unbounded sequences. If a sequence  $\{a_n\}$  is unbounded above, then  $\overline{\lim}_{n \rightarrow \infty} a_n = \infty$ . In case it is bounded below,  $\overline{\lim}_{n \rightarrow \infty} a_n = -\infty$ . Further,

if a sequence  $\{a_n\}$  is unbounded below and there is no subsequential limit,  $\underline{\lim}_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} a_n = \lim a_n = -\infty$ . Similarly, for a sequence which is unbounded above and has no subsequential limit,  $\underline{\lim}_{n \rightarrow \infty} a_n = \underline{\lim} a_n = \lim a_n = \infty$ .

An expression  $\sum_{n=1}^{\infty} u_n$  (or simply  $\sum u_n$ ) where the numbers  $u_n$  depend on the index  $n=1, 2, 3, \dots$  is called a (number) series. For each  $n$ , let  $S_n = u_1 + u_2 + \dots + u_n$ . The sequence  $\{S_n\}$  is called the sequence of

partial sums of the series  $\sum u_n$  and the number  $S_n$  is called the  $n$ th partial sum of the series  $\sum u_n$ . The series  $\sum u_n$  is convergent if the limit  $\lim_{n \rightarrow \infty} S_n = S$  exists. In this case, we write

$$S = u_1 + u_2 + \dots = \sum_{n=1}^{\infty} u_n,$$

and call  $S$ , the sum of the series. A series  $\sum u_n$  is **absolutely convergent** if the series  $\sum |u_n|$  is convergent. Every absolutely convergent series is convergent and a convergent series with terms as nonnegative real numbers is absolutely convergent. A series with arbitrary terms though being convergent may not be absolutely convergent. The series

$\sum u_n$ , where  $u_n = \frac{(-1)^{n+1}}{n^\alpha}$  ( $\alpha > 0$ ), is convergent for all  $\alpha > 0$  but is absolutely convergent only for  $\alpha > 1$ .

A sequence  $\{f_n\}$  of functions, all defined on  $D$  is said to **converge pointwise** on  $D$  if the sequence  $\{f_n(x)\}$  of real numbers converges for each  $x \in D$ . In other words, a sequence  $\{f_n\}$  of functions converges pointwise to  $f$  on  $D$  if for each  $x \in D$  and a given  $\epsilon > 0$ , there exists a positive integer  $N = N(x, \epsilon)$  such that

$$|f_n(x) - f(x)| < \epsilon \quad (n \geq N). \quad (1)$$

In this case, we write  $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ .

It is not always possible to find an  $N$  for which (1) holds good for all  $x \in D$  simultaneously. If for each  $\epsilon > 0$ , it is possible to find an  $N$  (only dependable on  $\epsilon$ ) such that (1) holds for all  $x \in D$  simultaneously, then we say that the sequence  $\{f_n\}$  **converges to  $f$  uniformly** on  $D$ . It is trivial that if the sequence is converging uniformly, it would certainly converge pointwise. The converse of this, however, may not be true always; for example, the sequence  $\{f_n\}$  of functions defined by  $f_n(x) = x^n$  on  $[0, 1]$  converges pointwise to the function  $f$  defined on  $[0, 1]$  as  $f(x) = 0$  ( $0 \leq x < 1$ ),  $f(1) = 1$ . However, the sequence does not converge to  $f$  uniformly on  $[0, 1]$ .

Just as the convergence of a series of real numbers is defined in terms of the convergence of the sequence of its partial sums, the convergence of a series of functions is also defined in terms of a sequence of its partial sums. Accordingly, a series  $\sum u_n$  of real-valued functions defined on a set  $D$  is said to **converge pointwise** (or simply converge) to the function  $f$  on  $D$  if the sequence  $\{f_n\}$  of functions converge pointwise to  $f$  on  $D$ , where  $f_n = u_1 + u_2 + \dots + u_n$ . In this case we write  $\sum_{n=1}^{\infty} u_n = f$  or  $\sum_{n=1}^{\infty} u_n(x) = f(x)$  ( $x \in D$ ). Further, if the sequence  $\{f_n\}$

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converges to  $f$  uniformly on  $D$ , we say that the series  $\sum u_n$  converges uniformly to  $f$  on  $D$ .

### 7 CONTINUITY AND DIFFERENTIABILITY

Let  $f$  be a function defined for all points in a nbd of a point  $c$ , except possibly at the point  $c$  itself. Then the function  $f$  is said to tend to limit  $l$  as  $x$  tends (or approaches) to  $c$  if for each  $\epsilon > 0$ ,  $\exists a \delta > 0$  such that

$$|f(x) - l| < \epsilon \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

Further,  $f$  is said to tend to  $+\infty$  as  $x$  tends to  $c$ , written  $\lim_{x \rightarrow c} f(x) = +\infty$ , if for each  $k > 0$  (however large),  $\exists a \delta > 0$  such that

$$f(x) > k \quad \text{whenever} \quad |x - c| < \delta.$$

Similarly, the function  $f$  is said to tend to  $-\infty$  as  $x$  tends to  $c$ , written  $\lim_{x \rightarrow c} f(x) = -\infty$ , if for each  $k > 0$  (however large) there exists a  $\delta > 0$  such that

$$f(x) < -k \quad \text{whenever} \quad |x - c| < \delta.$$

A function  $f$  is said to tend to a limit  $l$  as  $x$  tends to  $c$  from the left, called the left-hand limit and written  $\lim_{x \rightarrow c-0} f(x) = l$ , if for each  $\epsilon > 0$ , there exists a  $\delta > 0$  such that

$$|f(x) - l| < \epsilon \quad \text{whenever} \quad c - \delta < x < c;$$

and is said to tend to a limit  $l$  as  $x$  tends to  $c$  from the right, called the right-hand limit and written  $\lim_{x \rightarrow c+0} f(x) = l$ , if for each  $\epsilon > 0$  there exists a  $\delta > 0$  such that

$$|f(x) - l| < \epsilon \quad \text{whenever} \quad c < x < c + \delta.$$

Obviously,  $\lim_{x \rightarrow c} f(x)$  exists if and only if both the limits, the left-hand and the right-hand, exist and are equal. It may further be noted that right-hand and left-hand limits may exist without being equal to each other. Monotone functions always have right-hand and left-hand limits.

A function  $f$  is said to be continuous at a point  $c$  if

$$\lim_{x \rightarrow c} f(x) = f(c).$$

A function  $f$  is said to be continuous from the left at  $c$  if

$$\lim_{x \rightarrow c-0} f(x) = f(c);$$

and continuous from the right at  $c$  if

$$\lim_{x \rightarrow c+0} f(x) = f(c).$$

Further, the function  $f$  is said to be **continuous in an interval** if it is continuous at every point of the interval. We cite below a few results which are used at places in the book.

1. If  $f$  and  $g$  are two functions continuous at a point  $c$ , the functions  $f+g$ ,  $f-g$ ,  $f \cdot g$  and  $\frac{f}{g}$  ( $g \neq 0$ ) are also continuous at  $c$ .
2. A function  $f$  defined on an interval  $I$  is continuous at a point  $c \in I$  if and only if for every sequence  $\{c_n\}$  converging to  $c$ , we have  $\lim_{n \rightarrow \infty} f(c_n) = f(c)$ .
3. If  $f$  is continuous in  $[a, b]$ , then  $f$  is bounded therein and attains bounds at least once in  $[a, b]$ .
4. If a function  $f$  is continuous on  $[a, b]$  and  $f(a)f(b) < 0$ , there exists at least one point  $c \in ]a, b[$  such that  $f(c) = 0$ . Consequently, if a function is continuous on  $[a, b]$ , it assumes every value between its bounds.
5. A function  $f$  is continuous if and only if the inverse image of an open (a closed) set is open (closed).

A function  $f$  defined on an interval  $I$  is said to be **uniformly continuous** on  $I$  if for each  $\epsilon > 0$ , there exists a  $\delta > 0$  such that

$$|f(x_2) - f(x_1)| < \epsilon \quad \text{whenever} \quad |x_1 - x_2| < \delta, \quad x_1, x_2 \in I.$$

It may be noted that if a function is uniformly continuous on an interval  $I$ , it is continuous therein but the converse may not be true. However, if the interval  $I$  is closed, the converse holds good.

Let  $f$  be a function defined on an interval  $[a, b]$ . It is said to be **derivable (differentiable)** at an interior point  $c$  ( $a < c < b$ ) if

$$\lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h},$$

or, equivalently

$$\lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}$$

exists. The limit in case it exists, is called the **derivative** of the function  $f$  at  $x = c$ , and is denoted by  $f'(c)$ . The left-hand limit

$$\lim_{x \rightarrow c-0} \frac{f(x) - f(c)}{x - c},$$

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denoted by  $f'(c-0)$ ,  $f'(c-)$  or  $\underline{D}f(c)$ , is called the **left-hand derivative**; while the right-hand limit

$$\lim_{x \rightarrow c+0} \frac{f(x) - f(c)}{x - c},$$

denoted by  $f'(c+0)$ ,  $f'(c+)$  or  $\overline{D}f(c)$ , is called the **right-hand derivative** of  $f$  at  $c$ . A function  $f$  is **derivable** at  $x=c$  if and only if both  $\overline{D}f(c)$  and  $\underline{D}f(c)$  exist and are equal. It may be noted that a function which is derivable at a point is necessarily continuous thereat, but the converse may not be true. For example, the function  $f$  defined by  $f(x) = |x|$  ( $-\infty < x < \infty$ ) is continuous at  $x=0$  but  $f'(0)$  does not exist. In fact, there exists a function which is continuous at each point but does not have a derivative at any point, see Appendix II.

A function  $f$  is said to be **increasing** or **decreasing** in an interval  $I$  according as  $f(x_2) \geq f(x_1)$  or  $f(x_2) \leq f(x_1)$ , for all  $x_2 \geq x_1$  and  $x_1, x_2 \in I$ . In case the strict inequality holds in the above relations,  $f$  is said to be **strictly increasing** or **strictly decreasing**. The function  $f$  is said to be **monotone (monotonic)** in  $I$  if it is either increasing or decreasing therein, and is said to be **strictly monotone** if it is either strictly increasing or strictly decreasing.

## II

# Infinite Sets

The act of counting is undoubtedly one of the oldest human activities. Even a child who is unable to count can nevertheless determine whether there are, for example, just as many chairs as persons in the room. He needs only to have each person take a seat on a chair. By this act, pairs are formed, each pair consisting of one person and one chair. One can also say that the chairs and persons are made to correspond to each other in a one-to-one manner, i.e., in such a way that precisely one chair corresponds to each person, and exactly one person corresponds to each chair. The primitive procedure, however, can be carried over to arbitrary sets too, and then it leads to a concept which corresponds to that of the 'same number' of elements in the case of finite sets. This method of comparison is more powerful since it can be applied even when the sets to be compared are finite. For instance, consider the sets

$\mathbf{N}$  = set of all natural numbers

and  $S = \{x : x = \frac{1}{n}, n \in \mathbf{N}\}.$

The method of comparison shows at once that the number of elements (in some sense) is the same in both the sets because we may arrange these sets as follows:

$\mathbf{N}$		1	2	3	4	...	$n$	...
$S$		1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	...	$\frac{1}{n}$	...

### 1 EQUIVALENT SETS

**1.1 Definition.** A set  $A$  is said to be equivalent to a set  $B$  if  $\exists$  a function  $f : A \rightarrow B$  which is both one-one and onto.

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In symbol, we write  $A \sim B$ . It can easily be verified that the relation ' $\sim$ ' in a family of sets is an equivalence relation. Thus, any two sets are equivalent if and only if there is a one-to-one correspondence between them.

### 1.2 Examples

1. The sets  $\mathbf{N}$  and  $E$  (all even natural numbers) are equivalent because the function

$$f: \mathbf{N} \rightarrow E$$

defined by

$$f(n) = 2n, n \in \mathbf{N}$$

is one-one from  $\mathbf{N}$  onto  $E$ .

2. Let  $A = \{1, 2, 3, 4\}$  and  $B = \{x, y, z, t\}$ . These sets are equivalent because  $\exists$  a function  $f: A \rightarrow B$  defined as

$$f(1) = x, \quad f(2) = y, \quad f(3) = z, \quad f(4) = t$$

which is one-one and onto.

3. The sets  $]0, 1[$  and  $[0, 1]$  are equivalent as  $\exists$  a function  $f: [0, 1] \rightarrow ]0, 1[$  defined by

$$f(x) = \begin{cases} \frac{1}{2} & \text{if } x = 0 \\ \frac{x}{2x+1} & \text{if } x = \frac{1}{n}, n \in \mathbf{N} \\ x & \text{if } x \neq 0, \frac{1}{n}, n \in \mathbf{N} \end{cases}$$

which is clearly one-one and onto.

4. Let  $A = \{1, 2, 3\}$  and  $B = \{x, y\}$ . If we list all the functions defined from  $A$  to  $B$ , we see that none of them is one-to-one. Hence  $A$  is not equivalent to  $B$ , i.e.  $A \not\sim B$ .
5. The finite interval  $]-\frac{\pi}{2}, \frac{\pi}{2}[$  is equivalent to the set  $\mathbf{R}$  (set of all real numbers) since  $\exists$  a function  $f: ]-\frac{\pi}{2}, \frac{\pi}{2}[ \rightarrow \mathbf{R}$ , defined by  $f(x) = \tan x$ , which is one-one and onto.
6. The function

$$f: \mathbf{I} \rightarrow \mathbf{N}$$

defined by

$$f(0) = 1, \quad f(n) = 2n \quad \text{and} \quad f(-n) = 2n + 1 \quad \text{for } n \in \mathbf{N}$$

is clearly a one-one function from  $\mathbf{I}$  onto  $\mathbf{N}$ . Hence  $\mathbf{I} \sim \mathbf{N}$ .

It is often quite hard to exhibit a one-to-one correspondence between two sets explicitly. It is rather easier to put one set into one-to-one correspondence with a subset of the other set. The Schroeder-Bernstein Theorem which is one of the most important and powerful theorems in set theory is very useful in such situations.

**1.3 Theorem.** *Let  $A \supset B \supset A_1$  and  $A \sim A_1$ , then  $A \sim B$ .*

*Proof.* Since  $A \sim A_1$ ,  $\exists$  a function  $f: A \rightarrow A_1$  which is one-to-one. Furthermore, since  $A \supset B$ , the restriction of  $f$  to  $B$ , which we shall also denote by  $f$ , is one-one. This gives that the set  $B$  is equivalent to a subset of  $A_1$ , i.e.,  $B \sim B_1$  where

$$A \supset B \supset A_1 \supset B_1,$$

and that the function  $f: B \rightarrow B_1$  is one-to-one.

Now  $A_1 \subset B$ , and for similar reasons  $A_1 \sim A_2$  where

$$A \supset B \supset A_1 \supset B_1 \supset A_2,$$

and the mapping  $f: A_1 \rightarrow A_2$  is one-to-one. Consequently, there exist equivalent sets  $A, A_1, A_2, \dots$  and equivalent sets  $B, B_1, B_2, \dots$  such that

$$A \supset B \supset A_1 \supset B_1 \supset A_2 \supset B_2 \supset \dots$$

Let

$$T = A \cap B \cap A_1 \cap B_1 \cap A_2 \cap B_2 \cap \dots$$

Then

$$A = (A - B) \cup (B - A_1) \cup (A_1 - B) \cup \dots \cup T$$

and

$$B = (B - A_1) \cup (A_1 - B_1) \cup (B_1 - A_2) \cup \dots \cup T.$$

Note that

$$(A - B) \sim (A_1 - B_1) \sim (A_2 - B_2) \sim \dots$$

Specifically, the function

$$f: A_n - B_n \rightarrow A_{n+1} - B_{n+1}$$

is one-to-one.

Consider the function  $g: A \rightarrow B$  defined by Fig. 2.1.

In other words

$$g(x) = \begin{cases} f(x) & \text{if } x \in A_l - B_l \text{ or } A - B \\ i(x) = x & \text{if } x \in B_m - A_n (n > m) \text{ or } T. \end{cases}$$

The function  $g$  is obviously one-to-one. Hence  $A \sim B$ . ■

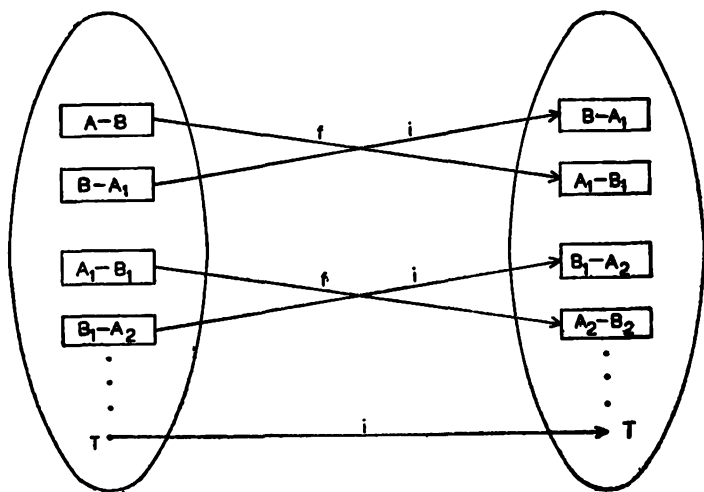


Fig. 2.1

**1.4 Theorem (Schroeder-Bernstein Theorem).** *Let  $A$  and  $B$  be two sets. If each of them is equivalent to a subset of the other, then  $A \sim B$ .*

*Proof.* Since  $A$  is equivalent to a subset  $B_1$  of  $B$ ,  $\exists$  a function  $f: A \rightarrow B_1$  which is one-to-one. Similarly,  $\exists$  a function  $g$  from  $B$  to  $A_1$ , a subset of  $A$ , which is also one-to-one. To each element of  $B$  there corresponds some element of  $A_1$  under  $g$ . In particular, those elements of  $A_1$  which correspond to elements of the set  $B_1$  form a set  $A_2$ . It is obvious that

$$A \supset A_1 \supset A_2 \quad \text{and} \quad A \sim A_2.$$

The sets  $A$  and  $A_2$  are equivalent because  $A \sim B_1$  and  $B_1 \sim A_2$ . In view of Theorem 1.3, it is concluded that  $A \sim A_1$ . Since  $A_1 \sim B$  and the relation ' $\sim$ ' is an equivalence relation, it follows that  $A \sim B$ . ■

*Note.* The Schroeder-Bernstein Theorem has great theoretical and practical significance. With its help we can prove equivalence with a minimum of effort for many specific sets. We shall encounter numerous applications of it in this chapter.

### Problems

1. Show that the open interval  $]0, 1[$  is equivalent to the set  $\mathbb{R}^+$  of all positive real numbers. [Hint: Consider a function  $f: ]0, 1[ \rightarrow \mathbb{R}^+$  given by

$$f(x) = \begin{cases} x & \text{if } 0 < x \leq 1/2 \\ \frac{1}{4(1-x)} & \text{if } 1/2 < x < 1. \end{cases}$$

2. For any two sets  $X$  and  $Y$ , prove that  $(X \times Y) \sim (Y \times X)$ .
3. Let  $X$  be a subset of the Euclidean plane  $\mathbb{R}^2$  defined by

$$X = \{(x, y) : 0 \leq x, y \leq 1\}$$

and  $I$  be the closed-open interval defined by

$$I = \{(x, y) : 0 \leq x < 1 \text{ and } y = 0\}$$

Show that  $X \sim I$ . [Hint: Apply the Schroeder-Bernstein Theorem.]

4. Show that the set  $\mathbb{R}$  of all real numbers is equivalent to the set  $\mathbb{R}^+$  of all positive real numbers. [Hint: Consider the function

$$f: \mathbb{R} \rightarrow \mathbb{R}^+ \text{ defined by } f(x) = e^x.]$$

## 2 FINITE AND INFINITE SETS

A set which is either empty or has a one-to-one correspondence with the set  $\{1, 2, 3, \dots, n\}$ , for some natural number  $n$ , is said to be **finite**. A set which is not finite is called an **infinite set**.

In view of Examples 2 and 4 in 1.2 it can be seen that two finite sets are equivalent only if they consist of the same number of elements.

**2.1 Theorem.** *Every set equivalent to a finite (infinite) set is finite (infinite).*

*Proof.* Every set equivalent to a set of  $n$  elements, where  $n$  is a natural number, is also a set of  $n$  elements. Thus, it follows that every set equivalent to a finite set is a finite set. This further proves that every set equivalent to an infinite set is infinite. ■

**Problem 5.** Give an example of two infinite sets which are not equivalent. [Hint: The sets  $\mathbb{R}$  and  $\mathbb{N}$ .]

## 3 COUNTABLE SETS

**3.1 Definition.** An infinite set is said to be **denumerable** or **enumerable** if it is equivalent to the set  $\mathbb{N}$ , the set of all natural numbers.

**3.2 Definition.** A set which is either finite or denumerable is called a **countable set**.

Sometime, denumerable sets are referred to as **countably infinite** sets.

*Remark.* There is no general agreement in English usage governing the meaning attached to the words 'denumerable' and 'countable'. Both of these terms are used by various authors to mean finite or equivalent to  $\mathbb{N}$ . However, we will follow the convention that 'denumerable' means only equivalent to  $\mathbb{N}$  and that 'countable' means finite or equivalent to  $\mathbb{N}$ .

If  $A$  is a finite set with  $n$  elements and  $f$  is a one-to-one function from  $\{1, 2, 3, \dots, n\}$  to  $A$ , then by letting  $f(i) = a_i$  for  $i = 1, 2, \dots, n$ , the set  $A$  can be written as  $\{a_i; i = 1, 2, \dots, n\}$ . Similarly, if  $A$  is a denumerable set, we can write  $A$  as the indexed set  $\{a_i; i \in \mathbb{N}\}$ , where  $a_i \neq a_j$  if  $i \neq j$ . The process of writing a denumerable set in this form is called *enumeration*. This leads to

**3.3 Theorem.** *Every countable set is equivalent to a set of natural numbers.*

In the following, we shall make use of Axiom of choice indirectly.

**3.4 Theorem.** *A set is infinite if and only if it contains a denumerable subset.*

*Proof.* It is sufficient to prove that every infinite set contains a denumerable set. Let  $A$  be an infinite set. The set  $A$  will not exhaust if we take out one element from it. Let us denote this element by  $a_1$ . Thus, the set  $A - \{a_1\}$  is a nonempty set. We again can extract an element, say  $a_2$ , from  $A - \{a_1\}$  leaving behind a non-empty set  $A - \{a_1, a_2\}$ . Since the set  $A$  is infinite, we could continue this process indefinitely, and as a result a sequence of elements

$$a_1, a_2, \dots, a_n, \dots$$

is extracted from the set  $A$ . Let these elements form a set  $B$ . Clearly  $B$  is a denumerable subset of  $A$ . ■

**3.5 Theorem.** *The family of all finite subsets of a countable set is countable.*

*Proof.* It is sufficient to prove that the family  $\mathcal{F}$  of all finite subsets of  $\mathbb{N}$  is countable.

In the binary system every natural number  $x$  can be uniquely represented by

$$x = \sum_{n=1}^{\infty} \frac{x_n}{2^{n-1}},$$

where  $x_n$  is either 0 or 1. The integers  $x_n, n \in \mathbb{N}$  are called the *binary digits* of  $x$ . For each  $x \in \mathbb{N}$ , only a finite number of the binary digits can be equal to 1.

For each  $x \in \mathbb{N}$ , define a set

$$P_x = \{n \in \mathbb{N} : x_n = 1\}$$

The set  $P_x$  is clearly a finite set and the correspondence  $x \rightarrow P_x$  defines a function

$$f: \mathbb{N} \rightarrow \mathcal{F}$$

which is one-to-one. This gives that the family  $\mathcal{F}$  is countable. ■

In Examples 1, 3 and 6 in 1.2 we came across infinite sets which are equivalent to one of its proper subsets. In the following, we establish it for all the infinite sets.

**3.6 Theorem.** *Every infinite set is equivalent to one of its proper subsets*

*Proof.* Let  $E$  be an infinite set. Let  $x_0 \in E$ . Then consider a set  $F$  which is obtained from  $E$  by deleting the point  $x_0$ , i.e.  $F = E - \{x_0\}$ . We will show that  $E$  is equivalent to  $F$ .

Since  $E$  is an infinite set, by Theorem 3.4, it contains a denumerable set, say,  $P = \{x_1, x_2, \dots, x_n, \dots\}$ . Consider a function  $f: E \rightarrow F$  given by

$$f(x) = \begin{cases} x_1 & \text{if } x = x_0 \\ x_{i+1} & \text{if } x = x_i, i \in \mathbb{N} \\ x & \text{if otherwise.} \end{cases}$$

Then the function  $f$  is one-one and onto. Hence  $E \sim F$ . This proves the theorem. ■

*Remark.* Since this property is not possessed by any finite set, some writers use it even as the definition of an infinite set. Richard Dedekind (1831-1916) was the first who used it as the definition for an infinite set.

**3.7 Theorem.** *A subset of a countable set is countable.*

*Proof.* Let  $A$  be a countable set. By definition,  $A$  is either finite or denumerable. If  $A$  is a finite set, then obviously every subset of it is also finite and so countable. On the other hand, if  $A$  is denumerable then  $A$  can be written as range of a sequence. Let

$$A = \{a_1, a_2, a_3, \dots, a_n, \dots\} \quad (1)$$

and let  $B$  be a subset of  $A$ .

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If  $B$  is an empty set, then  $B$  is finite and hence countable. Suppose  $B \neq \phi$ . Let  $a_{n_1}$  be the first element in the range (1) such that  $a_{n_1} \in B$ . Further, let  $a_{n_2}$  be the first element which follows  $a_{n_1}$  in the range (1) such that  $a_{n_2} \in B$ , and so on. Thus

$$B = \{a_{n_1}, a_{n_2}, a_{n_3}, \dots\}$$

If the set of integers  $n_1, n_2, n_3, \dots$  is bounded,  $B$  is finite; otherwise denumerable. Hence in either case the set  $B$  is countable. ■

**3.8 Theorem.** *The union of a finite set and a countable set is a countable set.*

*Proof.* Let  $A$  be a finite set and  $B$  be a countable set. If  $B$  is finite, then obviously  $A \cup B$  is a finite set and hence countable. On the other hand, if  $B$  is denumerable then there are two possibilities: (i)  $A \cap B = \phi$  and (ii)  $A \cap B \neq \phi$ .

*Case (i)* (when  $A \cap B = \phi$ ). Let the sets  $A$  and  $B$ , respectively, be written as

$$A = \{a_1, a_2, \dots, a_p\}$$

$$B = \{b_1, b_2, \dots, b_n, \dots\}.$$

This implies

$$A \cup B = \{a_1, a_2, \dots, a_p, b_1, b_2, \dots, b_n, \dots\}.$$

Define a function  $f: \mathbb{N} \rightarrow A \cup B$  by

$$f(n) = \begin{cases} a_n & \text{if } 1 \leq n \leq p \\ b_{n-p} & \text{if } n \geq p+1. \end{cases}$$

One can easily see that  $f$  is one-one and onto. Hence  $A \cup B$  is denumerable and so countable.

*Case (ii)* (when  $A \cap B \neq \phi$ ). Let  $C = A \cap B$ . Set  $B^* = B - C$ . It is obvious that  $B^*$  is a denumerable set disjoint from  $A$ . In view of case (i), the set  $A \cup B^*$  is countable. Further,  $A \cup B = A \cup B^*$ . Hence  $A \cup B$  is a countable set. ■

**3.9 Corollary.** *The set  $\mathbb{I}^+$  of all non-negative integers is a countable set.*

*Proof.* Writing  $\mathbb{I}^+ = \mathbb{N} \cup \{0\}$ , the result follows. ■

**3.10 Theorem.** *The union of two countable sets is a countable set.*

*Proof.* Let  $A$  and  $B$  be any two countable sets. Then there are three possibilities:

- (i) Both the sets are finite.
- (ii) One is a finite set and the other is denumerable.
- (iii) Both the sets are denumerable.

Cases (i) and (ii) above are disposed of in view of Theorem 3.8.

Assume that both the sets  $A$  and  $B$  are denumerable. Let

$$A = \{a_1, a_2, a_3, \dots, a_n, \dots\}$$

$$B = \{b_1, b_2, b_3, \dots, b_n, \dots\}.$$

Then

$$A \cup B = \{a_1, b_1, a_2, b_2, a_3, b_3, \dots\}.$$

Now, define a function  $f: \mathbb{N} \rightarrow A \cup B$  by

$$f(n) = \begin{cases} a_{n+1/2} & \text{if } n \text{ is odd} \\ b_{n/2} & \text{if } n \text{ is even.} \end{cases}$$

Clearly  $f$  defines a one-to-one correspondence. This proves that  $A \cup B$  is denumerable and hence countable. ■

*Remarks.* 1. In the proof of Theorem 3.10 it has been assumed that both the sets are disjoint. If the sets are not disjoint, even then the result can be proved as done in Theorem 3.8, case (ii).

2. The result can be extended to any finite number of sets.

**3.11 Corollary.** *The set  $I$  of all integers is countable.*

*Proof.* Writing  $I = \mathbb{N} \cup \{0\} \cup \mathbb{N}^-$ , where  $\mathbb{N}^-$  is the set of all negative integers, the result follows. ■

*Note.* For an alternative proof of Corollary 3.11, see Example 6 in 1.2.

**3.12 Theorem.** *If  $A$  and  $B$  are countable sets, then  $A \times B$  is countable.*

*Proof.* If  $A$  or  $B$  is empty, then the result is trivial. Assume that both  $A$  and  $B$  are nonempty sets. Since  $A$  and  $B$  are countable, we can write them as

$$A = \{a_i: i \in J_1\}$$

and

$$B = \{b_j: j \in J_2\},$$

where  $J_1$  and  $J_2$  are subsets of  $\mathbb{N}$ . Define a function  $f: A \times B \rightarrow \mathbb{N}$  by

$$f(a_i, b_j) = 2^i 3^j, \quad i \in J_1 \quad \text{and} \quad j \in J_2.$$

The function  $f$  is clearly one-to-one on the sets  $A \times B$  and

$$C = \{x: x = 2^i 3^j, i, j \in \mathbb{N}\}.$$

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But  $C \subset \mathbb{N}$  and so  $C$  is a countable set, in view of Theorem 3.7. This proves that  $A \times B$  is a countable set. ■

**3.13 Corollary.** *The set  $\mathbb{N} \times \mathbb{N}$  is countable.*

**3.14 Theorem.** *The union of a denumerable collection of denumerable sets is denumerable.*

*Proof.* Let  $\mathcal{F}$  be a denumerable collection of denumerable sets. The collection  $\mathcal{F}$  may be expressed as

$$\mathcal{F} = \{A_1, A_2, A_3, \dots, A_n, \dots\},$$

where each of the sets  $A_1, A_2, A_3, \dots$  is denumerable. Since each  $A_i$ ,  $i \in \mathbb{N}$  is denumerable, we have

$$\begin{aligned} A_1 &= \{a_{11}, a_{12}, a_{13}, a_{14}, \dots, a_{1n}, \dots\} \\ A_2 &= \{a_{21}, a_{22}, a_{23}, a_{24}, \dots, a_{2n}, \dots\} \\ &\vdots \\ A_n &= \{a_{n1}, a_{n2}, a_{n3}, a_{n4}, \dots, a_{nn}, \dots\} \\ &\vdots \end{aligned}$$

Now, list the elements of  $\bigcup_{i \in \mathbb{N}} A_i$  as follows:

$$\begin{array}{ccccccc} a_{11} & & & & & & \\ a_{21} & a_{12} & & & & & \\ a_{31} & a_{22} & a_{13} & & & & \\ a_{41} & a_{32} & a_{23} & a_{14} & & & \\ \vdots & & & & & & \\ a_{n1} & a_{n-1 2} & a_{n-2 3} & \dots & a_{1n} & & \\ \vdots & & & & & & \end{array}$$

In the above, the first row consists of all the elements  $a_{pq}$  where  $p+q=2$ . The second row consists of all those  $a_{pq}$  for which  $p+q=3$ ; and so on. In this process, an element  $a_{jk}$  will be removed if it has been listed already. In this manner, a definite place is given to each element of the collection  $\mathcal{F}$ . For example,  $a_{ij}$  is the  $j^{\text{th}}$  element of the  $(i+j-1)^{\text{th}}$  row. This proves that the collection  $\mathcal{F}$  is denumerable.

*Alternative proof.* Let  $\{A_i : i \in P \subset \mathbb{N}\}$  be a denumerable collection of denumerable sets. Then each  $A_i$  can be written as

$$A_i = \{a_{ij} : j \in Q_i \subset \mathbb{N}\}.$$

Assume that  $\bigcup_{i \in P} A_i$  is nonempty, otherwise the result is trivial. If  $x \in A_i$  then  $x = a_{ij}$  for at least one pair  $(i, j)$  with  $i \in P$  and  $j \in Q_i$ . Let

$f(x) = (i, j)$  where  $i$  is the smallest positive integer such that  $x = a_{ij}$ . This gives a one-one function  $f: \bigcup_{i \in P} A_i \rightarrow \mathbb{N} \times \mathbb{N}$ . In view of the Corollary 3.13, the set  $\bigcup_{i \in P} A_i$  is countable. ■

**3.15 Theorem.** *The set  $\mathbb{Q}$  of all rational numbers is a denumerable set.*

*Proof.* Let  $\mathbb{Q}^+$  and  $\mathbb{Q}^-$ , respectively, denote the sets of positive and negative rational numbers. Then

$$\mathbb{Q} = \mathbb{Q}^- \cup \{0\} \cup \mathbb{Q}^+.$$

Consider a function  $f: \mathbb{Q}^+ \rightarrow \mathbb{N} \times \mathbb{N}$  given by

$$f\left(\frac{p}{q}\right) = (p, q).$$

Since  $p/q$  is a member of  $\mathbb{Q}^+$  expressed as the ratio of two relatively prime positive integers, the function  $f$  is one-one from  $\mathbb{Q}^+$  into  $\mathbb{N} \times \mathbb{N}$ . Thus  $\mathbb{Q}^+$  is equivalent to a subset of  $\mathbb{N} \times \mathbb{N}$  and hence denumerable.

The sets  $\mathbb{Q}^+$  and  $\mathbb{Q}^-$  being in one-to-one correspondence,  $\mathbb{Q}^-$  is also denumerable. Hence it follows, from Theorem 3.10, that the set  $\mathbb{Q}$  which is the union of  $\mathbb{Q}^+$ ,  $\mathbb{Q}^-$  and  $\{0\}$  is denumerable. ■

*Note.* The assertion that the set  $\mathbb{Q}^+$  is denumerable can also be established by observing that the set

$$A_q = \left\{ \frac{1}{q}, \frac{2}{q}, \frac{3}{q}, \dots \right\}$$

is denumerable for each fixed  $q \in \mathbb{N}$  and

$$\mathbb{Q}^+ = \bigcup_{q \in \mathbb{N}} A_q.$$

**3.16 Theorem.** *The set  $P$  of all polynomials*

$$p(x) = a_0 + a_1x + a_2x^2 + \dots + a_mx^m \tag{2}$$

*with integral coefficients is denumerable.*

*Proof.* For each pair of natural numbers  $(m, n)$ , let  $P_{mn}$  be the set of polynomials of the form (2) each of degree  $n$  in which

$$|a_0| + |a_1| + \dots + |a_n| = m.$$

Since  $m$  and  $n$  are finite, the set  $P_{mn}$  is finite. Also

$$\begin{aligned} P &= \bigcup \{P_{mn} : m, n \in \mathbb{N}\} \\ &= \bigcup \{P_k : k \in \mathbb{N} \times \mathbb{N}\}. \end{aligned}$$

Hence  $P$ , in view of Theorem 3.14 is a denumerable set. ■

**3.17 Definition.** A real number is called an **algebraic number** if it is a solution of a polynomial equation

$$p(x) = a_0 + a_1x + \dots + a_nx^n = 0 \quad (a_n \neq 0)$$

with integral coefficients.

*Remark.* The set of all algebraic numbers contains the set of all rational numbers (e.g.  $3/4$  is the root of  $4x - 3 = 0$ ) and many other numbers besides (the square root of 3 is a root of  $x^2 - 3 = 0$ ).

**3.18 Definition.** A real number which is not "algebraic" is called a **transcendental number**.

*Note.* The numbers  $e$  and  $\pi$  are the best known transcendental numbers. The numbers  $e$  and  $\pi$  were proved to be transcendental, respectively, by Hermite in 1873 and by Lindemann in 1892.

*Remark.* Every transcendental number is irrational but the converse is not true.

**3.19 Theorem.** *The set  $A$ , of all real algebraic numbers is denumerable (countable).*

*Proof.* By Theorem 3.16, the set  $P$  of all polynomial equations is denumerable. We, therefore, can write

$$P = \{P_i(x) = 0 : i \in \mathbf{N}\}.$$

Let

$$A_i = \{x : x \text{ is a solution of } P_i(x) = 0\},$$

for a fixed  $i \in \mathbf{N}$ . Since each polynomial of degree  $n$  can have at most  $n$  roots, each  $A_i$  is finite and therefore

$$A = \bigcup_{i \in \mathbf{N}} A_i,$$

being a countable union of countable sets, is countable. Further, since  $A$  is not finite, it is denumerable. ■

### Problems

6. Using the function  $f: \mathbf{N} \times \mathbf{N} \rightarrow \mathbf{N}$  defined by

$$f(m, n) = 2^m(2^n + 1) - 1,$$

prove that  $\mathbf{N} \times \mathbf{N}$  is a countable set.

7. Show that every quotient set of a countable set is countable.
8. Prove that the set of all polynomials with rational coefficients is countable.
9. The set of all complex numbers which are algebraic over the field of rational numbers is countable.
10. Let  $f$  denote a function with domain  $A$  and range  $B$ . Show that if  $A$  is countable, then so is  $B$ .
11. Let  $A$  be a countable set. Then prove that the set of all finite sequences from  $A$  is also countable. [Hint: It is sufficient to prove that the set of all finite sequences of natural numbers is countable.]
12. Applying the Schroeder-Bernstein Theorem, show that there are just as many sets of positive integers as there are real numbers.
13. Point out in Theorem 3.4, the use of axiom of choice.
14. The family of all finite subsets of a countable set is countable.
15. A set  $E$  is called isolated if  $E \cap E' = \phi$ , where  $E'$  denotes the set of all limit points of  $E$ . Prove that every isolated set of  $\mathbb{R}$  is countable.
16. Prove that the continuous image of a countable set is countable.
17. Let  $X$  be a subset of the coordinate plane  $\mathbb{R}^2$  defined by

$$X = \{(x, y) : 0 \leq x, y < 1\},$$

and  $I$  be the closed-open interval defined by

$$I = \{(x, y) : 0 \leq x < 1 \text{ and } y = 0\}.$$

Show that  $X \sim I$ . [Hint: Apply the Schroeder-Bernstein Theorem.]

18. Let  $F$  denote the set of all functions defined on the set  $I_n = \{1, 2, \dots, n\}$  with range in  $I_+$ . Prove that  $F$  is a denumerable set.
19. Prove that the set of all finite sequences whose terms are algebraic numbers is countable.
20. Prove that the set of all straight lines in a plane each of which passes through (at least) two different points with rational coordinates is countable.
21. A point  $x \equiv (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$  is said to be a rational point if each  $x_i$  ( $i = 1, 2, \dots, n$ ) is a rational number. Show that the set of all rational points in  $\mathbb{R}^n$  is denumerable.
22. Prove that the set of points of discontinuity of a monotonic function is countable.
23. Prove Theorem 3.15 by using Theorem 3.7 and Problem 11.

#### 4 UNCOUNTABLE SETS

On the basis of the previous section we may form an opinion that every set is countable (perhaps!) but in fact it is not so. It has been shown therein that each of the sets  $\mathbf{N}$ ,  $\mathbf{I}$ ,  $\mathbf{Q}$  and  $\mathbf{A}$  (the set of all algebraic numbers) is countable though one represents, loosely speaking, a "higher type of infinity" than the previous one because of the proper inclusive relation  $\mathbf{N} \subset \mathbf{I} \subset \mathbf{Q} \subset \mathbf{A}$ .

Now the question arises whether the set  $\mathbf{R}$  of all real numbers which is a superset of  $\mathbf{A}$  is also countable? The answer to this question is *no*. For, Cantor discovered that the set  $\mathbf{R}$  is "not" countable—or, as we phrase it,  $\mathbf{R}$  is "uncountable" or "uncountably infinite". Since, as a geometrical representation, we identified the elements of  $\mathbf{R}$  with the points on the real line, this means that the set of all points on the real line represents, again loosely speaking, a "higher type of infinity" than that of only the integral points or rational points. Further, one may think that the set of all points on the real line  $\mathbf{R}$  is uncountable because  $\mathbf{R}$  is infinitely long. But surprisingly, we will show in the following that any open interval in  $\mathbf{R}$ , no matter how small it may be, has precisely as many points as on  $\mathbf{R}$  itself.

**4.1 Definition.** A set which is not countable is called **uncountable**.

*Note.* Since an uncountable set is necessarily an infinite set, we can call that as **nondenumerable set**.

**4.2 Theorem.** *The open interval  $]0, 1[$  is uncountable.*

*Proof.* Let, if possible,  $]0, 1[$  be a countable set. Then the elements of  $]0, 1[$  can be written as

$$\{a_1, a_2, a_3, \dots\}.$$

Expand each  $a_n$  in the form of an infinite decimal. Let

$$\begin{aligned} a_1 &= \cdot b_{11}b_{12}b_{13} \dots b_{1n} \dots \\ a_2 &= \cdot b_{21}b_{22}b_{23} \dots b_{2n} \dots \\ &\vdots \\ a_n &= \cdot b_{n1}b_{n2}b_{n3} \dots b_{nn} \dots \\ &\vdots \end{aligned}$$

where  $b$ 's may have any integral value from 0 to 9. In representing the elements by decimals, we avoid the infinite chain of 9's; for instance, we write  $1/2$  as  $\cdot 5000 \dots$  and not as  $\cdot 4999 \dots$ . In this way,

we guarantee that each element in  $]0, 1[$  has one and only one decimal representation. Now construct a number

$$a = .b_1b_2b_3 \dots b_n \dots$$

where each  $b$  is an integer from 0 to 8 but  $b_1 \neq b_{11}, b_2 \neq b_{22}, b_3 \neq b_{33}, \dots, b_n \neq b_{nn}, \dots$

The above construction shows that the number  $a$  belongs to the interval  $]0, 1[$  and is different from any one of the numbers  $a_n$  for it is different from  $a_1$  in at least the first digit and so on. This contradicts our assumption that we can list all the real numbers between 0 and 1. This completes the proof. ■

**4.3 Theorem.** *The closed interval  $A = [0, 1]$  is uncountable.*

*Proof.* Let us assume, on the contrary, that the interval  $A$  is a countable set. Then all its points can be arranged in a sequence

$$x_1, x_2, \dots, x_n, \dots \tag{3}$$

We assume, then, that every point  $x \in A$  occurs in the sequence (3). Divide  $A$  into three equal parts by means of the points  $1/3$  and  $2/3$ . It is clear that the point  $x_1$  cannot belong to all three subintervals

$$\left[0, \frac{1}{3}\right], \left[\frac{1}{3}, \frac{2}{3}\right], \left[\frac{2}{3}, 1\right] \tag{4}$$

and that at least one of these subintervals fails to contain  $x_1$ . We denote this interval by  $A_1$  (if there are two such subintervals, then take either of them). We now divide the interval  $A_1$  into three subintervals of equal length and denote by  $A_2$ , one of the new intervals which does not contain the point  $x_2$ . We further divide the segment  $A_2$  into three equal intervals and designate by  $A_3$  the one interval which does not contain  $x_3$ , and so on.

As a result, we obtain an infinite sequence of nested intervals

$$A_1 \supset A_2 \supset A_3 \supset A_4 \supset \dots$$

which possesses the property that

$$x_n \notin A_n.$$

Since the length of the interval  $A_n$  is  $3^{-n}$ , it is clear that this length tends to zero as  $n \rightarrow \infty$ . Then, in accordance with a well-known limit theorem, there exists exactly one point belonging to all the intervals. Suppose

$$\eta \in A_n, \text{ for all } n.$$

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Being a point of the interval  $A$ , the point  $\eta$  must appear in the sequence (3). However, this is clearly impossible because, for every  $n$ , we have

$$x_n \notin A_n \quad \text{and} \quad \eta \in A_n, \quad \forall n \in \mathbb{N}.$$

It gives that

$$\eta \neq x_n, \quad \forall n \in \mathbb{N},$$

i.e.  $\eta$  is not a point of the sequence (3). This contradicts our assumption and hence  $[0, 1]$  is uncountable. ■

*Remark.* In view of Example 3 in 1.2, either of the Theorems 4.2 and 4.3 can be obtained from the other.

*Note.* The interval  $[0, 1]$  is often designated as the **continuum**.

**4.4 Theorem.** Any open interval  $]a, b[$  is equivalent to any other open interval  $]c, d[$ .

*Proof.* Let  $x \in ]a, b[$ . Consider a function  $f: ]a, b[ \rightarrow ]c, d[$  given by

$$f(x) = c + \frac{d-c}{b-a}(x-a).$$

It is easy to verify that  $f$  is one-one and onto. Hence the result follows. ■

**4.5 Corollary.** Let  $a$  and  $b$  be any two real numbers with  $a < b$ . Then  $]a, b[$  is an uncountable set.

*Proof.* The result is obtained in view of Theorems 4.2 and 4.4. ■

**4.6 Theorem.** Any closed interval  $[a, b]$  is equivalent to any other closed interval  $[c, d]$ .

*Proof.* It follows by using the same function as in the proof of Theorem 4.4. ■

Using the Schroeder-Bernstein Theorem, we now prove that any two intervals are equivalent.

**4.7 Theorem.** Any two intervals are equivalent.

*Proof.* It is enough to prove this result only for the case when one of the intervals is open and the other interval is closed. A similar argument will hold if either of these is a semi-open interval.

Suppose  $I_1 = [a, b]$  and  $I_2 = ]c, d[$ . Since  $a < b$ ,  $\exists$  two real numbers  $a_1$  and  $b_1$  such that

$$a \leq a_1 < b_1 \leq b.$$

Similarly, there exist two real numbers  $c_1$  and  $d_1$  such that

$$c < c_1 < d_1 < d.$$

The interval  $]a_1, b_1[$  is contained in  $[a, b]$  and the interval  $[c_1, d_1]$  in  $]c, d[$ . By Theorems 4.5 and 4.6, it may be noted that

$$]a_1, b_1[ \sim ]c, d[$$

and 
$$]c_1, d_1[ \sim [a, b].$$

Hence, by the Schroeder-Bernstein Theorem,  $[a, b] \sim ]c, d[$ . ■

*Note.* The result proved above is true for any two intervals, regardless of their lengths.

**4.8 Theorem.** *The set  $\mathbb{R}$  of all real numbers is uncountable.*

*Proof.* To prove this, it is sufficient to show that  $]0, 1[ \sim \mathbb{R}$ . Consider the function  $f : ]0, 1[ \rightarrow \mathbb{R}$  given by

$$f(x) = \begin{cases} \frac{2x-1}{x} & \text{if } 0 < x < \frac{1}{2} \\ \frac{2x-1}{1-x} & \text{if } \frac{1}{2} \leq x < 1 \end{cases}$$

Clearly,  $f$  defines a one-to-one correspondence between  $]0, 1[$  and  $\mathbb{R}$ . This proves that  $]0, 1[ \sim \mathbb{R}$ . Hence, in view of Theorem 4.2,  $\mathbb{R}$  is uncountable.

*Geometrical proof.* Let  $]a, b[$  be an open interval. We bend  $]a, b[$  into a semi-circle and rest this semi-circle tangentially on the real line  $\mathbb{R}$  as shown in Fig. 2.2.

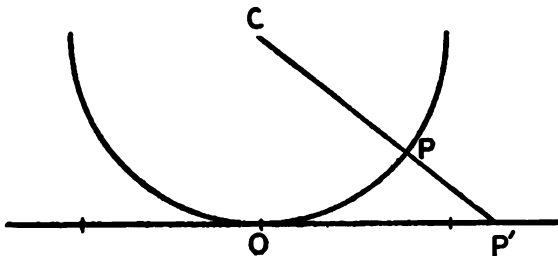


Fig. 2.2

Let  $P$  be a point of  $]a, b[$ . Then, by joining  $P$  with the centre of the semi-circle and producing it to meet the real line at  $P'$ , we can locate a point on the real line corresponding to each point of  $]a, b[$ ; and conversely. This establishes a one-to-one correspondence between the sets  $]a, b[$  and  $\mathbb{R}$ . Hence  $\mathbb{R}$  is uncountable. ■

*Note.* The result can be obtained by setting a one-to-one correspondence between  $] -\pi/2, \pi/2[$  and  $\mathbb{R}$ , see Example 5 in 1.2.

**4.9 Corollary.** *Any interval is equivalent to the set  $\mathbb{R}$ .*

**4.10 Corollary.** *The set of irrational numbers is uncountable.*

*Proof.* Let, on the contrary, the set of irrational numbers be countable. Then the union of the set of rational numbers and the set of irrational numbers which is nothing but the set  $\mathbb{R}$  of real numbers would be countable. In view of Theorem 4.8, this contradicts our assumption. Hence the result follows. ■

**4.11 Theorem\*** *The set of all real transcendental numbers is uncountable.*

*Proof.* Let, on the contrary, the set of all real transcendental numbers be countable. In Theorem 3.19, we have proved that the set of algebraic numbers is countable. So, the union of the sets of algebraic numbers and transcendental numbers which is the set of real numbers is countable, in view of Theorem 3.10. This contradicts the fact that the set  $\mathbb{R}$  of all real numbers is uncountable. ■

**4.12 Theorem** *Every subset  $X$  of  $\mathbb{R}$  containing an open interval  $I$  is equivalent to  $\mathbb{R}$ .*

*Proof.* The proof of this result makes use of the Schroeder-Bernstein Theorem and the result that  $I$  is equivalent to  $\mathbb{R}$  (see Theorem 4.8).

The set  $X$  is equivalent to itself and so equivalent to a subset of  $\mathbb{R}$ . Also  $\mathbb{R}$  is equivalent to a subset  $I$  of  $\mathbb{R}$ . So, by the Schroeder-Bernstein Theorem, the sets  $X$  and  $\mathbb{R}$  are equivalent. ■

### Problems

24. Prove that the set of all sequences of natural numbers is uncountable.

\*The result is also valid for complex transcendental numbers.

25. Let  $A$  be the set of all sequences whose elements are the digits 0 and 1. Prove that  $A$  is uncountable. [Hint: The proof can be obtained by applying the method known as the Cantor diagonal process.]
26. Prove that the set of all points of the plane is uncountable.
27. Prove that the family of all subsets of a denumerable set is uncountable.
28. If  $f: A \rightarrow B$  and the range of  $f$  is uncountable, prove that the domain of  $f$  is uncountable.

## 5 CARDINALITY OF SETS

Up to now, we have classified sets into finite, denumerable and non-denumerable sets. In § 4, there are simply all those sets that remain after the first two classes have been split off, i.e. all sets which are not countable. One can raise the question whether or not it is possible to further subdivide the class of uncountable sets. Motivated by the fact that elements from the set  $\mathbb{N}$  of natural numbers are used for the purpose of counting the elements in sets having finite number of elements,  $G$ . Cantor gave this question the following form:

*Can the concept of natural numbers be generalised in such a manner that every set is assigned one of these generalised "numbers" for the "number of elements" in the set?*

Should this be possible, there would result immediately a classification of infinite sets, too, according to the number of their elements.

Cantor himself tried to define the concept of the power of a set, finite or infinite, by the aid of an expression: "The power of a given set  $A$  is that general idea which remains with us when thinking of this set, we abstract from it all properties of its elements as well as from their order." He used the symbol  $\overline{\overline{A}}$  to denote the power of the set  $A$  (the two lines indicate double abstraction). The term **cardinal number** is commonly used as synonymous with the term power. More precisely we have:

Let all the sets be divided into families such that two sets fall into one family if and only if they are equivalent which is possible since the relation  $\sim$  between the sets is an equivalence relation. To every such family of sets, we assign some arbitrary symbol and call this symbol the cardinal number (or the power) of each set of the given family. If the cardinal number of a set  $A$  is  $\alpha$ , we write

$$\overline{\overline{A}} = \alpha$$

or

$$\text{card}(A) = \alpha.$$

The way cardinal number of a set is defined it is obvious that two cardinal numbers  $\alpha$  and  $\beta$  are equal if  $\exists$  sets  $A$  and  $B$  with  $\overline{A} = \alpha$  and  $\overline{B} = \beta$  such that  $A \sim B$ .

The cardinal number of the empty set is defined to be 0 (zero). We designate the number of elements of a nonempty finite set as the cardinal number of the finite set, i.e., 1 as the cardinal number of all singleton sets; 2 as the cardinal number of all sets which are equivalent to the set  $\{a, b\}$ ; and so on. Thus, we can easily assign a cardinal number to a finite set from the set of nonnegative integers. We shall assign  $\aleph_0$  (read as aleph nought, being the first letter of the Hebrew alphabet) to the class of all denumerable sets and as such  $\aleph_0$  is the cardinal number of any denumerable set. We denote by  $c$ , the first letter of the word *continuum*, the cardinal number of the set  $[0, 1]$ .

Clearly, the set of cardinal numbers

$$\{0, 1, 2, 3, \dots, \aleph_0, c\}$$

is a superset of the set of natural numbers and thus we may consider the set of all cardinal numbers as an extension of the set  $\mathbb{N}$ . Everyone is tempted to extend the order relation and the operations of addition and multiplication of  $\mathbb{N}$  to the set of cardinal numbers in a natural way. This has been discussed in the sections that follow.

## 6 ORDER RELATION IN CARDINAL NUMBERS

**6.1 Definition.** Let  $\alpha$  and  $\beta$  be two cardinal numbers. Let  $A$  and  $B$  be two sets having cardinal numbers  $\alpha$  and  $\beta$ , respectively. We say  $\alpha \leq \beta$  or  $\beta \geq \alpha$  if the set  $A$  is equivalent to a subset of  $B$ .

We write  $\alpha < \beta$  if  $\alpha \leq \beta$  and  $\alpha \neq \beta$ . In other words,  $\alpha < \beta$  if the set  $A$  is equivalent to a proper subset of  $B$  and  $A \not\sim B$ .

*Note.* The definition given above and other definitions and results involving cardinal numbers are independent of the specific choice of the sets  $A$  and  $B$ . In other words, if we select two other sets  $A'$  and  $B'$  such that  $A \sim A'$  and  $B \sim B'$ , then  $A'$  and  $B'$  serve just as well in describing the relationship between  $\alpha$  and  $\beta$  and  $A \sim B$ .

**6.2 Theorem.** *If  $\alpha$ ,  $\beta$  and  $\gamma$  are cardinal numbers and if  $\alpha < \beta$  and  $\beta < \gamma$ , then  $\alpha < \gamma$ .*

*Proof.* Let  $A$ ,  $B$  and  $C$  be the sets having cardinal numbers  $\alpha$ ,  $\beta$  and  $\gamma$ , respectively. Since  $\alpha < \beta$ , the set  $A$  is equivalent to  $B^*$ , a proper

subset of  $B$ . Similarly  $\beta < \gamma$  implies  $B$  is equivalent to  $C^*$ , a proper subset of  $C$ . From the relations

$$A \sim B^* \quad \text{and} \quad B \sim C^*,$$

it can easily be obtained that  $A \sim C^{**}$ , a subset of  $C^*$ . Hence  $\alpha < \gamma$ . ■

*Remark.* Since the asymmetry of the relation ‘ $<$ ’ in the set of cardinal numbers follows trivially from the definition, we conclude, in view of the transitivity available from Theorem 6.2, that this relation defines a “linear order” in the set of cardinal numbers.

**6.3 Definition.** The cardinal number of a finite set is called **finite cardinal number** and that of an infinite set is called **transfinite cardinal number**.

The cardinal numbers  $\aleph_0$  and  $c$  are the examples of transfinite cardinal numbers.

**6.4 Theorem.** *If  $\alpha$  is any transfinite cardinal number and  $\beta$  is any finite cardinal numbers, then  $\alpha > \beta$ .*

*Proof.* Let  $\bar{A} = \alpha$  and  $\bar{B} = \beta$ . Since the set  $A$  is an infinite set and  $B$  is a finite set containing  $\beta$  elements,  $A$  contains a subset (any subset containing  $\beta$  elements) equivalent to  $B$  but  $A$  is not equivalent to any subset of  $B$  since  $B$  is finite. Hence  $\alpha > \beta$ . ■

**6.5 Corollary.**  $n < \aleph_0$ , for any natural number  $n$ .

Now the question is whether there is any cardinal number lying between  $n$  and  $\aleph_0$ . The answer is in the negative since as a consequence of Theorem 3.4, we have  $\aleph_0$  as the smallest transfinite cardinal number.

**6.6 Theorem.**  $\aleph_0 < c$ .

*Proof.* Since  $\aleph_0$  is the smallest transfinite cardinal number,  $\aleph_0 \leq c$ . Further  $\aleph_0 \neq c$  because  $\aleph_0$  is the cardinality of a denumerable set and  $c$  is the cardinal number of  $[0, 1]$  which is an uncountable set. Hence  $\aleph_0 < c$ . ■

We have, so far, two transfinite cardinal numbers  $\aleph_0$  and  $c$ , and we know that  $\aleph_0 < c$ . The question now arises: “Is there any cardinal number greater than  $c$ ?” The answer is in the affirmative as a consequence of the following theorem.

**6.7 Theorem.**  $\aleph_0 < c < \mathfrak{f}$ , where  $\mathfrak{f}$  is the cardinal number of the set  $\mathcal{F}$  of all real-valued functions defined on the interval  $[0, 1]$ .

*Proof.* In view of Theorem 6.6, we are left to prove that  $c < \mathfrak{f}$ . Let  $a$  be any element of  $[0, 1]$ . Define a function  $\varphi_a : [0, 1] \rightarrow \mathbb{R}$  by

$$\varphi_a(x) = a, \quad \forall x \in [0, 1].$$

Let  $A = \{\varphi_a : a \in [0, 1]\}$ . Then  $A \subset \mathcal{F}$  and  $A \sim [0, 1]$ . This verifies that  $c \leq \mathfrak{f}$ .

We shall now prove that  $c \neq \mathfrak{f}$ . Let, if possible,  $c = \mathfrak{f}$ . Then there is one-to-one correspondence between  $[0, 1]$  and  $\mathcal{F}$ . Let  $y \in [0, 1]$ , and let us denote by  $f_y$  the corresponding element in  $\mathcal{F}$ .

Write  $f_y(x) = F(x, y)$ . Clearly  $F(x, y)$  is a function of two variables  $x$  and  $y$  defined for all  $x$  and  $y$  in  $[0, 1]$ . Further, writing

$$\begin{aligned} \psi(x) &= F(x, x) + \frac{1}{2} \\ &= f_x(x) + \frac{1}{2}, \quad x \in [0, 1] \end{aligned}$$

we note that  $\psi : [0, 1] \rightarrow \mathbb{R}$  and so  $\psi \in \mathcal{F}$ . Also,  $\psi \neq f_x$  for any  $x \in [0, 1]$ . Thus,  $\exists a \psi \in \mathcal{F}$  which has no pre-image in  $[0, 1]$ . This verifies that  $[0, 1] \not\sim \mathcal{F}$ . Hence  $c \neq \mathfrak{f}$ .

This completes the proof.  $\blacksquare$

Again, the natural problem is to find if there is a cardinal number greater than  $\mathfrak{f}$ . The answer to this is also affirmative. In fact, we can show, starting from a set of arbitrary cardinal number, that it is possible to construct a set of greater cardinal number. This has been shown in the following famous theorem known as Cantor's Theorem.

**6.8 Theorem (Cantor's Theorem).** *Let  $A$  be any arbitrary set and  $\mathcal{P}(A)$  be the power set of  $A$ . Then*

$$\overline{A} < \overline{\mathcal{P}(A)}.$$

*Proof.* Let  $B^*$  be the set of all singleton subsets of  $A$ . Then obviously  $B^* \sim A$  and  $B^* \subset \mathcal{P}(A)$ . The one-to-one correspondence  $f$  between  $A$  and  $B^*$  is

$$f(x) = \{x\}, \quad \text{for all } x \in A.$$

Thus  $\overline{A} \leq \overline{\mathcal{P}(A)}$ . The result will be proved on showing  $\overline{A} \neq \overline{\mathcal{P}(A)}$ . Let, if possible,  $\overline{A} = \overline{\mathcal{P}(A)}$ . Then  $A \sim \mathcal{P}(A)$ . Let  $\varphi$  be a one-to-one correspondence carrying  $A$  onto  $\mathcal{P}(A)$ . To every  $a \in A$ , there corresponds a definite element of  $\mathcal{P}(A)$  under the correspondence  $\varphi$  which we

designate by  $\varphi(a)$ , and every element of  $\mathcal{P}(A)$  is  $\varphi(a)$  for one and only one  $a \in A$ .

Let us call an element  $a \in A$  a "bad" element if  $a$  is not a member of the set which is its image, i.e. if  $a \notin \varphi(a)$ , and let  $T$  be the set of bad elements. Specifically.

$$T = \{a : a \in A, a \notin \varphi(a)\}.$$

Note that  $T$  is a subset of  $A$ , i.e.  $T \in \mathcal{P}(A)$ . Hence, since  $\varphi$  is onto,  $\exists$  an element  $t \in A$  with the property that  $\varphi(t) = T$ .

Is  $t$  a "bad" or "good" element?

If  $t \in T$ , then by the definition of  $T$ ,  $t \notin \varphi(t) = T$ , which is untrue. Thus the assumption  $A \sim \mathcal{P}(A)$  leads to a contradiction. Hence the result is true. ■

**6.9 Corollary.** *Given a cardinal number, there exists a larger cardinal number. In other words, there is no largest cardinal number.*

## 7 ADDITION OF CARDINAL NUMBERS

**7.1 Definition.** Let  $\alpha$  and  $\beta$  be two cardinal numbers and let  $A$  and  $B$  be two disjoint sets with cardinal numbers  $\alpha$  and  $\beta$ , respectively. We then define  $\alpha + \beta$  as the cardinal number of  $A \cup B$ .

*Remark.* We hasten to point out that  $\alpha + \beta$  is always defined since it is always possible to find appropriate sets  $A$  and  $B$  which are disjoint. In fact, if  $A \cap B \neq \phi$ , then define

$$A_0 = \{a, 0\} : a \in A\}$$

and

$$B_0 = \{(b, 1) : b \in B\}$$

to get  $A \sim A_0$ ,  $B \sim B_0$  and  $A_0 \cap B_0 = \phi$ .

Since the relations

$$A \cup B = B \cup A \quad \text{and} \quad (A \cup B) \cup C = A \cup (B \cup C)$$

hold good for arbitrary sets, we have

$$\alpha + \beta = \beta + \alpha \quad (\text{commutative property})$$

$$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma) \quad (\text{associative property}).$$

Further, if  $\alpha \leq \beta$  and  $\gamma \leq \delta$ , then  $\alpha + \gamma \leq \beta + \delta$ .

*Remarks.* 1. We cannot always infer from  $\alpha < \beta$  and  $\gamma \leq \delta$  that  $\alpha + \gamma \leq \beta + \delta$ . For instance, given any finite cardinal number  $n$ , then  $n < \aleph_0$  and  $\aleph_0 \leq \aleph_0$ , but  $n + \aleph_0 = \aleph_0 + \aleph_0$  (see Theorem 7.2(a) and (b)).

2. The cancellation law may not hold good in cardinal numbers. Consider for instance the sets  $A$  and  $B$ , respectively, of the even and odd natural numbers. The cardinality of each of these sets is  $\aleph_0$  and the cardinality of  $A \cup B$  which in fact is the set  $\mathbb{N}$  is also  $\aleph_0$ . So

$$\aleph_0 + \aleph_0 = \aleph_0 = 0 + \aleph_0.$$

If the cancellation law is permissible, we must have  $\aleph_0 = 0$  which is not true.

## 7.2 Theorem

(a)  $n + \aleph_0 = \aleph_0, \quad n \in \mathbb{N}$

(b)  $\aleph_0 + \aleph_0 = \aleph_0$

(c)  $\aleph_0 + \aleph_0 + \dots = \aleph_0$

(d)  $\aleph_0 + c = c$

(e)  $c + c = c$

(f)  $c + c + c + \dots = c.$

*Proof.* The proofs of (a), (b) and (c) are obtained by using Theorems 3.8, 3.10 and 3.14, whereas that of (d) is obtained by using Theorems 3.15, 4.8 and Corollary 4.10. However, we leave the details to the reader.

To prove (e), consider the disjoint intervals  $]0, 1]$  and  $]1, 2[$ . Each of these has cardinal number  $c$ . Also their union is  $]0, 2[$ , the cardinal number of which is  $c$ . Hence  $c + c = c$ .

The result in (f) can easily be obtained on the lines of (d) by considering the pairwise disjoint intervals

$$\left[0, \frac{1}{2}\right], \left[\frac{1}{2}, \frac{2}{3}\right], \left[\frac{2}{3}, \frac{3}{4}\right], \dots \blacksquare$$

## 8 MULTIPLICATION OF CARDINAL NUMBERS

**8.1 Definition.** Let  $\alpha$  and  $\beta$  be two cardinal numbers and let  $A$  and  $B$  be any two sets having cardinal numbers  $\alpha$  and  $\beta$ , respectively. Then the product  $\alpha \cdot \beta$  is defined to be the cardinal number of  $A \times B$ . In symbol,  $\alpha \cdot \beta = \overline{A \times B}$ .

*Note.* We can alternatively define  $\alpha \cdot \beta$  by “adding  $\alpha$  to itself  $\beta$  times”; this refers to the formation of the sum  $\sum_{i \in A} \alpha_i$ , where, the index set  $A$  has cardinal number  $\beta$  and  $\alpha_i = \alpha$  for each  $i \in A$ .

**8.2 Theorem.** *The operation of multiplication in the set of cardinal numbers is commutative, associative and distributive over addition, i.e.*

- (a)  $\alpha \cdot \beta = \beta \cdot \alpha$
- (b)  $(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$
- (c)  $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$

where  $\alpha$ ,  $\beta$  and  $\gamma$  are cardinal numbers.

*Proof.* Let  $A$ ,  $B$  and  $C$  be the sets such that  $\bar{A} = \alpha$ ,  $\bar{B} = \beta$  and  $\bar{C} = \gamma$ . We know that

$$A \times B \sim B \times A$$

and

$$A \times (B \times C) \sim (A \times B) \times C,$$

for the arbitrary sets  $A$ ,  $B$  and  $C$ . These, by definition, imply

$$\alpha \cdot \beta = \beta \cdot \alpha$$

and

$$(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma).$$

This proves (a) and (b).

Further, we have

$$\begin{aligned} \alpha \cdot (\beta + \gamma) &= \overline{\overline{A \times (B \cup C)}} \\ &= \overline{(\overline{A \times B}) \cup (\overline{A \times C})} \\ &= \overline{\overline{A \times B} + \overline{A \times C}} \\ &= \alpha \cdot \beta + \alpha \cdot \gamma. \end{aligned}$$

This proves (c) and completes the proof of the theorem. ■

*Remark.* As in the case of addition, the cancellation law does not hold good for the operation of multiplication also.

### 8.3 Theorem

- (a)  $n \cdot \aleph_0 = \aleph_0, \quad \forall n \in \mathbb{N}$
- (b)  $\aleph_0 \aleph_0 = \aleph_0$
- (c)  $\aleph_0 \cdot \aleph_0 \cdot \aleph_0 \dots = \aleph_0$
- (d)  $N_0 \cdot c = c$
- (e)  $c \cdot c = c$
- (f)  $c \cdot c \cdot c \dots = c.$

*Proof.* We prove the result (e) only and leave the others to the reader.

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Let  $A = [0, 1[$  and  $B = A \times A = \{(x, y) : 0 \leq x < 1, 0 \leq y < 1\}$ . In order to establish (e), we have to prove that the cardinal number of  $B$  is the same as that of  $A$  for which we need to prove that  $A \sim B$ . It is easy to see that  $\exists$  a one-one function of  $A$  into  $B$  given by  $f(x) = (x, 0)$ . It remains only to show that  $\exists$  a one-one function of  $B$  into  $A$  since then the result would follow by the Schroeder-Bernstein Theorem.

For every  $(x, y) \in B$ , we can write

$$x = \cdot a_1 a_2 a_3 \dots, \quad y = \cdot b_1 b_2 b_3 \dots$$

where  $a_1, a_2, \dots; b_1, b_2, \dots$  are all digital numbers. Let us assume that none of these fractions has recurring 9's.

Now consider the function  $\varphi : B \rightarrow A$  defined by

$$\varphi(x, y) = \cdot a_1 b_1 a_2 b_2 \dots$$

This is clearly a one-one function of  $B$  into  $A$ . Hence the result is proved. ■

## 9 EXPONENTIATION OF CARDINAL NUMBERS

**9.1 Definition.** Let  $\alpha (\neq 0)$  and  $\beta (\neq 0)$  be any two cardinal numbers. Let  $A$  and  $B$  be two sets having cardinal numbers  $\alpha$  and  $\beta$ , respectively. We define  $\alpha^\beta$  as the cardinal number of the set  $A^B$ , the set of all functions on  $B$  with range in  $A$ .

*Note.* We can proceed from product to exponent in the same way as we proceeded from sum to product. In other words the definition of  $\alpha^\beta$ , for cardinal numbers  $\alpha$  and  $\beta$ , can be given through the concept of repeated multiplication as well. More precisely, we may define, alternatively,  $\alpha^\beta$  by "multiplying  $\alpha$  by itself  $\beta$  times"; this refers to the formation of the product  $\prod_{i \in A} \alpha_i$ , where the index set  $A$  has cardinal number  $\beta$ , and  $\alpha_i = \alpha$  for each  $i \in A$ .

To illustrate the exponentiation of cardinal numbers, we take the following example.

**9.2 Example.** Let  $A = \{a, b, c\}$  and  $B = \{1, 2\}$ . The following functions of  $B$  into  $A$  can be defined

$$\begin{array}{lll} \{(1, a), (2, a)\}, & \{(1, a), (2, b)\}, & \{(1, b), (2, a)\} \\ \{(1, c), (2, c)\}, & \{(1, b), (2, c)\}, & \{(1, c), (2, b)\} \\ \{(1, b), (2, b)\}, & \{(1, c), (2, a)\}, & \{(1, a), (2, c)\}. \end{array}$$

We notice that there are 9 functions of  $B$  into  $A$ . In other words, the set  $A^B$  consists of 9 elements. So  $\overline{A^B} = 9$ . This implies that there are  $3^2 (= 3 \cdot 3)$  functions of  $B$  into  $A$ .

**9.3 Theorem.** For any cardinal numbers  $\alpha$ ,  $\beta$  and  $\gamma$ ,

- (a)  $\alpha^\beta \cdot \alpha^\gamma = \alpha^{\beta+\gamma}$
- (b)  $(\alpha^\beta)^\gamma = \alpha^{\beta\gamma}$
- (c)  $(\alpha\beta)^\gamma = \alpha^\gamma \beta^\gamma$
- (d)  $\alpha^\gamma \leq \beta^\gamma$ , if  $\alpha \leq \beta$
- (e)  $\gamma^\alpha \leq \gamma^\beta$ , if  $\alpha \leq \beta$ .

*Proof.* (a) Let  $\alpha = \overline{A}$ ,  $\beta = \overline{B}$  and  $\gamma = \overline{C}$ , where  $B \cap C = \phi$ . Then

$$\overline{B \cup C} = \beta + \gamma.$$

So, we have

$$\alpha^{\beta+\gamma} = (\overline{A^{B \cup C}}), \quad \alpha^\beta = (\overline{A^B}) \quad \text{and} \quad \alpha^\gamma = (\overline{A^C}).$$

Since  $\alpha^\beta \cdot \alpha^\gamma = \overline{A^B \times A^C}$ , we are left to prove that

$$A^{B \cup C} \sim A^B \times A^C.$$

Suppose  $f \in A^{B \cup C}$  corresponds to the ordered pair of functions  $(f|_B, f|_C)$ , where  $f|_B$  is the restriction of  $f$  to  $B$  and  $f|_C$  is the restriction of  $f$  to  $C$ . Evidently  $(f|_B, f|_C)$  belongs to  $A^B \times A^C$ . Now, we define a mapping  $\psi : A^{B \cup C} \rightarrow A^B \times A^C$  given by

$$\psi(f) = (f|_B, f|_C).$$

Clearly  $\psi$  is one-to-one between  $A^{B \cup C}$  and  $A^B \times A^C$ . Hence

$$A^{B \cup C} \sim A^B \times A^C.$$

(b) Here, we require to show that

$$(A^B)^C \sim A^{B \times C}.$$

To do this, we define a function  $\psi$  on  $(A^B)^C$  as follows:

$$\psi(f) = g \in A^{B \times C},$$

where  $g(y, z) = (f(z)(y)) \in A$  for  $(y, z) \in B \times C$ . Now  $\psi$  is onto, since if  $g \in A^{B \times C}$ , we define  $f$  to be that function on  $C$  whose value at  $z \in C$  is that function on  $B$  which assigns to each  $y \in B$ , the value  $g(y, z) \in A$ . Then  $\psi(f) = g$ . To see that  $\psi$  is one-one, suppose  $f_1 \neq f_2$  in  $(A^B)^C$ . Then there is a  $z_0 \in C$  such that  $f_1(z_0) \neq f_2(z_0)$ . Since these two functions on  $B$  are different, there must be a  $y_0 \in B$  such that

$$f_1(z_0)(y_0) \neq f_2(z_0)(y_0).$$

Thus  $[\psi(f_1)](y_0, z_0) \neq [\psi(f_2)](y_0, z_0)$ , so  $\psi(f_1)$  and  $\psi(f_2)$  are different functions. Hence  $\psi$  is one-one. This proves that  $(A^B)^C \sim A^{B \times C}$ . This gives the result in (b).

(c) Here, we shall prove that  $(A \times B)^C \sim A^C \times B^C$ . Let  $f \in (A \times B)^C$ . Then the values of the functions  $f$  are ordered pairs belonging to  $A \times B$ . Thus we can write

$$f(C) = (g(C), h(C)),$$

where  $g(C) \in A$  and  $h(C) \in B$ . Therefore,  $g \in A^C$  and  $h \in B^C$ . We have thus assigned to every function  $f$ , a pair of functions  $(g, h)$ , i.e. an element of the set  $A^C \times B^C$ . It can be easily verified that this correspondence is one-to-one. Hence

$$(A \times B)^C \sim A^C \times B^C.$$

Thus

$$(\alpha \cdot \beta)^\gamma = \alpha^\gamma \cdot \beta^\gamma.$$

The proofs of the results in (d) and (e) are left as an exercise. ■

**9.4 Theorem.** *Let  $A$  be a set with cardinal number  $\alpha$ . Then the cardinal number of the set  $\mathcal{P}(A)$ , the set of all subsets of  $A$ , is  $2^\alpha$ .*

*Proof.* It is sufficient to prove that

$$\mathcal{P}(A) \sim \{0, 1\}^A,$$

since the set  $\{0, 1\}^A$  has  $2^\alpha$  as its cardinal number. Define a function  $\varphi : \mathcal{P}(A) \rightarrow \{0, 1\}^A$  as follows:

$$\varphi(E) = \zeta_E, \quad \text{for } E \subseteq A$$

where  $\zeta : A \rightarrow \{0, 1\}$  is given by

$$\zeta_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \in A \cap E^c. \end{cases}$$

The function  $\varphi$  is clearly one-to-one. Hence  $\mathcal{P}(A) \sim \{0, 1\}^A$ . ■

**9.5 Corollary.** *Let  $\alpha$  be any cardinal number. Then  $2^\alpha > \alpha$ .*

*Proof.* The result follows in view of Theorems 6.6 and 8.3. ■

**9.6 Corollary.**  $2^{\aleph_0} > \aleph_0$ .

[Hint: Take  $\alpha = \aleph_0$  in Corollary 9.5.]

*Remark.* Given a cardinal number  $\alpha$ , there corresponds a set  $A$  with cardinal number  $\alpha$ . But then the set  $\mathcal{P}(A)$ , the set of all subsets of  $A$

has cardinal number  $2^\alpha (> \alpha)$ . Thus, there is no largest cardinal number. Consequently, we have a chain of cardinal numbers

$$1 < 2 < 3 < \dots < \aleph_0 < 2^{\aleph_0} < 2^{2^{\aleph_0}} < \dots$$

in which there are infinitely many transfinite cardinal numbers.

**9.7 Theorem.**  $2^{\aleph_0} = c$ .

*Proof.* Let  $A = \{0, 1\}$ . Then  $\{0, 1\}^{\mathbb{N}}$  denotes the family of all functions from  $\mathbb{N}$  into the set  $\{0, 1\}$ . The result will be proved, if we show that

$$\{0, 1\}^{\mathbb{N}} \sim [0, 1].$$

Let  $f \in \{0, 1\}^{\mathbb{N}}$  be any element, i.e.

$$f: \mathbb{N} \rightarrow \{0, 1\}.$$

Define a function  $\varphi: \{0, 1\}^{\mathbb{N}} \rightarrow [0, 1]$  by

$$\varphi(f) = \sum_{n=1}^{\infty} \frac{f(n)}{2^n}.$$

Note that  $f(n)$  is either 0 or 1 as such  $\varphi(f) \in [0, 1], \forall f$ . The function  $\varphi$  is also well defined. Further, any element of  $[0, 1]$  has a binary expansion (expansion in the scale of 2) of the form

$$\sum_{n=1}^{\infty} \frac{b_n}{2^n}, \tag{5}$$

where each  $b_n$  is either 0 or 1 and we assume that not all  $b_k$ 's after a certain term are zero (i.e., the series is non-terminating).

Thus, the correspondence  $\varphi$  is one-to-one, since corresponding to each  $f \in \{0, 1\}^{\mathbb{N}}$ , there exists an element of  $[0, 1]$ ; and conversely, given an element of  $[0, 1]$ , it can be expressed in the binary form as in (5) which corresponds to some  $f \in \{0, 1\}^{\mathbb{N}}$ . Hence

$$\{0, 1\}^{\mathbb{N}} \sim [0, 1].$$

This proves that  $2^{\aleph_0} = c$ . ■

**9.8 Corollary.**  $c^{\aleph_0} = c$ .

*Proof:* We have

$$\begin{aligned} c^{\aleph_0} &= (2^{\aleph_0})^{\aleph_0} \\ &= 2^{\aleph_0 \cdot \aleph_0} \\ &= 2^{\aleph_0} \\ &= c. \blacksquare \end{aligned}$$

**9.9 Corollary.**  $c^c = 2^c$ .

*Proof.* We have

$$\begin{aligned} c^c &= (2^{\aleph_0})^c \\ &= 2^{\aleph_0^c} \\ &= 2^c. \blacksquare \end{aligned}$$

*Remark.* The inequality  $c < f$  in Theorem 6.7 can be established in view of the fact that  $f = c^c$  and  $2^c > c$  since

$$c < 2^c = c^c = f.$$

**10 CANTOR-LIKE SETS**

Cantor-like sets and functions defined on them are quite useful, particularly for the construction of Counter-examples. One such set is Cantor's ternary set or simply the Cantor set, which was exhibited by G. Cantor (1845-1918) as an illustration of certain curious things which can occur with point sets on the real line. Some of the properties of this set, as we would see, defy geometric intuition.

The **Cantor set**, denoted by  $C$ , is a subset of the interval  $[0, 1]$  which is left after the removal of a certain specified countable (infinite) collection of open intervals from  $[0, 1]$ . To construct the set  $C$ , we proceed as follows.

Let  $C_0$  denote the interval  $[0, 1]$ . Remove from  $C_0$  in succession:

1. the open interval  $\left] \frac{1}{3}, \frac{2}{3} \right[$ , the middle third of the interval in  $C_0$ , leaving behind the set

$$C_1 = \left[ 0, \frac{1}{3} \right] \cup \left[ \frac{1}{3}, 1 \right];$$

2. the open intervals  $\left] \frac{1}{9}, \frac{2}{9} \right[$  and  $\left] \frac{7}{9}, \frac{8}{9} \right[$ , the middle thirds of the two closed intervals  $\left[ 0, \frac{1}{3} \right]$  and  $\left[ \frac{1}{3}, 1 \right]$  in  $C_1$ , leaving behind the set

$$C_2 = \left[ 0, \frac{1}{9} \right] \cup \left[ \frac{2}{9}, \frac{1}{3} \right] \cup \left[ \frac{2}{3}, \frac{7}{9} \right] \cup \left[ \frac{8}{9}, 1 \right];$$

3. the open intervals  $\left] \frac{1}{27}, \frac{2}{27} \right[$ ,  $\left] \frac{7}{27}, \frac{8}{27} \right[$ ,  $\left] \frac{19}{27}, \frac{20}{27} \right[$  and  $\left] \frac{25}{27}, \frac{26}{27} \right[$ , the middle thirds of the four ( $2^2$ ) closed intervals in  $C_2$ , leaving behind the set

$$C_3 = \left[0, \frac{1}{27}\right] \cup \left[\frac{2}{27}, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{7}{27}\right] \cup \left[\frac{8}{27}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{19}{27}\right] \\ \cup \left[\frac{20}{27}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, \frac{25}{27}\right] \cup \left[\frac{26}{27}, 1\right]; \text{ and}$$

4. continue this process generating a sequence  $\{C_n\}$  of sets, where  $C_{n+1}$  is obtained from  $C_n$  by removing the (open) middle thirds of the  $2^n$  disjoint closed intervals of which  $C_n$  is composed of.

Figure 2.3 depicts  $C_0, C_1$  and  $C_2$ .

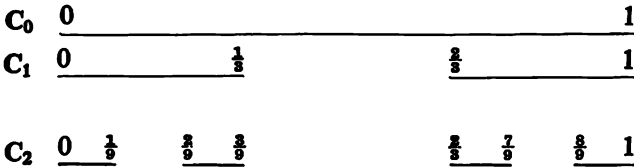


Fig. 2.3

The points of the interval  $[0, 1]$  which are never removed in the process constitute the Cantor set  $C$ , more precisely, the points common to all the sets  $C_n$ , i.e.

$$C = \bigcap_{n=1}^{\infty} C_n.$$

Each of the sets  $C_n$  is nonempty, closed and bounded. Also,  $C_{n+1} \subset C_n$  for all  $n$ . Hence the set  $C$  is nonempty, closed and bounded.

Let  $E_n$  denote the set composed of all the open intervals removed at the  $n^{\text{th}}$  stage; for instance

$$E_1 = \left] \frac{1}{2}, \frac{2}{3} \right[ \\ E_2 = \left] \frac{1}{9}, \frac{2}{9} \right[ \cup \left] \frac{7}{9}, \frac{8}{9} \right[ \\ E_3 = \left] \frac{1}{27}, \frac{2}{27} \right[ \cup \left] \frac{7}{27}, \frac{8}{27} \right[ \cup \left] \frac{19}{27}, \frac{20}{27} \right[ \cup \left] \frac{25}{27}, \frac{26}{27} \right[ \\ \vdots \\ \vdots$$

Thus, it follows that the Cantor set  $C$  can also be expressed as the complement of the union  $\bigcup_{n=1}^{\infty} E_n$  with respect to the set  $[0, 1]$ , i.e.

$$C = [0, 1] - \bigcup_{n=1}^{\infty} E_n.$$

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Note that in forming the Cantor set  $C$ , the length of the open intervals removed at different stages are given by

$$l(E_1) = \frac{1}{3}$$

$$l(E_2) = \frac{2}{3^2}$$

$$l(E_3) = \frac{4}{3^3}$$

and in general,  $l(E_n) = \frac{1}{3} \left(\frac{2}{3}\right)^{n-1}$ . Thus, the sum of the lengths of all open intervals removed upto the  $n^{\text{th}}$  stage is given by

$$\begin{aligned} S_n &= \sum_{i=1}^n l(E_i) \\ &= 1 - \left(\frac{2}{3}\right)^{n-1} \end{aligned}$$

$$\Rightarrow \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left[ 1 - \left(\frac{2}{3}\right)^{n-1} \right] = 1.$$

Hence the sum of lengths of all intervals removed is the length of the original interval  $[0, 1]$ . As such the set remaining on  $[0, 1]$  which, in fact, is the Cantor set may seem so sparse as to be insignificant.

Intuitively, it may appear that the only points left in the Cantor set are the endpoints

$$0, 1, \frac{1}{3}, \frac{2}{3}, \frac{1}{9}, \frac{2}{9}, \frac{7}{9}, \frac{8}{9}, \dots$$

which are denumerable in number. Closer scrutiny reveals that this impression is *wrong*, and that the Cantor set is actually nondenumerable. The set  $C$  has many other remarkable intuitive properties.

We give below a characterization of the Cantor set which is very useful in obtaining many a properties of the set.

**10.1 Theorem.** *Each point  $x$  of the Cantor set  $C$  can be represented uniquely by a series of the form*

$$x = \sum_{n=1}^{\infty} a_n/3^n,$$

where each  $a_n$  is either 0 or 2; and conversely.

*Proof.* Just as with decimal representation, every  $x \in [0, 1]$  has a representation of the form

$$x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}, \quad (6)$$

where  $a_n = 0, 1$  or  $2$ ; and each number thus expressed is in  $[0, 1]$ . Representation in this general case, is not always unique, for instance  $\frac{1}{3}$  has two representations,

$$\frac{1}{3} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{2}{3^n}.$$

In such situations, for the sake of uniqueness, we shall adopt the one which is nonrecurring.

The points of  $E_1$  can be expressed as

$$x = \frac{1}{3} + \sum_{n=2}^{\infty} \frac{a_n}{3^n};$$

the points of  $E_2$  as

$$x = \frac{1}{9} + \left(0 \text{ or } \frac{2}{3}\right) + \sum_{n=3}^{\infty} \frac{a_n}{3^n};$$

the points of  $E_3$  as

$$x = \frac{1}{27} + \left(0, \frac{2}{3}, \frac{2}{9} \text{ or } \frac{2}{3} + \frac{2}{9}\right) + \sum_{n=4}^{\infty} \frac{a_n}{3^n};$$

and in general, the point of  $E_n$  as

$$x = \frac{1}{3^n} + \sum_{i=1}^{n-1} \frac{b_i}{3^i} + \sum_{i=n+1}^{\infty} \frac{a_i}{3^i},$$

where  $b_i = 0$  or  $2$  and  $a_i = 0, 1$  or  $2$ , but  $a_i$ 's are neither all 0's nor all 2's. Thus  $x \in E_n$  if and only if

$$x = \sum_{i=1}^{\infty} \frac{a_i}{3^i},$$

where  $a_i = 0$  or  $2$  for  $i < n$ ;  $a_n = 1$ ; and for  $i > n$ ,  $a_i$  has no restriction except that these  $a_i$ 's are neither all 0's nor all 2's. Hence, each point in  $C$  can be represented by a series of the form (6).

We now show that there cannot be more than one such representation, for if

$$\sum_{n=1}^{\infty} \frac{a_n}{3^n} = \sum_{n=1}^{\infty} \frac{b_n}{3^n},$$

where each  $b_n$  also is either 0 or 2. We shall show that  $a_n = b_n$  for every  $n$ . Let, if possible,  $a_n \neq b_n$  for some  $n$ . Let  $N$  be the smallest natural

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number  $n$  such that  $a_n \neq b_n$ . Then  $|a_N - b_N| = 2$  and  $|a_n - b_n| \leq 2$  for every  $n$ . Now, we have

$$\begin{aligned} 0 &= \left| \sum_{n=N}^{\infty} \frac{a_n - b_n}{3^n} \right| \geq \frac{1}{3^N} \left\{ |a_N - b_N| - \sum_{n=N+1}^{\infty} \frac{|a_n - b_n|}{3^{n-N}} \right\} \\ &\geq \frac{1}{3^N} \left\{ 2 - \sum_{n=1}^{\infty} \frac{2}{3^n} \right\} = \frac{1}{3^N}, \end{aligned}$$

which is absurd. Hence  $a_n = b_n$  for each  $n$ . ■

### 10.2 Theorem. Each point of $C$ is a limit point of $C$ .

*Proof.* Let  $x_0$  be any point of  $C$ . We may write  $x_0$  in its ternary expression as

$$x = {}_3a_1a_2a_3 \dots a_n \dots$$

where  $a_n = 0$  or  $2$ . The symbol  $\hat{3}$  emphasizes that it is the ternary (not the decimal) representation. We now construct a sequence  $\{x_n\}$  of points as follows:

$$x_1 = {}_3a'_1a_2a_3 \dots a_n \dots,$$

$$x_2 = {}_3a_2a'_2a_3 \dots a_n \dots,$$

$$x_3 = {}_3a_1a_2a'_3 \dots a_n \dots,$$

.

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$$x_n = {}_3a_1a_2a_3 \dots a'_n \dots,$$

.

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.

where  $a'_n = 0$  if  $a_n = 2$  and  $a'_n = 2$ , if  $a_n = 0$ . Clearly,  $\{x_n\}$  is a sequence of points in  $C$  such that  $x_n \neq x_0$  for any  $n$ . Also, note that  $\lim_{n \rightarrow \infty} x_n = x_0$ .

Hence  $x_0$  is a limit point of  $C$ . ■

As a consequence of Theorem 10.2 we have the following.

### 10.3 Corollary. The Cantor set is perfect. (A set is perfect if it is both closed and dense in itself.)

### 10.4 Theorem. The Cantor set is uncountable.

*Proof.* Let, if possible, the Cantor set  $C$  be countable. Then, we may write  $C$  as

$$C = \{x_1, x_2, x_3, \dots, x_n, \dots\}.$$

Write the elements in  $C$  in their ternary expansion as

$$x_1 = {}_3a_{11}a_{12}a_{13} \dots a_{1n} \dots$$

$$x_2 = {}_3a_{21}a_{22}a_{23} \dots a_{2n} \dots$$

⋮

⋮

⋮

$$x_n = {}_3a_{n1}a_{n2}a_{n3} \dots a_{nn} \dots$$

⋮

⋮

⋮

where  $a_{ij} = 0$  or  $2$ . Consider a sequence  $\{a_n\}$ , where

$$a_n = \begin{cases} 0 & \text{if } a_{nn} = 2 \\ 2 & \text{if } a_{nn} = 0. \end{cases}$$

Clearly the element

$$x = {}_3a_1a_2a_3 \dots a_n \dots$$

is in  $C$ . But  $x \neq x_n$  since it differs from  $x_n$  at least in the  $n$ th place. This is true for each  $n$  and as such  $x$  should not be in  $C$ . Hence the result is proved by contradiction.

*Alternative proof.* Let  $f$  be a function defined from  $C$  into  $[0, 1]$  by

$$f(x) = {}_2a_1a_2a_3 \dots$$

where  $x = {}_3b_1b_2b_3 \dots$  is in  $C$  and  $2a_i = b_i$ . It is easy to verify that the function  $f$  defined above is onto  $[0, 1]$ . The set  $C$  is uncountable in view of Problem 28 and Theorem 4.3. ■

**10.5 Corollary.** *The cardinal number of the Cantor set is  $c$ .*

**Problem 29.** Prove that:

- $C$  contains no interior point.
- $C$  does not contain any open interval.
- Between any two points of  $C$ ,  $\exists$  an open interval which belongs to the complement of  $C$ .
- $C$  is nowhere dense.

*Note.* It should be emphasized that the Cantor set and its properties are not given to destroy geometric intuition or to instill fear of using geometric intuition, but rather to guide the use of geometric intuition in analysis to a supporting role in discovering what to suspect and how then to prove or disprove the suspected. Hence, the construction of the Cantor set should be placed, and kept, clearly in mind.

The idea involved in the characterization of the elements in  $C$  as a ternary expansion may be generalized to generate several such sets, viz. the Cantor  $n$ -ary set. Let us describe it as follows:

Let  $n$  be a natural number, and let  $k$  be a natural number such that  $0 < k < n-1$ . The Cantor  $n$ -ary set is composed of by the points  $x$  in  $[0, 1]$  which has an expansion, called  $n$ -ary expansion, of the form

$$x = \sum_{i=1}^{\infty} \frac{a_i}{n^i},$$

where  $a_i$  are the natural numbers  $0, 1, 2, \dots, k-1, k+1, \dots, (n-1)$ . One can easily verify that the Cantor  $n$ -ary set has several of those properties satisfied by the Cantor set  $C$ . (Verify!)

We further modify the construction of the Cantor set and obtain a set, called the generalized Cantor set. Let  $0 < \alpha < 1$  be given. Let  $\{a_n\}$  be a sequence of positive real numbers such that  $\sum_{n=1}^{\infty} a_n = \alpha$ .

Remove from the middle of the interval  $[0, 1]$ , the open interval of length  $a_1$  leaving behind two closed intervals. From the middle of each of these two intervals, remove an open interval of length  $a_2/2$  leaving behind  $2^2$  closed intervals. Repeating this process, at the  $n^{\text{th}}$  stage, from the middle of each of the  $2^{n-1}$  intervals, remove an open interval of length  $a_n/n$  leaving behind  $2^n$  closed intervals; and so on. Then the set left after the process is repeated infinitely many times is the desired set, called the generalized Cantor set and is denoted by  $C(\alpha)$ .

**Problem 30.** Prove that the set  $C(\alpha)$ ,  $0 < \alpha < 1$ , is a nonempty, closed and uncountable set.

**10.6 Definition.** Let  $C$  be the Cantor set. Then a function  $f: C \rightarrow \mathbb{R}$  defined by

$$f(x) = \sum_{n=1}^{\infty} \frac{a_n}{2^{n+1}},$$

where  $x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$  is called the Cantor function.

**Problem 31.** Prove that the Cantor function is monotone increasing and continuous.

## 11 CONTINUUM HYPOTHESIS

Motivated by the fact that for any finite cardinal number  $n > 1$  there is always a cardinal number between  $n$  and  $2^n$ ; and otherwise also, it is a natural question whether there is any cardinal number  $\mu$  such

that  $\aleph_0 < \mu < 2^{\aleph_0} (= c)$ . This problem is yet unsolved and is called **Cantor's continuum problem**. However, many important results in mathematics have been proved by assuming that there is no cardinal number between  $\aleph_0$  and  $c$ . This assumption is called the **continuum hypothesis**.

According to this hypothesis,  $c$  is the second transfinite cardinal number.

**Generalised Continuum Hypothesis.** The assumption that, no matter what the infinite set  $A$  is, there is no set with cardinal number  $\mu$  such that

$$\overline{\overline{A}} < \mu < \overline{\overline{\mathcal{P}(A)}},$$

is called the *generalised continuum hypothesis* (g.c.h). K. Gödel has proved that the g.c.h. is consistent with generally accepted axioms of the theory of sets, provided they are themselves consistent.

It can easily be seen that the continuum hypothesis is a particular case of the g.c.h., namely the case when  $A$  is the set of all natural numbers.

### Problems

32. If  $\text{card}(A) = \text{card}(C)$  and  $\text{card}(B) = \text{card}(D)$ , show that  $\text{card}(A \times B) = \text{card}(C \times D)$ .
33. Show that  $\mathbb{R}^2$  has cardinal number  $c$ .
34. Show that the set of all sequences of real numbers has cardinal number  $c$ .
35. Show that the set of all continuous real valued functions defined on the closed interval  $[0, 1]$  has power  $c$ .
36. Prove that  $f = 2^c$ .  
[Hint: There are at least as many such functions as there are "characteristic functions" defined on the closed unit interval.]
37. Prove Theorem 7.2(f).
38. Prove Theorem 8.3(d) and (e).
39. Show that (i)  $\aleph_0^{\aleph_0} = c$  and (ii)  $\aleph_0^c = 2^c$ .
40. Prove that if  $\alpha$  and  $\beta$  are any two cardinal numbers such that  $\alpha < \beta$ , then  $\exists$  a cardinal number  $\gamma$  ( $\gamma > 0$ ) such that  $\beta = \alpha + \gamma$ . Does the converse hold true? Justify your answer in either case.
41. Let  $\alpha$  be a transfinite cardinal number. Prove that the cardinal number  $2^\alpha$  is simultaneously even and odd, i.e.,  $\exists$  cardinal numbers  $\gamma$  and  $\delta$  such that  $2^\alpha = 2\gamma = 2\delta + 1$ .

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42. Let  $\alpha$  be a cardinal number. Prove that

$$\alpha \geq \aleph_0 \Rightarrow \alpha + \aleph_0 = \alpha$$

and

$$\alpha \geq c \Rightarrow \alpha + c = \alpha.$$

[Hint:  $\alpha \geq \aleph_0 \Rightarrow \exists \beta$  such that  $\alpha = \aleph_0 + \beta$  in view of Problem 40. Then  $\alpha + \aleph_0 = (\aleph_0 + \beta) + \aleph_0 = \beta + \aleph_0 = \alpha$ .]

43. Prove that for any transfinite cardinal number  $\alpha$ ,  $\alpha + \alpha = \alpha \cdot \alpha = \alpha$ .  
 44. Let  $\alpha$  and  $\beta$  be two transfinite cardinal numbers. Show that

$$\alpha + \beta = \alpha \cdot \beta = \max \{ \alpha, \beta \}.$$

45. Show that for finite cardinal numbers  $m, m_1, n$  and  $n_1$

$$m \leq n, m_1 < n_1 \Rightarrow mm_1 < nn_1.$$

Show, by means of examples, that the result is not true in the case of transfinite cardinal numbers.

46. Prove that a cardinal number  $m$  is finite if and only if

$$2^{2^m} + 1 \neq 2^{2^m}$$

47. Prove that a set of cardinality  $c$  has  $2^c$  different subsets, each of cardinality  $c$ .

48. Prove that  $1 + 2^{2^{\aleph_0}} = 2^{2^{\aleph_0}}$ .

49. Prove that (i)  $1^{\aleph_0} = 1 = 1^c$ .

(ii)  $(n+1)^{\aleph_0} = c, n \geq 1$ .

50. Show that the subset  $X$  of  $\mathbb{R}^3$  defined by

$$X = \{ (x_1, x_2, x_3) : 0 \leq x_i \leq 1 \text{ for } i = 1, 2, 3 \}$$

has cardinal number  $c$ .

51. If  $A = \bigcup A_n$  and  $\overline{A} = c$ , then prove that at least one of the sets  $A_n$  has cardinality  $c$ .

### III

## Measurable Sets

The length of an interval  $I$ , written  $l(I)$ , is defined to be the difference of the endpoints of the interval  $I$ . Thus, irrespective of whether an interval  $I$  with  $a$  and  $b$  as its endpoints is closed, open, open-closed or closed-open, the length  $l(I)$  is  $b - a$ , where  $a < b$ . In case  $a = b$ , the interval  $[a, b]$  degenerates to a point and has length zero while an infinite interval has length infinity. Thus, length is an example of a set function, i.e., a function which associates an extended real number to each set in some collection of sets. In the case of length, the domain is the collection of all intervals. The set function  $l$  clearly satisfies the following:

1.  $l(I) \geq 0$ , for all intervals  $I$ .
2. If  $\{I_i\}$  is a countable collection of mutually disjoint intervals, then

$$l\left(\bigcup_{i=1}^{\infty} I_i\right) = \sum_{i=1}^{\infty} l(I_i).$$

3. If  $x$  is any fixed real number, then

$$l(I) = l(I + x).$$

In the above, we have said that in the case of length, the domain is the collection of all intervals. We would now like to extend the notion of length to more complicated and arbitrary sets than intervals.

#### 1 LENGTH OF SETS

Let  $O$  be an open set in  $\mathbb{R}$ . Then  $O$  can be written as a countable union of mutually disjoint open intervals  $\{I_i\}$ , unique except so far as order is concerned; i.e.,

$$O = \bigcup_{i=1}^{\infty} I_i.$$

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The length of the open set  $O$  is defined by

$$l(O) = \sum_{i=1}^{\infty} l(I_i).$$

The length  $l(O)$  is well defined since the sum on the right does not depend on the order of the terms used in the summing process. Thus, the length of an open set is the sum of the lengths of the intervals (of course open and mutually disjoint) comprising  $O$ .

It is easy to verify that if  $O_1$  and  $O_2$  are two open sets in  $\mathbb{R}$  such that  $O_1 \subset O_2$ , then

$$l(O_1) \leq l(O_2).$$

Hence, for any open set  $O$  contained in  $[a, b]$ , we have

$$0 \leq l(O) \leq b - a.$$

Further, let  $F$  be a closed set contained in some interval  $[a, b]$ . Then the length of the closed set  $F$  is defined by

$$l(F) = b - a - l(F^c),$$

where  $F^c = [a, b] - F$ . It can easily be seen that  $l(F) \geq 0$ .

So far, we have extended the concept of length to open and closed sets. And since the classes of these sets are too restricted, we would like to extend the concept of length to a wider class of sets in  $\mathbb{R}$ , if possible, to the class of all sets in  $\mathbb{R}$ . In this regard, we imagine a function  $m$  which assigns to each set  $E$  in  $\mathbb{R}$ , a nonnegative extended real number, written  $m(E)$ , called the *measure* of  $E$  (an extension of the notion of length function), satisfying the following properties:

1.  $m(E)$  is defined for all sets  $E \in \mathcal{P}(\mathbb{R})$ .
2.  $m(I) = l(I)$ , for an interval  $I$ .
3. If  $\{E_i\}$  is a sequence of disjoint sets, then

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m(E_i).$$

(This property is known as **countable additivity**.)

4.  $m(E + y) = m(E)$ , where  $y$  is any fixed number. (This property is known as **translation invariance**.)

Unfortunately, it is impossible to construct a set function which satisfies all the above four properties (1) to (4). In fact, if the continuum hypothesis "any uncountable set of all real numbers is equivalent to the set of all real numbers" is assumed, one cannot construct such a measure satisfying the properties (1) to (4). As a result one of these

four properties must be sacrificed or weakened at least. Following Henri Lebesgue (1875-1941), who made many contributions to measure theory and integration, it is most useful to retain the last three properties, i.e., (2) to (4), and to weaken the condition given in (1) so that  $m(E)$  need not be defined for all sets  $E$  in  $\mathbf{R}$ . Still, of course, we shall be interested in defining  $m(E)$  for as many sets as possible.

Weakening property (1) is not the only approach; it is also possible to replace property (3) of countable additivity by the weaker property of finite additivity: for each finite sequence  $\{E_i\}$  of disjoint sets, we have  $m(\cup E_i) = \sum m(E_i)$ . Another possible alternative to property (3) is countable subadditivity which is satisfied by the outer measure. Thus it is convenient to introduce first a set function, the outer measure, defined for all sets in  $\mathbf{R}$  and is related to the measure of the set (when it exists).

## 2 OUTER MEASURE

All the sets considered in this chapter are contained in  $\mathbf{R}$ , unless stated otherwise. We shall be concerned particularly with intervals  $I$  of the form  $]a, b[$  unless otherwise specified.

Let us consider the family  $\mathcal{F}$  of all countable collections of open intervals. For any arbitrary  $\mathcal{J} \in \mathcal{F}$ , the sum  $\sum_{I \in \mathcal{J}} l(I)$  is a non-negative

extended real number. Further, this sum depends only on  $\mathcal{J}$  and not on the order used in the summing process.

Now, let  $E$  be an arbitrary set. Consider the subfamily  $\mathcal{C}$  of  $\mathcal{F}$  consisting of countable collections  $\mathcal{J}$  of open intervals  $\{I_i\}$  such that  $E \subset \bigcup_i I_i$ ; i.e.

$$\mathcal{C} = \{\mathcal{J} : \mathcal{J} \in \mathcal{F} \text{ and } \mathcal{J} \text{ covers } E\}.$$

The subfamily  $\mathcal{C}$  is obviously nonempty. Thus we obtain a well defined number  $m^*(E)$  in the set of all nonnegative extended real numbers given by

$$m^*(E) = \inf \left\{ \sum_{I \in \mathcal{J}} l(I) : \mathcal{J} \in \mathcal{C} \right\}.$$

For a more precise understanding, see Definition 2.1.

**2.1 Definition.** *The Lebesgue outer measure or briefly the outer measure  $m^*(E)$  of an arbitrary set  $E$  is given by*

$$m^*(E) = \inf \sum_i l(I_i),$$

where the infimum is taken over all countable collections  $\{I_i\}$  of open intervals such that  $E \subset \bigcup_i I_i$ .

*Remark.* The outer measure  $m^*$  is a set function which is defined from the power set  $\mathcal{P}(\mathbb{R})$  into the set of all non-negative extended real numbers.

## 2.2 Theorem

- (a)  $m^*(A) \geq 0$ , for all sets  $A$ .
- (b)  $m^*(\phi) = 0$ .
- (c) If  $A$  and  $B$  are two sets with  $A \subset B$ , then  $m^*(A) \leq m^*(B)$ .  
(This property is known as **monotonicity**.)
- (d)  $m^*(A) = 0$ , for every singleton set  $A$ .
- (e) The function  $m^*$  is translation invariant, i.e.,  $m^*(A+x) = m^*(A)$ , for every set  $A$  and for every  $x \in \mathbb{R}$ .

*Proof.* The proofs of (a) and (b) are obvious.

(c) Let  $\{I_n\}$  be a countable collection of disjoint open intervals such that  $B \subset \bigcup_n I_n$ . Then  $A \subset \bigcup_n I_n$  and therefore

$$m^*(A) \leq \sum_{n=1}^{\infty} l(I_n).$$

This inequality is true for any coverings  $\{I_n\}$  of  $B$ . Hence the result follows.

(d) Let  $A = \{x\}$  be an arbitrary singleton set. Since

$$I_n = \left] x - \frac{1}{n}, x + \frac{1}{n} \right[$$

is an open covering of  $A$  for each  $n \in \mathbb{N}$ , and  $l(I_n) = \frac{2}{n}$ , the result follows in view of (a).

(e) Given any interval  $I$  with end points  $a$  and  $b$ , the set  $I+x$  defined by

$$I+x = \{y+x : y \in I\}$$

is clearly an interval with endpoints  $a+x$  and  $b+x$ . Moreover,

$$l(I+x) = l(I).$$

Now, let  $\epsilon > 0$  be given. Then there is a countable collection  $\{I_n\}$  of open intervals such that  $A \subset \bigcup_n I_n$  and satisfies

$$\sum_{n=1}^{\infty} l(I_n) < m^*(A) + \epsilon.$$

Clearly  $A+x \subset \cup(I_n+x)$ . Therefore

$$m^*(A+x) \leq \sum_{n=1}^{\infty} l(I_n+x) = \sum_{n=1}^{\infty} l(I_n) < m^*(A) + \epsilon.$$

Since  $\epsilon > 0$  is arbitrary, we have  $m^*(A+x) \leq m^*(A)$ . The reverse inequality follows by considering  $A = (A+x) - x$  and using the above.

To answer affirmatively the question whether  $m^*$  is a generalization of the length function defined for the intervals, we prove the following theorem.

**2.3 Theorem.** *The outer measure of an interval is its length.*

*Proof. Case 1:* Suppose  $I$  is a closed finite interval, (say)  $[a, b]$ . Since, for each  $\epsilon > 0$ , the open interval  $\left] a - \frac{\epsilon}{2}, b + \frac{\epsilon}{2} \right[$  contains  $[a, b]$ , we have

$$m^*(I) \leq l\left(\left] a - \frac{\epsilon}{2}, b + \frac{\epsilon}{2} \right[\right) = b - a + \epsilon.$$

This being true for each  $\epsilon > 0$ , we must have

$$m^*(I) \leq b - a = l(I).$$

To complete the proof of the result, we need to show that

$$m^*(I) \geq b - a. \quad (1)$$

Let  $\epsilon > 0$  be given. Then there exists a countable collection  $\{I_n\}$  of open intervals covering  $[a, b]$  such that

$$m^*(I) > \sum_n l(I_n) - \epsilon. \quad (2)$$

By the Heine-Borel Theorem (cf. I-1.1), any collection of open intervals covering  $[a, b]$  contains a finite subcollection which also covers  $[a, b]$ , and since the sum of the lengths of the finite subcollection is not greater than the sum of the lengths of the original collection, it suffices to establish the inequality (2) for finite collections  $\{I_n\}$  which cover  $[a, b]$ .

Since  $a \in [a, b]$  implies that  $a \in \cup I_n$ , there must be one of the intervals  $I_n$  which contains  $a$ . Let it be  $]a_1, b_1[$ . Then  $a_1 < a < b_1$ . If  $b_1 \leq b$ , then  $b_1 \in [a, b]$ , and since  $b_1 \notin ]a_1, b_1[$ , there must be an interval  $]a_2, b_2[$  in the finite collection  $\{I_n\}$  such that  $b_1 \in ]a_2, b_2[$ ; that is  $a_2 < b_1 < b_2$ . Continuing in this manner, we get intervals  $]a_1, b_1[$ ,  $]a_2, b_2[$ ,  $\dots$  from the collection  $\{I_n\}$  such that

$$a_i < b_{i-1} < b_i, \quad i = 1, 2, \dots$$

where  $b_0 = a$ . Since  $\{I_n\}$  is a finite collection, this process must terminate with some interval  $]a_k, b_k[$  in the collection which is possible only when  $b \in ]a_k, b_k[$ . Thus

$$\begin{aligned} \sum_n l(I_n) &\geq \sum_{i=1}^k l(a_i, b_i) \\ &= (b_k - a_k) + (b_{k-1} - a_{k-1}) + \dots + (b_1 - a_1) \\ &= b_k - (a_k - b_{k-1}) - \dots - (a_2 - b_1) - a_1 \\ &> b_k - a_1 \\ &> b - a, \end{aligned}$$

since  $a_i - b_{i-1} < 0$ ,  $b_k > b$  and  $a_1 < a$ . This, in view of (2), verifies that

$$m^*(I) > b - a - \epsilon.$$

Hence  $m^*(I) \geq b - a$ .

*Case 2:* Suppose  $I$  is any finite interval. Then given an  $\epsilon > 0$ , there exists a closed finite interval  $J \subset I$  such that

$$l(J) > l(I) - \epsilon.$$

Therefore,

$$\begin{aligned} l(I) - \epsilon &< l(J) = m^*(J) \leq m^*(I) \leq m^*(I) = l(I) = l(I) \\ \Rightarrow \quad & l(I) - \epsilon < m^*(I) \leq l(I). \end{aligned}$$

This is true for each  $\epsilon > 0$ . Hence  $m^*(I) = l(I)$ .

*Case 3:* Suppose  $I$  is an infinite interval. Then given any real number  $K > 0$ , there exists a closed finite interval  $J \subset I$  such that  $l(J) = K$ . Thus  $m^*(I) \geq m^*(J) = l(J) = K$ , that is  $m^*(I) \geq K$  for any arbitrary real number  $K > 0$ . Hence  $m^*(I) = \infty = l(I)$ . ■

The next theorem asserts that  $m^*$  has the property of **countable sub-additivity**.

**2.4 Theorem.** *Let  $\{E_n\}$  be a countable collection of sets. Then*

$$m^*\left(\bigcup_n E_n\right) \leq \sum_n m^*(E_n).$$

*Proof.* If  $m^*(E_n) = \infty$  for some  $n \in \mathbb{N}$ , the inequality holds trivially. Let us assume that  $m^*(E_n) < \infty$ , for each  $n \in \mathbb{N}$ . Then, for each  $n$ , and for a given  $\epsilon > 0$ ,  $\exists$  a countable collection  $\{I_{n,i}\}_i$  of open intervals such that  $E_n \subset \bigcup_i I_{n,i}$  satisfying

$$\sum_i l(I_{n,i}) < m^*(E_n) + 2^{-n}\epsilon.$$

Then

$$\bigcup_n E_n \subset \bigcup_n \bigcup_i I_{n,i}.$$

However, the collection  $\{I_{n,i}\}_{n,i}$  forms a countable collection of open intervals, as the countable union of countable sets is countable (c.f. II-3.14), and covers  $\bigcup_n E_n$ . Therefore

$$\begin{aligned} m^*(\bigcup_n E_n) &\leq \sum_{n,i} l(I_{n,i}) \\ &< \sum_n (m^*(E_n) + 2^{-n}\epsilon) \\ &= \sum_n m^*(E_n) + \epsilon. \end{aligned}$$

But  $\epsilon > 0$  being arbitrary, the result follows. ■

**2.5 Corollary.** *If  $E$  is a countable set, then  $m^*(E) = 0$ .*

*Proof.* Since the set  $E$  is countable, we may express it as

$$E = \{a_1, a_2, \dots, a_n, \dots\}.$$

Let  $\epsilon > 0$  be given. Then enclosing each  $a_i$  in an open interval  $I_i$  with  $l(I_i) = 2^{-i}\epsilon$ , ( $i = 1, 2, \dots$ ) we get

$$m^*(E) \leq \sum_{i=1}^{\infty} l(I_i) = \sum_{i=1}^{\infty} 2^{-i}\epsilon = \epsilon.$$

Letting  $\epsilon \rightarrow 0$ , the result follows. ■

*Note.* Each of the sets  $\mathbf{N}$ ,  $\mathbf{I}$ ,  $\mathbf{Q}$  and  $A$  (the set of all algebraic numbers) has outer measure zero since each one of these is countable.

**2.6 Corollary.** *The set  $[0, 1]$  is uncountable.*

*Proof.* Assume on the contrary, that the set  $[0, 1]$  is countable. Then, in view of Corollary 2.5,  $m^*([0, 1]) = 0$  and so  $l([0, 1]) = 0$ , by Theorem 2.3. This is absurd. Hence the set  $[0, 1]$  is uncountable. ■

*Remark.* The result in Corollary 2.6 is true for any interval which does not degenerate.

As a negation of Corollary 2.5, we have the following.

**2.7 Corollary.** *Any set with the outer measure different from zero is uncountable.*

The converse of Corollary 2.5, that is “a set with outer measure zero is countable”, is not always true.

**2.8 Example.** The Cantor set  $C$  is uncountable with outer measure zero.

Let  $C_n$  denote the union of the intervals left at the  $n$ th stage while constructing the Cantor set  $C$  (cf. II-10). One may note that  $C_n$  consists of  $2^n$  closed intervals, each of length  $3^{-n}$ . Therefore

$$m^*(C_n) \leq 2^n \cdot 3^{-n},$$

by Theorems 2.3 and 2.4. But any point of  $C$  must be in one of the intervals comprising the union  $C_n$ , for each  $n \in \mathbb{N}$ , and as such  $C \subset C_n$ , for all  $n \in \mathbb{N}$ . Hence

$$m^*(C) \leq \left(\frac{2}{3}\right)^n.$$

This being true for each  $n \in \mathbb{N}$ , letting  $n \rightarrow \infty$  gives  $m^*(C) = 0$ .

**Problem 1.** If  $m^*(A) = 0$ , then  $m^*(A \cup B) = m^*(B)$ . In particular, if  $B \subset A$ , then  $m^*(B) = 0$ .

*Solution.* By Theorem 2.4, we have

$$m^*(A \cup B) \leq m^*(A) + m^*(B) = m^*(B).$$

But  $B \subset A \cup B$  verifies (cf. Theorem 2.2(c))

$$m^*(B) \leq m^*(A \cup B).$$

Hence the result follows. ■

**2.9  $\mathcal{F}_\sigma$  and  $\mathcal{G}_\delta$  Sets.** Although the intersection of any collection (countable or uncountable) of sets in  $\mathbb{R}$  is closed and the union of a finite collection of closed sets is closed, the union of a countable (infinite) collection of closed sets need not be closed. Similarly, the intersection of a countable (infinite) collection of open sets need not be open (cf. I-5). Thus we are motivated to define two new classes of sets,  $\mathcal{F}_\sigma$ -sets and  $\mathcal{G}_\delta$ -sets, different from those of open sets and closed sets, and consequently many others.

**2.10 Definition.** A set which is a countable (finite or infinite) union of closed sets is called an  $\mathcal{F}_\sigma$ -set.

*Note.* The class of all  $\mathcal{F}_\sigma$ -sets is denoted by  $\mathcal{F}_\sigma$ . In  $\mathcal{F}_\sigma$ ,  $\mathcal{F}$  stands for *ferme* (closed) and  $\sigma$  for *summe* (sum).

**2.11 Example.** Each of the following is an  $\mathcal{F}_\sigma$ -set:

- (i) A closed set.
- (ii) A countable set.

- (iii) A countable union of  $\mathcal{F}_\sigma$ -sets.
- (iv) An open interval  $]a, b[$  since

$$]a, b[ = \bigcup_{n=1}^{\infty} \left[ a + \frac{1}{n}, b - \frac{1}{n} \right],$$

and hence an open set.

**2.12 Definition.** A set which is a countable intersection of open sets is a  $\mathcal{G}_\delta$ -set.

*Note.* The class of all  $\mathcal{G}_\delta$ -sets is denoted by  $\mathcal{G}_\delta$ . In  $\mathcal{G}_\delta$ ,  $\mathcal{G}$  stands for *gebiet* (region) and  $\delta$  for *durchschnitt* (intersection).

*Remark.* The complement of an  $\mathcal{F}_\sigma$ -set is a  $\mathcal{G}_\delta$ -set, and conversely.

**2.13 Example.** Each of the following is a  $\mathcal{G}_\delta$ -set:

- (i) An open set, and, in particular, an open interval.
- (ii) A closed interval  $[a, b]$  since

$$[a, b] = \bigcap_{n=1}^{\infty} \left[ a - \frac{1}{n}, b + \frac{1}{n} \right].$$

- (iii) A closed set.
- (iv) A countable intersection of  $\mathcal{G}_\delta$ -sets.

*Remark.* Each of the classes  $\mathcal{F}_\sigma$  and  $\mathcal{G}_\delta$  of sets is wider than the classes of open and closed sets.

*Note.* We can also consider a class  $\mathcal{F}_{\sigma\delta}$  of  $\mathcal{F}_{\sigma\delta}$ -sets: An  $\mathcal{F}_{\sigma\delta}$ -set is one which is obtained by taking the intersection of a countable collection of sets each of which is an  $\mathcal{F}_\sigma$ -set. Similarly, we can construct the classes  $\mathcal{G}_{\delta\sigma}$ ,  $\mathcal{F}_{\sigma\delta\sigma}$ ,  $\mathcal{G}_{\delta\sigma\delta}$ , etc.

We now turn to a result which gives useful relations between the outer measure of an arbitrary set and the outer measure of open sets and  $\mathcal{G}_\delta$ -sets which contain the set.

**2.14 Theorem.** *Let  $E$  be any set. Then:*

- (a) Given  $\epsilon > 0$ ,  $\exists$  an open set  $O \supset E$  such that

$$m^*(O) < m^*(E) + \epsilon;$$

and hence  $m^*(E) = \inf m^*(O)$ , where the infimum is taken over all open sets  $O \supset E$ .

- (b)  $\exists$  a  $\mathcal{G}_\delta$ -set  $G \supset E$  such that

$$m^*(E) = m^*(G).$$

*Proof.* (a) There exists a countable collection  $\{I_n\}$  of open intervals such that  $E \subset \bigcup_n I_n$  and

$$\sum_{n=1}^{\infty} l(I_n) < m^*(E) + \epsilon.$$

Set  $O = \bigcup_{n=1}^{\infty} I_n$ . Clearly  $O$  is an open set and

$$\begin{aligned} m^*(O) &= m^*\left(\bigcup_{n=1}^{\infty} I_n\right) \\ &\leq \sum_{n=1}^{\infty} m^*(I_n) = \sum_{n=1}^{\infty} l(I_n) < m^*(E) + \epsilon. \end{aligned}$$

(b) Choose  $\epsilon = \frac{1}{n}$ ,  $n \in \mathbb{N}$  in (a). Then, for each  $n \in \mathbb{N}$ ,  $\exists$  an open set  $O_n \supset E$  such that

$$m^*(O_n) < m^*(E) + \frac{1}{n}.$$

Define  $G = \bigcap_{n=1}^{\infty} O_n$ . Clearly,  $G$  is a  $\mathcal{G}_\delta$ -set and  $G \supset E$ . Moreover, we observe that

$$m^*(E) \leq m^*(G) \leq m^*(O_n) < m^*(E) + \frac{1}{n}, \quad n \in \mathbb{N}.$$

Hence letting  $n \rightarrow \infty$  verifies  $m^*(G) = m^*(E)$ . ■

*Remark.* The significance of Theorem 2.14(b) is that any arbitrary set  $E$  can be included in a set which is of relatively simple type, namely a  $\mathcal{G}_\delta$ -set, with the same outer measure.

In defining the notion of outer measure of a set  $E$ , we have used the collections  $\{I_n\}$  of open intervals that cover  $E$ . Now the question arises whether or not the restriction of the intervals in the collection being open can be dropped. The answer is yes. In fact, we prove the following.

**2.15 Theorem.** *The function  $m^* : \mathcal{P}(\mathbb{R}) \rightarrow \mathbb{R} \cup \{0\}$  (outer measure) is obtained whether, for all  $n$ , we stipulate (i)  $I_n$  open, (ii)  $I_n$  closed-open, (iii)  $I_n$  open-closed, (iv)  $I_n$  closed or (v)  $I_n$  mixtures of various types of intervals.*

*Proof.* In case (i), we obtain  $m^*$  by Definition 2.1. Write the corresponding  $m^*$  as  $m_{co}^*$  in case (ii),  $m_{oc}^*$  in case (iii),  $m_c^*$  in case (iv) and  $m_m^*$

in case (v). We shall show that each of these equals  $m^*$ . It is sufficient to prove that  $m_m^* = m^*$ , the other cases being similar.

Let  $E$  be any set. By definition,

$$m_m^*(E) \leq m^*(E).$$

To prove the reverse inequality, let  $\epsilon > 0$  be given. Then, by the definition of  $m_m^*$ ,  $\exists$  a sequence  $\{J_n\}$  of intervals (of course  $J_n$  are mixtures of various types of intervals) such that  $E \subset \bigcup_n J_n$  and

$$\sum_{n=1}^{\infty} l(J_n) < m_m^*(E) + \epsilon.$$

But for a given  $\epsilon > 0$  and any  $J_n$ , there is an open interval  $I_n$  such that  $J_n \subset I_n$  and

$$l(I_n) = (1 + \epsilon)l(J_n)$$

$$\Rightarrow l(J_n) = (1 + \epsilon)^{-1}l(I_n).$$

Thus,  $\{I_n\}$  is a sequence of open intervals with  $E \subset \bigcup_n I_n$  and satisfies

$$(1 + \epsilon)^{-1} \sum_{n=1}^{\infty} l(I_n) < m_m^*(E) + \epsilon$$

$$\Rightarrow \sum_{n=1}^{\infty} l(I_n) < (1 + \epsilon)m_m^*(E) + (1 + \epsilon)\epsilon.$$

This verifies that  $m^*(E) \leq m_m^*(E)$ . ■

**Problems**

2. Let  $A$  be the set of rational numbers between 0 and 1, and  $\{I_n\}$  be a finite collection of open intervals covering  $A$ . Prove that  $\sum l(I_n) \geq 1$ .
3. Prove that the outer measure of the generalized Cantor set  $C(\alpha)$ ,  $0 < \alpha < 1$ , is  $1 - \alpha$ .

**3 LEBESGUE MEASURE**

The outer measure, although defined for all sets in  $\mathbb{R}$ , does not satisfy, in general, the countable additivity (Theorem 2.4). In order to have the property of countable additivity satisfied, we have to restrict the domain of definition for the function  $m^*$  to some suitable subset,

$\mathcal{M}$ , of the power set  $\mathcal{P}(\mathbb{R})$ . The members of  $\mathcal{M}$  are called **measurable sets** which are defined as follows.

**3.1 Definition.\*** A set  $E$  is said to be **Lebesgue measurable** or briefly **measurable** if for each set  $A$ , we have

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c). \quad (3)$$

*Remark.* The definition of measurability says that the measurable sets are those (bounded or unbounded) which split every set (measurable or not) into two pieces that are additive with respect to the outer measure.

Since  $A = (A \cap E) \cup (A \cap E^c)$  and  $m^*$  is subadditive, we always have

$$m^*(A) \leq m^*(A \cap E) + m^*(A \cap E^c).$$

Thus, in order to establish that  $E$  is measurable, we need only to show, for any set  $A$ , that

$$m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c). \quad (4)$$

*Note.* The inequality (4) is often used in practice to show that a given set  $E$  is measurable and the set  $A$  in reference is called **test set** since it is used to test the measurability.

*Remark.* H. Lebesgue, in his investigation, did not actually use the definition given above to define measurable sets. Instead, he considered set  $E$  in the bounded interval  $[a, b]$  and first defined the **interior** (or **inner**) **measure** of the set  $E$  as

$$m_*(E) = b - a - m^*(E^c).$$

He, then, called the set  $E$  to be measurable if

$$m_*(E) = m^*(E).$$

In other words,  $E$  is measurable if

$$m^*(E) = b - a - m^*(E^c). \quad (5)$$

If we let  $A = [a, b]$ , the equality (5) becomes

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c),$$

which is the same as (3). Thus the actual definition which Lebesgue used is a special case of (3). Since Lebesgue started with sets contained in  $[a, b]$ , i.e. bounded sets, appropriate modifications had to be

\*This definition is due to Carathéodory.

made for unbounded sets. Such modifications, however, are not needed if Definition 3.1 is used.

**3.2 Definition.** The set function  $m : \mathcal{M} \rightarrow \mathbf{R}^*$ , obtained by restricting the set functions  $m^*$  to the subset  $\mathcal{M}$  of the domain of definition  $\mathcal{P}(\mathbf{R})$  of  $m^*$ ; that is,  $m = m^*|_{\mathcal{M}}$ , is called Lebesgue measure function for the sets in  $\mathcal{M}$ .

For each  $E \in \mathcal{M}$ ,  $m(E) = m^*(E)$ . The extended real number  $m(E)$  is called the Lebesgue measure or simply measure of the set  $E$ .

Having selected the measurable sets by the test given in (3) and the measure of a measurable set to be the same as its outer measure, we can deduce the properties of measurable sets from the properties of outer measure already established, and erect a theory of integration on this foundation. This is the method originated by Carathéodory. We shall now show, in the following sections, that the measure  $m$  satisfies all the properties as desired in §1; namely,  $m$  is countably additive (Theorem 4.14),  $m$  is invariant under translation (Theorem 6.1),  $m$  is a generalization of the "length" (Problem 13) and many others.

## 4 PROPERTIES OF MEASURABLE SETS

### 4.1 Theorem

- (a) If  $E$  is a measurable set, then so is  $E^c$ .
- (b) The sets  $\phi$  and  $\mathbf{R}$  are measurable sets.

*Proof.* The proof is evident from Definition 3.1. ■

**4.2 Theorem** If  $E$  has the outer measure zero, then  $E$  is a measurable set. Furthermore, every subset of  $E$  is measurable.

*Proof.* Let  $A$  be any set. Then

$$m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c),$$

since

$$A \cap E \subset E \Rightarrow m^*(A \cap E) \leq m^*(E) = 0$$

and

$$A \cap E^c \subset A \Rightarrow m^*(A \cap E^c) \leq m^*(A).$$

Hence  $E$  is measurable. The other part follows from Problem 1. ■

**4.3 Corollary.** Every countable set is measurable and its measure is zero.

*Proof.* It follows from Corollary 2.5 and Theorem 4.2. ■

**4.4 Corollary.** *The Cantor set  $C$  and all its subsets are measurable, and each of them has measure zero.*

*Proof.* It follows from Example 2.8 and Theorem 4.2. ■

**4.5 Theorem.** *If  $E_1$  and  $E_2$  are measurable sets, then so is  $E_1 \cup E_2$ .*

*Proof.* Using the fact that  $E_1$  and  $E_2$  are measurable sets, for any set  $A$ , we have

$$\begin{aligned} m^*(A) &= m^*(A \cap E_1) + m^*(A \cap E_1^c) \\ &= m^*(A \cap E_1) + m^*([A \cap E_1^c] \cap E_2) + m^*([A \cap E_1^c] \cap E_2^c) \\ &= m^*(A \cap E_1) + m^*([A \cap E_2] \cap E_1^c) + m^*(A \cap E_1^c \cap E_2^c) \\ &= m^*(A \cap E_1) + m^*(A \cap E_2 \cap E_1^c) + m^*(A \cap [E_1 \cup E_2]^c) \\ &\geq m^*(A \cap [E_1 \cup E_2]) + m^*(A \cap [E_1 \cup E_2]^c), \end{aligned}$$

since  $A \cap (E_1 \cup E_2) = [A \cap E_1] \cup [A \cap E_2 \cap E_1^c]$ . Hence  $E_1 \cup E_2$  is a measurable set. ■

### Problems

4. Prove that the intersection and difference of two measurable sets are measurable. [Hint: For two sets  $E_1$  and  $E_2$ , we can write  $(E_1 \cap E_2)^c = E_1^c \cup E_2^c$  and  $E_1 - E_2 = E_1 \cap E_2^c$ .]
5. Prove that the symmetric difference of two measurable sets is measurable. [Hint: The symmetric difference of two sets  $E_1$  and  $E_2$  is given by  $E_1 \Delta E_2 = (E_1 - E_2) \cup (E_2 - E_1)$ ; apply the result in Problem 4.]

**4.6 Definition.** A collection  $\mathcal{A}$  of subsets of a set  $S$  is called an **algebra** (or **Boolean algebra**) of sets in  $\mathcal{P}(S)$  if

- (a)  $A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}$
- (b)  $A \in \mathcal{A} \Rightarrow A^c$ , that is  $S - A \in \mathcal{A}$ .

By DeMorgan's law it follows that if  $\mathcal{A}$  is an algebra of sets in  $\mathcal{P}(S)$ , then

- (c)  $A, B \in \mathcal{A} \Rightarrow A \cap B \in \mathcal{A}$ ,

while, on the other hand, if any collection  $\mathcal{A}$  of subsets of  $S$  satisfies (b) and (c), then it also satisfies (a) and hence  $\mathcal{A}$  is an algebra of sets in  $\mathcal{P}(S)$ .

By induction, it is easy to check that an algebra  $\mathcal{A}$  is closed under any finite union and finite intersection.

As an immediate consequence of Theorems 4.1 and 4.5, we have the following result.

**4.7 Corollary.** *The family  $\mathcal{M}$  of all measurable sets (subsets of  $\mathbf{R}$ ) is an algebra of sets in  $\mathcal{P}(\mathbf{R})$ . In particular, if  $\{E_1, E_2, \dots, E_n\}$  is any finite collection of measurable sets, then so are  $\bigcup_{i=1}^n E_i$  and  $\bigcap_{i=1}^n E_i$ .*

**4.8 Theorem.** *Let  $E_1, E_2, \dots, E_n$  be a finite sequence of disjoint measurable sets. Then, for any set  $A$ ,*

$$m^* \left( A \cap \left[ \bigcup_{i=1}^n E_i \right] \right) = \sum_{i=1}^n m^*(A \cap E_i).$$

*Proof.* We prove it by induction on  $n$ . For  $n = 1$ , the result is trivially true. Assuming it to be true for  $(n - 1)$  sets, we would have

$$m^* \left( A \cap \left[ \bigcup_{i=1}^{n-1} E_i \right] \right) = \sum_{i=1}^{n-1} m^*(A \cap E_i).$$

Adding  $m^*(A \cap E_n)$  on both the sides, we get

$$\begin{aligned} m^* \left( A \cap \left[ \bigcup_{i=1}^{n-1} E_i \right] \right) + m^*(A \cap E_n) &= \sum_{i=1}^{n-1} m^*(A \cap E_i) \\ \Rightarrow m^* \left( A \cap \left[ \bigcup_{i=1}^n E_i \cap E_n^c \right] \right) + m^* \left( A \cap \left[ \bigcup_{i=1}^n E_i \cap E_n \right] \right) & \\ &= \sum_{i=1}^n m^*(A \cap E_i), \end{aligned}$$

since the sets  $E_i$  ( $i = 1, 2, \dots, n$ ) are disjoint. But the measurability of the set  $E_n$ , by taking  $A \cap \left[ \bigcup_{i=1}^n E_i \right]$  as a test set, implies

$$m^* \left( A \cap \left[ \bigcup_{i=1}^n E_i \right] \right) = m^* \left( A \cap \left[ \bigcup_{i=1}^n E_i \cap E_n \right] \right) + m^* \left( A \cap \left[ \bigcup_{i=1}^n E_i \cap E_n^c \right] \right)$$

Hence

$$m^* \left( A \cap \left[ \bigcup_{i=1}^n E_i \right] \right) = \sum_{i=1}^n m^*(A \cap E_i). \blacksquare$$

**4.9 Corollary.** *If  $E_1, E_2, \dots, E_n$  is a finite sequence of disjoint measurable sets, then*

$$m \left( \bigcup_{i=1}^n E_i \right) = \sum_{i=1}^n m(E_i).$$

*Proof.* Taking  $A = \mathbb{R}$  in Theorem 4.8, the result follows in view of Corollary 4.7. ■

### Problems

6. If  $E_1$  and  $E_2$  are measurable sets such that  $E_1 \supset E_2$  and  $m(E_2) < \infty$ , then  $m(E_1 - E_2) = m(E_1) - m(E_2)$ . [Hint: Express  $E_1$  as a union of two disjoint measurable sets  $E_1 - E_2$  and  $E_2$ ; apply the result in Corollary 4.9.]
7. If  $E_1$  and  $E_2$  are any measurable sets, then prove that

$$m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2).$$

*Solution.* Let  $A$  be any set. Since  $E_1$  is a measurable set, we have

$$m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_1^c).$$

Taking  $A = E_1 \cup E_2$ , and adding  $m(E_1 \cap E_2)$  on both sides, we get

$$m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m([E_1 \cup E_2] \cap E_1^c) + m(E_1 \cap E_2).$$

Since

$$[(E_1 \cup E_2) \cap E_1^c] \cup [E_1 \cap E_2] = E_2$$

is a union of disjoint measurable sets, by Corollary 4.9, we note that

$$m([E_1 \cup E_2] \cap E_1^c) + m(E_1 \cap E_2) = m(E_2).$$

Hence the result follows. ■

The result in Corollary 4.9 verifies that the measure function  $m$  on  $\mathcal{M}$  is finitely additive. However, we would extend it to a more general situation to the case of countably additive. But before we do so we prove that the family  $\mathcal{M}$  is closed under countable union.

**4.10 Theorem.** *A countable union of measurable sets is a measurable set.*

The proof of Theorem 4.10 requires a result on algebra of sets which we state and prove below in the form of a lemma.

**4.11 Lemma.** *Let  $\mathcal{A}$  be an algebra of subsets of a set  $S$ . If  $\{A_i\}$  is a sequence of sets in  $\mathcal{A}$ , then there exists a sequence  $\{B_i\}$  of mutually disjoint sets in  $\mathcal{A}$  such that*

$$\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} A_i.$$

*Proof.* If the sequence  $\{A_i\}$  is finite, the result follows trivially. Let

us assume  $\{A_i\}$  to be an infinite sequence. Set  $B_1 = A_1$ , and for each  $n \in \mathbb{N}$ , define

$$\begin{aligned} B_n &= A_n - \left[ \bigcup_{i=1}^{n-1} A_i \right] \\ &= A_n \cap A_1^c \cap A_2^c \cap \dots \cap A_{n-1}^c \end{aligned}$$

Now, we note the following facts:

- (i)  $B_n \in \mathcal{A}$ , for each  $n \in \mathbb{N}$ , since  $\mathcal{A}$  is closed under the complementation and finite intersection of sets in  $\mathcal{A}$ .
- (ii)  $B_n \subset A_n$ , for each  $n \in \mathbb{N}$ .
- (iii) The sets  $B_n$  are mutually disjoint. Consider  $B_n$  and  $B_m$  to be two such sets and suppose  $m < n$ . Then, because  $B_m \subset A_m$ , we have

$$\begin{aligned} B_m \cap B_n &\subset A_m \cap B_n \\ &= A_m \cap [A_n \cap A_1^c \cap \dots \cap A_m^c \cap \dots \cap A_{n-1}^c] \\ &= [A_m \cap A_m^c] \cap \dots \\ &= \phi \cap \dots \\ &= \phi. \end{aligned}$$

$$(iv) \bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} A_i.$$

Since  $B_i \subset A_i$ , for each  $i \in \mathbb{N}$ , we have

$$\bigcup_{i=1}^{\infty} B_i \subset \bigcup_{i=1}^{\infty} A_i.$$

To prove the reverse inclusion, let  $x \in \bigcup_{i=1}^{\infty} A_i$ . Clearly,  $x$  must be in at least one of the sets  $A_i$ 's. Let  $n$  be the least value of  $i$  such that  $x \in A_i$ .

Then  $x \in B_n$ , and so  $x \in \bigcup_{n=1}^{\infty} B_n$ . Hence

$$\bigcup_{i=1}^{\infty} A_i \subset \bigcup_{i=1}^{\infty} B_i.$$

This completes the proof of the lemma.

*Proof of Theorem 4.10.* Let  $\{E_i\}$  be a sequence of measurable sets and let  $E = \bigcup_{i=1}^{\infty} E_i$ . To prove  $E$  to be a measurable set, by Lemma 4.11, we may assume, without any loss of generality, that the sets  $E_i$  are mutually disjoint.

For each  $n \in \mathbb{N}$ , define  $F_n = \bigcup_{i=1}^n E_i$ . Since  $\mathcal{M}$  is an algebra of sets

and  $E_1, E_2, \dots, E_n$  are in  $\mathcal{M}$ , the sets  $F_n$  are measurable. Therefore, for any set  $A$ , we have

$$\begin{aligned} m^*(A) &= m^*(A \cap F_n) + m^*(A \cap F_n^c) \\ &\geq m^*(A \cap F_n) + m^*(A \cap E^c), \end{aligned}$$

since

$$F_n^c = \left[ \bigcup_{i=n+1}^{\infty} E_i \right] \cup E^c \supset E^c.$$

But, by Theorem 4.8, we observe that

$$m^*(A \cap F_n) = \sum_{i=1}^n m^*(A \cap E_i).$$

Therefore,

$$m^*(A) \geq \sum_{i=1}^n m^*(A \cap E_i) + m^*(A \cap E^c).$$

This inequality holds for every  $n \in \mathbb{N}$  and since the left-hand side is independent of  $n$ , letting  $n \rightarrow \infty$ , we obtain

$$\begin{aligned} m^*(A) &\geq \sum_{i=1}^{\infty} m^*(A \cap E_i) + m^*(A \cap E^c) \\ &\geq m^*(A \cap E) + m^*(A \cap E^c), \end{aligned}$$

in view of the countable subadditivity of  $m^*$ , cf. Theorem 2.4. Hence  $E$  is a measurable set. ■

**4.12 Definition.** An algebra  $\mathcal{A}$  of sets is called a  $\sigma$ -algebra (or  $\sigma$ -Boolean algebra or Borel field) if it is closed under countable union of sets; that is,  $\bigcup_{i=1}^{\infty} A_i$  is in  $\mathcal{A}$  whenever the countable collection  $\{A_i\}$  of sets; is in  $\mathcal{A}$ .

*Note.* It follows, from DeMorgan's law, that a  $\sigma$ -algebra is also closed under countable intersection of sets.

As an outcome of Theorem 4.10, in view of Corollary 4.7, we have the following result.

**4.13 Corollary.** *The family  $\mathcal{M}$  of all measurable sets (subsets of  $\mathbb{R}$ ) is a  $\sigma$ -algebra of sets in  $\mathcal{P}(\mathbb{R})$ .*

**4.14 Theorem.** *Let  $\{E_i\}$  be an infinite sequence of disjoint measurable sets. Then*

$$m \left( \bigcup_{i=1}^{\infty} E_i \right) = \sum_{i=1}^{\infty} m(E_i).$$

*Proof.* By Corollary 4.9, for each  $n \in \mathbb{N}$ , we have

$$m \left( \bigcup_{i=1}^n E_i \right) = \sum_{i=1}^n m(E_i).$$

But

$$\bigcup_{i=1}^{\infty} E_i \supset \bigcup_{i=1}^n E_i, \quad \forall n \in \mathbb{N}.$$

Therefore, in view of Theorem 2.2(c) and the measurability of both the sets in the above inclusion (cf. Definition 4.12), we obtain

$$m \left( \bigcup_{i=1}^{\infty} E_i \right) \geq \sum_{i=1}^n m(E_i).$$

Since the left-hand side is independent of  $n$ , letting  $n \rightarrow \infty$ , we get

$$m \left( \bigcup_{i=1}^{\infty} E_i \right) \geq \sum_{i=1}^{\infty} m(E_i).$$

The reverse inequality is readily available from the countable sub-additivity property of  $m^*$  (Theorem 2.4). ■

*Remark.* The result in Theorem 4.14 verifies that the set function  $m : \mathcal{M} \rightarrow \mathbb{R}^*$  is countably additive.

Towards the close of this section we would like to examine: How big could the family  $\mathcal{M}$  of measurable sets be?

**4.15 Theorem.** *The cardinality of  $\mathcal{M}$  is  $2^c$*

*Proof.* Since  $\mathcal{M} \subset \mathcal{P}(\mathbb{R})$ , it follows that  $\overline{\mathcal{M}} \leq 2^c$  (cf. II-9.4). The reverse inequality is available from the fact that the family  $\mathcal{P}(C)$ , of all subsets of the Cantor set  $C$ , is contained in  $\mathcal{M}$  (cf. Corollary 4.4) and that  $\mathcal{P}(C) = 2^c$ , since the set  $C$  is uncountable having cardinality  $c$  (cf. II-10.5).

*Remark.* Though all the sets in  $\mathcal{P}(\mathbb{R})$  may not be measurable as we shall see in §8, still we observe that the cardinalities of each of the families  $\mathcal{M}$  and  $\mathcal{P}(\mathbb{R})$  of subsets of  $\mathbb{R}$  is the same. This reveals the fact, inspite of the inclusion  $\mathcal{M} \subset \mathcal{P}(\mathbb{R})$  being proper, that the family  $\mathcal{M}$  is very rich; so much so that there are as many measurable sets as in  $\mathcal{P}(\mathbb{R})$ .

### Problems

8. If  $E_1$ ,  $E_2$  and  $E_3$  are measurable sets, prove that

$$\begin{aligned} m(E_1) + m(E_2) + m(E_3) &= m(E_1 \cup E_2 \cup E_3) + m(E_1 \cap E_2) + m(E_2 \cap E_3) \\ &\quad + m(E_1 \cap E_3) - m(E_1 \cap E_2 \cap E_3). \end{aligned}$$

9. Construct a nowhere dense perfect set in the interval  $[0, 1]$  whose measure is  $\frac{1}{2}$ .
10. Find the measure of  $E$ , the set of points in  $]0, 1[$  such that the decimal expansion of  $x \in E$  does not contain the digits 5 and 7.
11. Prove that there are  $2^c$  measurable sets in every interval.
12. Let  $E$  be a measurable set. Prove that

$$m^*(A) = m^*(A \cap E) + m^*(A - [A \cap E]),$$

where  $A$  is any set.

## 5 BOREL SETS AND THEIR MEASURABILITY

Given any collection  $\mathcal{C}$  of subsets of a set  $S$ , consider the family  $\mathcal{F}$  of all  $\sigma$ -algebras each of which contains  $\mathcal{C}$ , and let

$$\mathcal{A} = \bigcap \{ \Sigma : \Sigma \in \mathcal{F} \}.$$

Then one can verify that  $\mathcal{A}$  is the smallest  $\sigma$ -algebra (unique) that contains  $\mathcal{C}$ , in the sense that  $\mathcal{A}$  is a  $\sigma$ -algebra containing  $\mathcal{C}$  and if  $\Sigma$  is any other  $\sigma$ -algebra containing  $\mathcal{C}$ , then  $\mathcal{A} \subset \Sigma$ . Such a collection  $\mathcal{A}$  is called the  $\sigma$ -algebra generated by  $\mathcal{C}$ .

Since the intersection of a countable collection of open sets in  $\mathbb{R}$  need not be open, the collection of all open sets in  $\mathbb{R}$  is not a  $\sigma$ -algebra. This motivates us to introduce the following notion.

**5.1 Definition.** The  $\sigma$ -algebra generated by the family of all open sets in  $\mathbb{R}$ , denoted by  $\mathcal{B}$ , is called the class of Borel sets in  $\mathbb{R}$ . The sets in  $\mathcal{B}$  are called Borel sets in  $\mathbb{R}$ .

**5.2 Examples.** Each of the open sets, closed sets,  $G_\delta$ -sets,  $F_\sigma$ -sets,  $G_{\delta\sigma}$ -sets,  $F_{\sigma\delta}$ -sets, ... is a simple type of Borel set.

The class of Borel sets plays an important role in analysis in general, and in measure theory in particular. We now prove that the Borel sets are measurable (in sense of Lebesgue).

**5.3 Theorem.** *Every Borel set in  $\mathbb{R}$  is measurable; that is,  $\mathcal{B} \subset \mathcal{M}$ .*

*Proof.* We prove the theorem in several steps by using the fact that  $\mathcal{M}$  is a  $\sigma$ -algebra.

Step 1: *The interval  $]a, \infty[$  is measurable.*

It is enough to show, for any set  $A$ , that

$$m^*(A) \geq m^*(A_1) + m^*(A_2),$$

where  $A_1 = A \cap ]a, \infty[$  and  $A_2 = A \cap ]-\infty, a[$ .

If  $m^*(A) = \infty$ , our assertion is trivially true. Let  $m^*(A) < \infty$ . Then, for each  $\epsilon > 0$ ,  $\exists$  a countable collection  $\{I_n\}$  of open intervals that covers  $A$  and satisfies

$$\sum_{n=1}^{\infty} l(I_n) < m^*(A) + \epsilon.$$

Write  $I'_n = I_n \cap ]a, \infty[$  and  $I''_n = I_n \cap ]-\infty, a]$ . Then,

$$\begin{aligned} I'_n \cup I''_n &= \{I_n \cap ]a, \infty[ \cup \{I_n \cap ]-\infty, a]\} \\ &= I_n \cap ]-\infty, \infty[ \\ &= I_n, \end{aligned}$$

and  $I'_n \cap I''_n = \phi$ . Therefore,

$$\begin{aligned} l(I_n) &= l(I'_n) + l(I''_n) \\ &= m^*(I'_n) + m^*(I''_n). \end{aligned}$$

But

$$A_1 \subset [ \cup I_n \cap ]a, \infty[ = \cup (I_n \cap ]a, \infty[) = \cup I'_n,$$

so that  $m^*(A_1) \leq m^*(\cup I'_n) \leq \sum m^*(I'_n)$ . Similarly  $A_2 \subset \cup I''_n$  and so  $m^*(A_2) \leq \sum m^*(I''_n)$ . Hence,

$$\begin{aligned} m^*(A_1) + m^*(A_2) &\leq \sum \{m^*(I'_n) + m^*(I''_n)\} \\ &= \sum l(I_n) < m^*(A) + \epsilon. \end{aligned}$$

Since  $\epsilon > 0$  is arbitrary, this verifies the result.

Step 2: *The interval  $] - \infty, a]$  is measurable since*

$$] - \infty, a] = ]a, \infty[^c$$

Step 3: *The interval  $] - \infty, b[$  is measurable since it can be expressed as a countable union of the intervals of the form as in Step 2; that is,*

$$] - \infty, b[ = \bigcup_{n=1}^{\infty} ] - \infty, b - \frac{1}{n}].$$

Step 4: Since any open interval  $]a, b[$  can be expressed as

$$]a, b[ = ] - \infty, b[ \cap ]a, \infty[,$$

it is measurable.

Step 5: *Every open set is measurable.* It is so because it can be expressed as a countable union of open intervals (disjoint).

Hence, in view of Step 5, the  $\sigma$ -algebra  $\mathcal{M}$  contains all the open sets in  $\mathbb{R}$ . Since  $\mathcal{B}$  is the smallest  $\sigma$ -algebra containing all the open sets, we conclude that  $\mathcal{B} \subset \mathcal{M}$ . This completes the proof of the theorem. ■

**5.4 Corollary.** Each of the sets in  $\mathbb{R}$ : an open set, a closed set, an  $\mathcal{F}_\sigma$ -set and a  $\mathcal{G}_\delta$ -set is measurable.

**5.5 Corollary.** For any arbitrary set  $E$  (measurable or not), there exists a  $\mathcal{G}_\delta$ -set  $G$  such that  $E \subset G$  and  $m^*(E) = m(G)$ .

*Proof.* It follows by using Theorem 2.14(b) and the fact that  $\mathcal{G}_\delta$ -set is a measurable set. ■

*Note.* The set  $G$  of Corollary 5.5 is sometimes called an equimeasurable superset of  $E$ .

**Problem 13.** Prove that every interval is a measurable set and its measure is its length.

*Solution.* It follows in view of the fact that an interval is a Borel set and the outer measure of an interval is its length.

*Remark.* The Borel sets do not exhaust all measurable sets; that is, the inclusion  $\mathcal{B} \subset \mathcal{M}$  is proper. This assertion follows from the fact that the cardinality of  $\mathcal{M}$  is  $2^c$  (Theorem 4.15) whereas the cardinality\* of  $\mathcal{B}$  is  $c$ . However, there exist examples of sets which are measurable sets but not Borel sets.

**5.6 Example** (Non-Borel measurable set). Let  $\varphi$  be the Cantor function. Define a function  $\psi : [0, 1] \rightarrow [0, 2]$  by

$$\psi(x) = x + \varphi(x).$$

It is obvious that  $\psi$  is a homeomorphism of  $[0, 1]$  onto  $[0, 2]$ . Let  $D$  be a non-measurable subset\*\* of  $\psi(C)$ , where  $C$  is the Cantor set. Then  $\psi^{-1}(D)$  is a subset of  $C$  and hence is a measurable set of measure zero (cf. Corollary 4.4). However, since  $\psi^{-1}(D)$  is the image under a homeomorphism of a non-Borel set  $D$  (because  $D$  is nonmeasurable),  $\psi^{-1}(D)$  is not a Borel set.

*Note.* The first actual example of such a set (non-Borel measurable set) was constructed by the Russian mathematician M.Ya Sustin (1894-1919). In fact, Sustin discovered a very important and extensive class of sets, calling them  $\mathcal{A}$ -sets, some of which are measurable (in sense of Lebesgue) but not Borel sets.

\*The proof of this fact requires the concepts such as ordinal numbers and transfinite inductions, and is therefore out of the scope of the present book.

\*\*For the construction of nonmeasurable sets, refer to §8 of this chapter.

## 6 FURTHER PROPERTIES OF MEASURABLE SETS

In §1 of this chapter, we had mentioned the “translation invariance” of a measure as one of the desirable properties. This is quite true with the Lebesgue measure on  $\mathcal{M}$  as we establish in the following theorem.

**6.1 Theorem.** *Let  $E$  be a measurable set. Then any translate  $E+y$  is measurable, where  $y$  is a real number. Furthermore,*

$$m(E+y) = m(E).$$

*Proof.* Let  $A$  be any set. Since  $E$  is measurable, we have

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$$

$$\Rightarrow m^*(A+y) = m^*([A \cap E]+y) + m^*([A \cap E^c]+y),$$

in view of the fact that  $m^*$  is invariant under translation, cf. Theorem 2.2(e). But one can easily verify that

$$\begin{cases} [A \cap E]+y = (A+y) \cap (E+y) \\ [A \cap E^c]+y = (A+y) \cap (E^c+y). \end{cases}$$

Hence

$$m^*(A+y) = m^*([A+y] \cap [E+y]) + m^*([A+y] \cap [E^c+y]).$$

Since  $A$  is arbitrary, replacing  $A$  with  $A-y$ , we obtain

$$m^*(A) = m^*(A \cap E+y) + m^*(A \cap E^c+y).$$

The measurability of  $E+y$  follows by taking into account that  $(E+y)^c = E^c+y$ , and the proof is complete because  $m^*$  is translation invariant. ■

The following two theorems reveal a basic property of Lebesgue measure that concerns its behaviour for monotone sequence of sets.

**6.2 Theorem.** *Let  $\{E_i\}$  be an infinite decreasing sequence of measurable sets; that is, a sequence with  $E_{i+1} \subset E_i$  for each  $i \in \mathbb{N}$ . Let  $m(E_i) < \infty$  for at least one  $i \in \mathbb{N}$ . Then*

$$m\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n \rightarrow \infty} m(E_n).$$

*Proof.* Let  $p$  be the least integer such that  $m(E_p) < \infty$ . Then  $m(E_i) < \infty$ , for all  $i \geq p$ . Set  $E = \bigcap_{i=1}^{\infty} E_i$  and  $F_i = E_i - E_{i+1}$ . Then the sets  $F_i$  are pairwise disjoint, and

$$E_p - E = \bigcup_{i=p}^{\infty} F_i.$$

Therefore,

$$m(E_p - E) = \sum_{i=p}^{\infty} m(F_i) = \sum_{i=p}^{\infty} m(E_i - E_{i+1}).$$

But  $m(E_p) = m(E) + m(E_p - E)$  and  $m(E_i) = m(E_{i+1}) + m(E_i - E_{i+1})$ , for all  $i \geq p$ , since  $E \subset E_p$  and  $E_{i+1} \subset E_i$ . Further, using the fact that  $m(E_i) < \infty$ , for all  $i \geq p$ , it follows that

$$m(E_p - E) = m(E_p) - m(E)$$

and

$$m(E_i - E_{i+1}) = m(E_i) - m(E_{i+1}), \quad \forall i \geq p.$$

Hence,

$$\begin{aligned} m(E_p) - m(E) &= \sum_{i=p}^{\infty} (m(E_i) - m(E_{i+1})) \\ &= \lim_{n \rightarrow \infty} \sum_{i=p}^n (m(E_i) - m(E_{i+1})) \\ &= \lim_{n \rightarrow \infty} \{m(E_p) - m(E_n)\} \\ &= m(E_p) - \lim_{n \rightarrow \infty} m(E_n). \end{aligned}$$

Since  $m(E_p) < \infty$ , it gives

$$m(E) = \lim_{n \rightarrow \infty} m(E_n).$$

This proves the result. ■

**6.3 Corollary.** *Let  $\{E_i\}$  be an infinite decreasing sequence of measurable sets, and let  $m(E_1) < \infty$ . Then*

$$m\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n \rightarrow \infty} m(E_n).$$

*Remark.* The condition  $m(E_i) < \infty$ , for at least one  $i \in \mathbb{N}$ , in Theorem 6.2 (or  $m(E_1) < \infty$  in Corollary 6.3) cannot be dropped.

**6.4 Example.** Consider the sets  $E_n$  given by  $E_n = ]n, \infty[$ ,  $n \in \mathbb{N}$ . Then  $\{E_n\}$  is a decreasing sequence of measurable sets such that  $m(E_n) = \infty$  for each  $n \in \mathbb{N}$  and  $\bigcap_{n=1}^{\infty} E_n = \phi$ . Therefore,

$$\lim_{n \rightarrow \infty} m(E_n) = \infty, \quad \text{while} \quad m(\phi) = 0.$$

**6.5 Theorem.** *Let  $\{E_i\}$  be an infinite increasing sequence of measurable sets; that is, a sequence with  $E_{i+1} \supset E_i$  for each  $i \in \mathbb{N}$ . Then*

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) = \lim_{n \rightarrow \infty} m(E_n).$$

*Proof.* If  $m(E_p) = \infty$  for some  $p \in \mathbb{N}$ , then the result is trivially true, since

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) \geq m(E_p) = \infty,$$

and  $m(E_n) = \infty$ , for each  $n \geq p$ . Let  $m(E_i) < \infty$ , for each  $i \in \mathbb{N}$ . Set

$$E = \bigcup_{i=1}^{\infty} E_i, \quad F_i = E_{i+1} - E_i.$$

Then the sets  $F_i$  are pairwise disjoint, and

$$E - E_1 = \bigcup_{i=1}^{\infty} F_i$$

$$\Rightarrow m(E - E_1) = m\left(\bigcup_{i=1}^{\infty} F_i\right) = \sum_{i=1}^{\infty} m(F_i) = \sum_{i=1}^{\infty} m(E_{i+1} - E_i)$$

$$\begin{aligned} \Rightarrow m(E) - m(E_1) &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \{m(E_{i+1}) - m(E_i)\} \\ &= \lim_{n \rightarrow \infty} \{m(E_{n+1}) - m(E_1)\} \end{aligned}$$

$$\Rightarrow m(E) = \lim_{n \rightarrow \infty} m(E_n).$$

*Note.* Theorems 6.2 and 6.5 establish that the set function  $m$  is conditionally continuous from above as well as below.

We are always interested, whenever possible, in measurable sets. However, proving the measurability of a given set is sometimes difficult in practice. It may be desirable in that situation to apply theorems about outer measure. A particularly useful result is the following theorem which, in fact, improves Theorem 6.5.

**6.6 Theorem.** *Let  $\{E_i\}$  be an infinite increasing sequence of sets (not necessarily measurable). Then*

$$m^*\left(\bigcup_{i=1}^{\infty} E_i\right) = \lim_{n \rightarrow \infty} m^*(E_n).$$

*Proof.* For each  $E_i$ , there exists a  $\mathcal{G}_\delta$ -set  $G_i$  (of course measurable) such that  $E_i \subset G_i$  and  $m(G_i) = m^*(E_i)$ , cf. Corollary 5.5. For each  $n \in \mathbb{N}$ , define

$$F_n = \bigcap_{i=n}^{\infty} G_i.$$

Clearly  $\{F_n\}$  is an increasing sequence of measurable sets. Therefore, by Theorem 6.5, it follows that

$$m\left(\bigcup_{n=1}^{\infty} F_n\right) = \lim_{n \rightarrow \infty} m(F_n).$$

But  $E_i \subset E_{i+1}$  implies that  $E_n \subset G_i$ , for all  $i \geq n$  and as such  $E_n \subset F_n \subset G_n$ , for each  $n \in \mathbb{N}$ . Consequently

$$m^*(E_n) \leq m(F_n) \leq m(G_n) = m^*(E_n)$$

$$\Rightarrow m^*(E_n) = m(F_n), \quad \forall n \in \mathbb{N}.$$

Hence  $\lim_{n \rightarrow \infty} m^*(E_n)$  exists, and

$$\lim_{n \rightarrow \infty} m^*(E_n) = m\left(\bigcup_{n=1}^{\infty} F_n\right).$$

Also, we observe that

$$\bigcup_{n=1}^{\infty} F_n \supset \bigcup_{n=1}^{\infty} E_n,$$

and, therefore,

$$\lim_{n \rightarrow \infty} m^*(E_n) \geq m^*\left(\bigcup_{n=1}^{\infty} E_n\right).$$

The reverse inequality is obvious from the fact that

$$E_n \subset \bigcup_{n=1}^{\infty} E_n, \quad \forall n \in \mathbb{N}$$

and this completes the proof. ■

In analogy with the limit superior and the limit inferior of a sequence of real numbers (cf. I-6), let us define the concepts such as the limit superior and the limit inferior of a sequence of sets.

Let  $\{E_i\}$  be any sequence of sets. Define

$$\begin{cases} \limsup_{i \rightarrow \infty} E_i = \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} E_i \\ \liminf_{i \rightarrow \infty} E_i = \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} E_i. \end{cases}$$

If  $\limsup_{i \rightarrow \infty} E_i = \liminf_{i \rightarrow \infty} E_i$ , the sequence  $\{E_i\}$  of sets is said to be **convergent**, and the common limit is written by  $\lim E_i$ .

*Remark.* It is easy to see, from the definition, that  $\limsup E_i$  is the set of points which are in infinitely many of the sets  $E_i$  while

\*If no confusion is likely to occur we shall write  $\limsup E_i$ , etc. for  $\limsup_{i \rightarrow \infty} E_i$ , etc.

$\liminf E_i$  is the set of points which are in all but finitely many of the sets  $E_i$ . As a consequence of it, we note that  $\limsup E_i \supset \liminf E_i$ .

One can easily verify the following results.

**6.7 Theorem.** *If  $\{E_i\}$  is a decreasing sequence of sets, then  $\lim E_i = \bigcap E_i$ , and if  $\{E_i\}$  is an increasing one, then  $\lim E_i = \bigcup E_i$ .*

*Note.* The assertion in Theorem 6.7 is analogous to the result that a monotone sequence of real numbers has the limit.

**6.8 Theorem** *If  $\{E_i\}$  is a sequence of measurable sets, then the sets  $\limsup E_n$  and  $\liminf E_n$  are also measurable.*

**Problems 14.** Prove the results in Theorems 6.7 and 6.8.

**6.9 Theorem.** *Let  $\{E_i\}$  be a sequence of measurable sets. Then  $m(\liminf E_i) \leq \liminf m(E_i)$ . Further, if  $m\left(\bigcup_{i=p}^{\infty} E_i\right) < \infty$  for some  $p \in \mathbb{N}$ , then  $m(\limsup E_i) \geq \limsup m(E_i)$ .*

*Proof.* Set  $F_n = \bigcap_{i=n}^{\infty} E_i$ . Then, by definition, we have

$$\liminf E_i = \bigcup_{n=1}^{\infty} F_n.$$

But we observe that  $\{F_n\}$  is an increasing sequence of measurable sets. Therefore,

$$m\left(\bigcup_{n=1}^{\infty} F_n\right) = \lim_{n \rightarrow \infty} m(F_n)$$

$$\Rightarrow m(\liminf E_i) = \lim_{n \rightarrow \infty} m(F_n).$$

Let  $n \in \mathbb{N}$  be fixed. Then  $F_n \subset E_{n+k}$  and so  $m(F_n) \leq m(E_{n+k})$ , for all  $k \in \mathbb{N}$ . Thus, we have

$$m(F_n) \leq \liminf_{k \rightarrow \infty} m(E_{n+k}) = \liminf m(E_i).$$

This is true for each  $n \in \mathbb{N}$ . Hence

$$\lim_{n \rightarrow \infty} m(F_n) \leq \liminf m(E_i),$$

which verifies

$$m(\liminf E_i) \leq \liminf m(E_i).$$

The other part follows similarly. ■

**6.10 Corollary.** *If  $\{E_n\}$  is a convergent sequence of measurable sets and  $E_n \subset F$  for each  $n \in \mathbb{N}$ , where  $m^*(F) < \infty$ , then  $\lim_{n \rightarrow \infty} m(E_n)$  exists and*

$$m(\lim_{n \rightarrow \infty} E_n) = \lim_{n \rightarrow \infty} m(E_n).$$

**Problem 15.** Prove the second part of Theorem 6.9 and show that the additional assumption stated therein is necessary.

## 7 CHARACTERIZATIONS OF MEASURABLE SETS

We now give certain characterizations of the measurability of an arbitrary set  $E$  in  $\mathbb{R}$  in terms of certain relatively simple types of sets each of which is a Borel set.

**7.1 Theorem.** *Let  $E$  be a given set. Then the following statements are equivalent:*

- (a)  $E$  is measurable.
- (b) Given  $\epsilon > 0$ , there is an open set  $O \supset E$  such that  $m^*(O - E) < \epsilon$ .
- (c) There is a  $\mathcal{G}_\delta$ -set  $G \supset E$  such that  $m^*(G - E) = 0$ .
- (d) Given  $\epsilon > 0$ , there is a closed set  $F \subset E$  such that  $m^*(E - F) < \epsilon$ .
- (e) There is a  $\mathcal{F}_\sigma$ -set  $F \subset E$  such that  $m^*(E - F) = 0$ .

*Proof.* (a)  $\Rightarrow$  (b): Suppose first that  $m(E) < \infty$ . By Theorem 2.14(a), there is an open set  $O \supset E$  such that

$$m^*(O) < m^*(E) + \epsilon.$$

This, since both the sets  $O$  and  $E$  are measurable, verifies (cf. Problem 6).

$$m^*(O - E) = m^*(O) - m^*(E) < \epsilon.$$

Consider now the case when  $m(E) = \infty$ . Write the set  $\mathbb{R}$  of real numbers as a union of disjoint finite intervals; that is,  $\mathbb{R} = \bigcup_{n=1}^{\infty} I_n$ . Then, if  $E_n = E \cap I_n$ ,  $m(E_n) < \infty$ . We can, thus, find open sets  $O_n \supset E_n$  such that

$$m^*(O_n - E_n) < \frac{\epsilon}{2^n}.$$

Define  $O = \bigcup_{n=1}^{\infty} O_n$ . Clearly  $O$  is an open set such that  $O \supset E$  and satisfies

$$O - E = \bigcup_{n=1}^{\infty} O_n - \bigcup_{n=1}^{\infty} E_n \subset \bigcup_{n=1}^{\infty} (O_n - E_n).$$

Hence

$$m^*(O - E) \leq \sum_{n=1}^{\infty} m^*(O_n - E_n) < \epsilon.$$

(b)  $\Rightarrow$  (c): Given  $\epsilon = 1/n$ , there is an open set  $O_n \supset E$  with  $m^*(O_n - E) < 1/n$ . Define  $G = \bigcap_{n=1}^{\infty} O_n$ . Then  $G$  is a  $\mathcal{G}_\sigma$ -set such that  $G \supset E$  and

$$m^*(G - E) \leq m^*(O_n - E) < \frac{1}{n}, \quad \forall n \in \mathbb{N}.$$

This on letting  $n \rightarrow \infty$  proves (c).

(c)  $\Rightarrow$  (a): Write  $E = G - (G - E)$ . But the sets  $G$  and  $G - E$  are measurable since  $G$  is a Borel set and  $G - E$  is of outer measure zero. Hence  $E$  is measurable, cf. Problem 5.

(a)  $\Rightarrow$  (d):  $E^c$  is measurable and so, in view of (c), there is an open set  $O \supset E^c$  such that  $m^*(O - E^c) < \epsilon$ . But  $O - E^c = E - O^c$ . Taking  $F = O^c$ , the assertion (d) follows.

(d)  $\Rightarrow$  (e): Given  $\epsilon = 1/n$ , there is a closed set  $F_n \subset E$  with  $m^*(E - F_n) < 1/n$ . Define  $F = \bigcup_{n=1}^{\infty} F_n$ . Then  $F$  is a  $\mathcal{F}_\sigma$ -set such that  $F \subset E$  and

$$m^*(E - F) \leq m^*(E - F_n) < \frac{1}{n}, \quad \forall n \in \mathbb{N}.$$

Hence the result in (e) follows on letting  $n \rightarrow \infty$ .

(e)  $\Rightarrow$  (a): The proof is similar to (c)  $\Rightarrow$  (a).

This completes the proof of Theorem 7.1.  $\blacksquare$

*Remarks* 1. The assertions (b) and (d) in Theorem 7.1 state that the measurable sets are precisely those which can be approximated closely, in terms of  $m^*$ , by open or closed sets; whereas the assertions (c) and (e) state that any arbitrary measurable set differs from a Borel set, more precisely from a  $\mathcal{G}_\sigma$ -set and an  $\mathcal{F}_\sigma$ -set, by a set of measure zero.

2. Condition (b) in Theorem 7.1 is a necessary and sufficient condition for the measurability of a set  $E$ . This should not be confused with the conclusion of Theorem 2.14(a) which states that for any set  $E$  (measurable or not), there is an open set  $O \supset E$  such that  $m^*(O) < m^*(E) + \epsilon$ . In general, since  $O = E \cup (O - E)$  when  $O \supset E$ , we only have

$$m^*(O) \leq m^*(E) + m^*(O - E),$$

and we cannot conclude  $m^*(O - E) < \epsilon$  from  $m^*(O) < m^*(E) + \epsilon$ .

3. Similarly the condition (c) in Theorem 7.1 for the measurability of a set  $E$  should not be confused with the conclusion of Theorem 2.14(b).

**Problem 16.** If  $E$  is a measurable set, prove that there exist Borel sets  $B_1$  and  $B_2$  such that  $B_1 \subset E \subset B_2$  and  $m(B_1) = m(E) = m(B_2)$ . [Hint: Use (a), (c) and (e) in Theorem 7.1.]

Now, we prove a result which shows that a measurable set may be approximated by a finite number of intervals. It may be noted that the union of each intervals, in general, neither contains nor is contained in the set we are approximating.

**7.2 Theorem.** *Let  $E$  be a set with  $m^*(E) < \infty$ . Then  $E$  is measurable if and only if, given  $\epsilon > 0$ , there is a finite union  $B$  of open intervals such that*

$$m^*(E\Delta B) < \epsilon.$$

*Proof.* Suppose  $E$  be measurable, and let  $\epsilon > 0$  be given. Then, by Theorem 7.1(b), there exists an open set  $O \supset E$  with  $m^*(O - E) < \epsilon/2$ . As  $m^*(E)$  is finite, so is  $m^*(O)$ . Further, since the open set  $O$  can be expressed as the union of countable (disjoint) open intervals  $\{I_i\}$ , there exists an  $n \in \mathbb{N}$  such that

$$\sum_{i=n+1}^{\infty} l(I_i) < \frac{\epsilon}{2},$$

in view of Theorem 4.14 and the fact that  $m^*(O) < \infty$ .

Write  $B = \bigcup_{i=1}^n I_i$ . Then

$$E\Delta B = (E - B) \cup (B - E) \subset (O - B) \cup (O - E).$$

Hence

$$m^*(E\Delta B) \leq m^*\left(\bigcup_{i=n}^{\infty} I_i\right) + m^*(O - E) < \epsilon.$$

Conversely, assume that for a given  $\epsilon > 0$ , there is a finite union,  $B = \bigcup_{i=1}^n I_i$ , of open intervals with  $m^*(E\Delta B) < \epsilon$ . By Theorem 2.14(a), there is an open set  $O \supset E$  such that

$$m^*(O) < m^*(E) + \epsilon. \quad (6)$$

If we can show that  $m^*(O - E)$  is arbitrarily small, then by Theorem 7.1, it follows that  $E$  is a measurable set.

Write  $S = \bigcup_{i=1}^n (I_i \cap O)$ . Then  $S \subset B$  and so

$$S\Delta E = (E - S) \cup (S - E) \subset (E - S) \cup (B - E).$$

However,

$$E - S = (E \cap O^c) \cup (E \cap B^c) = E - B$$

since  $E \subset O$ . Therefore

$$SAE \subset (E - B) \cup (B - E) = E \Delta B,$$

and as such  $m^*(SAE) < \epsilon$ . However,  $E \subset S \cup (SAE)$  and so

$$m^*(E) < m^*(S) + \epsilon. \tag{7}$$

Also  $O - E \subset (O - S) \cup (SAE)$  gives

$$m^*(O - E) < m^*(O) - m^*(S) + \epsilon.$$

Hence, in view of (6) and (7), we get

$$m^*(O - E) < 2\epsilon.$$

This proves the theorem. ■

### 8 NONMEASURABLE SETS

We now turn to the question whether or not there exist sets which are nonmeasurable in the sense of Lebesgue.

Most of the sets we usually come across in analysis are measurable. However, there are several examples of nonmeasurable sets given by G. Vitali (1905), Van Vleck (1905), F. Bernstein (1908) and others. However, all these examples have been constructed on the assumption that the axiom of choice of set theory is valid, and it was not clear until recently whether a nonmeasurable set could be constructed without assuming the validity of the axiom of choice. Recently, Robert Solovay (1970) has solved this problem by proving that the existence of nonmeasurable sets cannot be established if the axiom of choice is disallowed.

In this section, we discuss an example of a nonmeasurable set which is a slight modification of the one given by Vitali. Before we do so we need certain preliminaries.

**8.1 Definition.** If  $x$  and  $y$  are real numbers in  $[0, 1[$ , then the sum modulo 1,  $\dot{+}$ , of  $x$  and  $y$  is defined by

$$x \dot{+} y = \begin{cases} x + y, & x + y < 1 \\ x + y - 1, & x + y \geq 1. \end{cases}$$

**8.2 Definition.** If  $E$  is a subset of  $[0, 1[$ , then the translate modulo 1 of  $E$  by  $y$  is defined to be the set given by

$$E \dot{+} y = \{z : z = x \dot{+} y, x \in E\}.$$

It is easy to verify that:

- (i)  $x, y \in [0, 1[ \Rightarrow x \dot{+} y \in [0, 1[$ .

- (ii) The operation  $\dot{+}$  is commutative and associative.  
 (iii) If we assign to each  $x \in [0, 1[$ , the angle  $2\pi x$ , then the sum modulo 1 corresponds to the addition of angles, and translate modulo 1 by  $y$  corresponds to rotation through an angle of  $2\pi y$ .

Below we prove that the measure (Lebesgue) is invariant under translate modulo 1.

**8.3 Theorem.** *Let  $E \subset [0, 1[$  be a measurable set and  $y \in [0, 1[$  be given. Then the set  $E \dot{+} y$  is measurable and  $m(E \dot{+} y) = m(E)$ .*

*Proof.* Define

$$\begin{cases} E_1 = E \cap [0, 1 - y[ \\ E_2 = E \cap [1 - y, 1[. \end{cases}$$

Clearly  $E_1$  and  $E_2$  are two disjoint measurable sets such that  $E_1 \cup E_2 = E$ . Therefore

$$m(E) = m(E_1) + m(E_2).$$

Now,  $E_1 \dot{+} y = E_1 + y$  and  $E_2 \dot{+} y = E_2 + y - 1$  and so  $E_1 \dot{+} y$  and  $E_2 \dot{+} y$  both are measurable sets with

$$\begin{cases} m(E_1 \dot{+} y) = m(E_1 + y) = m(E_1) \\ m(E_2 \dot{+} y) = m(E_2 + y - 1) = m(E_2), \end{cases}$$

since  $m$  is translation invariant (cf. Theorem 6.1). Also

$$\begin{aligned} E \dot{+} y &= (E_1 \cup E_2) \dot{+} y \\ &= (E_1 \dot{+} y) \cup (E_2 \dot{+} y) \end{aligned}$$

Hence  $E \dot{+} y$  is a measurable set with

$$\begin{aligned} m(E \dot{+} y) &= m(E_1 \dot{+} y) + m(E_2 \dot{+} y) \\ &= m(E_1) + m(E_2) \\ &= m(E). \blacksquare \end{aligned}$$

**8.4 Theorem.** *There exists a nonmeasurable set in the interval  $[0, 1[$ .*

*Proof.* We define an equivalence relation ' $\sim$ ' in the set  $I = [0, 1[$  by saying that  $x$  and  $y$  in  $I$  are equivalent, to be written  $x \sim y$ , if  $x - y$  is rational. Clearly, the relation  $\sim$  partitions the set  $I$  into mutually disjoint equivalent classes; that is, any two elements of the same class differ by a rational number while those of the different classes differ by an irrational number.

Construct a set  $P$  by choosing exactly one element from each equivalence class—this is possible by the axiom of choice. Clearly  $P \subset [0, 1[$ . We shall now show that  $P$  is a nonmeasurable set.

Let  $\{r_i\}$  be an enumeration of rational numbers in  $[0, 1[$  with  $r_0 = 0$ . Define

$$P_i = P \dot{+} r_i.$$

Then  $P_0 = P$ . We further observe that:

$$(a) P_m \cap P_n = \phi, m \neq n.$$

$$(b) \bigcup_n P_n = [0, 1[.$$

*Proof of (a).* Let, if possible,  $y \in P_m \cap P_n$ . Then there exist  $p_m$  and  $p_n$  in  $P$  such that

$$y = p_m \dot{+} r_m = p_n \dot{+} r_n$$

$$\Rightarrow p_m - p_n \text{ is a rational number}$$

$$\Rightarrow p_m = p_n, \text{ by the definition of the set } P$$

$$\Rightarrow m = n.$$

This is a contradiction.

*Proof of (b).* Let  $x \in [0, 1[$ . Then  $x$  lies in one of the equivalent classes and as such  $x$  is equivalent to an element  $y$  (say) of  $P$ . Suppose  $r_i$  is the rational number by which  $x$  differs from  $y$ . Then  $x \in P_i$  and hence  $[0, 1[ \subset \bigcup_n P_n$  while the reverse inclusion is obviously true.

Now, we turn towards the proof of the nonmeasurability of  $P$ . Assume that  $P$  is measurable. Then each  $P_i$  is measurable, and  $m(P_i) = m(P)$ , cf. Theorem 8.3. Therefore

$$\begin{aligned} m\left(\bigcup_i P_i\right) &= \sum_{i=1}^{\infty} m(P_i) \\ &= \sum_{i=1}^{\infty} m(P) \\ &= \begin{cases} 0 & \text{if } m(P) = 0 \\ \infty & \text{if } m(P) > 0. \end{cases} \end{aligned}$$

On the other hand

$$m\left(\bigcup_i P_i\right) = m([0, 1]) = 1.$$

These lead to contradictory statements. Hence  $P$  is a nonmeasurable set. ■

If we partition any measurable set with positive measure, instead of the interval  $[0, 1[$ , into equivalent classes and proceed exactly on the

lines of the proof of Theorem 8.4, we are led to the following result which reveals the fact that once the axiom of choice is accepted, there is no dearth of nonmeasurable sets in  $\mathbb{R}$ .

**8.5 Theorem.** *Every set of positive measure contains a nonmeasurable set.*

### Problems

17. Let  $E_1$  and  $E_2$  be measurable sets each having finite measure. Prove that the following statements are equivalent:
  - (a)  $m(E_1 \Delta E_2) = 0$ .
  - (b)  $m(E_1 - E_2) = 0 = m(E_2 - E_1)$ .
  - (c)  $m(E_1) = m(E_1 \cap E_2) = m(E_2)$ .
18. If  $E_1$  is a measurable set and  $m^*(E_1 \Delta E_2) = 0$ , then show that  $E_2$  is a measurable set.
19. Using the Heine-Borel theorem, show that every compact set in  $\mathbb{R}$  is measurable.
20. If  $\{E_i\}$  is a sequence of sets with  $m^*(E_i) = 0$  for each  $i \in \mathbb{N}$ , then prove that  $\bigcup_{i=1}^{\infty} E_i$  is a measurable set and has measure zero.
21. Show that each Cantor  $n$ -ary set\* has measure zero.
22. Prove that the generalized Cantor set\*  $C(\alpha)$ ,  $0 < \alpha < 1$ , is measurable and has measure  $1 - \alpha$ .
23. For any set  $E$  in  $\mathbb{R}$  and  $k > 0$ , let  $kE = \{x : k^{-1}x \in E\}$ . Show that
  - (a)  $m^*(kE) = km^*(E)$ .
  - (b)  $E$  is measurable if and only if  $kE$  is so.
24. Prove that the set of all irrational numbers in  $[0, 1]$  is measurable and find its measure. Is every set of the irrational numbers measurable? Justify your answer.
25. Show that a set  $E$  is measurable if and only if for each  $\epsilon > 0$ , there is an open set  $O$  and a closed set  $F$  such that  $F \subset E \subset O$  and  $m(O - F) < \epsilon$ .
26. Prove or disprove the result which states that the union of uncountably many measurable subsets of  $[a, b]$  is measurable set.
27. Work out the details of the proof of Theorem 8.5.
28. Let  $E$  be a measurable set such that  $E \subset P$ , where  $P$  is the non-measurable set constructed in Theorem 8.4. Prove that  $m(E) = 0$ .

\*For the definitions of Cantor  $n$ -ary set and generalized Cantor set, refer to II-10.

**Solution.** Set  $E_i = E \dot{-} r_i$ , where  $\{r_i\}$  is the enumeration of rational numbers in  $[0, 1[$  and  $r_0 = 0$ . Then, by using the arguments same as in proof of Theorem 8.4, one can show that  $\{E_i\}$  is a sequence of disjoint measurable sets such that  $m(E_i) = m(E)$  and  $\bigcup_{i=1}^{\infty} E_i = E$ . Thus

$$\sum_{i=1}^{\infty} m(E_i) = m\left(\bigcup_{i=1}^{\infty} E_i\right) = m(E) \leq m([0, 1]) = 1.$$

Since  $\sum_{i=1}^{\infty} m(E_i)$  is either 0 or  $\infty$ , this verifies  $m(E) = 0$ .

29. Show that if  $A$  is any set with  $m^*(A) > 0$ , then there is a non-measurable set  $E$  contained in  $A$ .
30. Give an example of a sequence  $\{E_i\}$  of disjoint sets such that

$$m^*\left(\bigcup_{i=1}^{\infty} E_i\right) < \sum_{i=1}^{\infty} m^*(E_i).$$

31. Give an example of a decreasing sequence  $\{E_i\}$  of sets with  $m^*(E_i) < \infty$  such that

$$m^*\left(\bigcap_{i=1}^{\infty} E_i\right) < \lim_{i \rightarrow \infty} m^*(E_i).$$

32. Suppose  $E$  to be the set of points in  $]0, 1[$  such that  $x \in E$  if and only if the decimal expansion of  $x$  does not contain the digit 7. Show that  $m(E) = 0$ .
33. Show that if  $m^*(E) = \infty$  and for each  $\epsilon > 0$ , there are intervals  $I_1, I_2, \dots, I_n$  such that

$$m^*\left(E \Delta \left[\bigcup_{i=1}^n I_i\right]\right) < \epsilon,$$

then at least one of the intervals  $I_i$  is finite.

## IV

# Measurable Functions

With the class  $\mathcal{M}$  of measurable sets in mind, we introduce a rich class of functions; namely, the class of measurable functions which includes the class of continuous functions as a proper subclass (Theorem 7.1 and Example 7.2). The class of measurable functions plays a role of central importance in Lebesgue theory of integration. It will assume a place comparable to that of the class of functions which are bounded and continuous almost everywhere in the Riemann theory of integration and of functions of bounded variation in the instance of Stieltjes integrals. Roughly speaking, a function is integrable if its behaviour is not too irregular, and if the values it takes are not too large too often. The second requirement is equivalent to the existence of the equality of the upper and lower integrals. We now introduce the notion of measurability which gives precisely the conditions required for the integrability, given that the function is not too large. In many cases, it is easier to examine the measurability of a function than to investigate its upper and lower integrals directly.

### 1 DEFINITION

In what follows, we shall make use of the following notations:

$$E(f \geq \alpha) = \{x \in E : f(x) \geq \alpha\}$$

$$E(f = \alpha) = \{x \in E : f(x) = \alpha\}$$

$$E(f \leq \alpha) = \{x \in E : f(x) \leq \alpha\}$$

$$E(f > \alpha) = \{x \in E : f(x) > \alpha\}$$

$$E(f < \alpha) = \{x \in E : f(x) < \alpha\}.$$

**1.1 Definition.** An extended real-valued function\*  $f$  defined on a measurable set  $E$  is said to be **Lebesgue-measurable** or, more briefly, **measurable** on  $E$ , if the set  $E(f > \alpha)$  is measurable for all real numbers  $\alpha$ .

*Note.* The measure of the set  $E(f > \alpha)$  may be finite or infinite.

*Justification of Definition 1.1.* As  $\alpha$  varies, the behaviour of the set  $E(f > \alpha)$  describes how the values of the function  $f$  are distributed. Intuitively, it is obvious that the smoother the function  $f$  is, the smaller the variety of the sets will be. For instance, if  $E = \mathbb{R}$  and  $f$  is continuous on  $\mathbb{R}$ , the set  $E(f > \alpha)$  is always open.

**Problem 1.** Show that a constant function with a measurable domain is measurable.

*Solution.* Let  $f: E$  (measurable)  $\rightarrow \mathbb{R}^*$  be a constant function defined by  $f(x) = K$ , where  $K$  is a constant. We clearly note, for any real number  $\alpha$ , that

$$E(f > \alpha) = \begin{cases} E & \text{if } \alpha < K \\ \phi & \text{if } \alpha \geq K. \end{cases}$$

This implies that  $E(f > \alpha)$  is a measurable set since both the sets  $E$  and  $\phi$  are measurable.

The following theorem gives various alternatives to the definition of measurability of a function.

**1.2 Theorem.** Let  $f$  be an extended real-valued function defined on a measurable set  $E$  (of finite or infinite measure). Then the following statements are equivalent:

- (a)  $E(f > \alpha)$  is measurable for all  $\alpha \in \mathbb{R}$ .
- (b)  $E(f \geq \alpha)$  is measurable for all  $\alpha \in \mathbb{R}$ .
- (c)  $E(f < \alpha)$  is measurable for all  $\alpha \in \mathbb{R}$ .
- (d)  $E(f \leq \alpha)$  is measurable for all  $\alpha \in \mathbb{R}$ .

*Proof.* Since the class  $\mathcal{M}$  of all measurable sets in  $\mathbb{R}$  is a  $\sigma$ -algebra of sets, cf. III-4.13, the implications

$$(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (a)$$

\*A function whose values are in the set of extended real numbers is called an **extended real-valued function**.

are consequences of the following relations:

$$E(f \geq \alpha) = \bigcap_{n=1}^{\infty} \left( f > \alpha - \frac{1}{n} \right)$$

$$E(f < \alpha) = E - E(f \geq \alpha)$$

$$E(f \leq \alpha) = \bigcap_{n=1}^{\infty} E \left( f < \alpha + \frac{1}{n} \right)$$

$$E(f > \alpha) = E - E(f \leq \alpha),$$

each of which holds good for each real number  $\alpha$ . ■

**1.3 Corollary.** *An extended real-valued function  $f$  defined on a measurable set  $E$  is measurable if and only if one of the statements (a), (b), (c) or (d) in Theorem 1.2 holds.*

**Problem 2.** Show that the function  $\psi$  defined on  $\mathbf{R}$  by

$$\psi(x) = \begin{cases} x+5 & \text{if } x < -1 \\ 2 & \text{if } -1 \leq x < 0 \\ x^2 & \text{if } x \geq 0 \end{cases}$$

is a measurable function.

*Solution.* Let  $\alpha$  be any real number. Then

$$\mathbf{R}(\psi \leq \alpha) = \begin{cases} ]-\infty, \alpha-5[ & \text{if } \alpha < 0 \\ ]-\infty, -5[ \cup \{0\} & \text{if } \alpha = 0 \\ ]-\infty, \alpha-5[ \cup [0, \sqrt{\alpha}] & \text{if } 0 < \alpha < 2 \\ ]-\infty, \alpha-5[ \cup [-1, \sqrt{\alpha}] & \text{if } 2 \leq \alpha < 4 \\ ]-\infty, \sqrt{\alpha}] & \text{if } 4 \leq \alpha. \end{cases}$$

Since each of the sets on the right-hand side is measurable, the set  $\mathbf{R}(\psi \leq \alpha)$  is measurable for all real  $\alpha$ . Thus  $\psi$  is a measurable function.

**1.4 Theorem.** *If  $f$  is a measurable function defined on  $E$ , then the set  $\mathbf{R}(f = \alpha)$  is measurable for each extended real number  $\alpha$ .*

*Proof.* For a real number  $\alpha$ , we have

$$E(f = \alpha) = E(f \geq \alpha) \cap E(f \leq \alpha).$$

Since  $f$  is a measurable function, the set  $E(f = \alpha)$  is measurable for each real number  $\alpha$ . For  $\alpha = \infty$ , we have

$$E(f = \infty) = \bigcap_{n=1}^{\infty} E(f > n);$$

and for  $\alpha = -\infty$ ,

$$E(f = -\infty) = \bigcap_{n=1}^{\infty} E(f < -n).$$

Since the countable intersection of measurable sets is again a measurable set, the result follows in both the cases, i.e. when  $\alpha = \pm\infty$ .

*Remark.* The converse of Theorem 1.4 need not be true.

**1.5 Example.** Let  $P$  be a nonmeasurable set in  $E = ]0, 1[$ . Consider the function  $f$  with domain  $E$  and defined by

$$f(x) = \begin{cases} x^2 & \text{if } x \in P \\ -x^2 & \text{if } x \in E - P. \end{cases}$$

Then, for each real number  $\alpha$ , the set  $E(f = \alpha)$  consists of at most two elements and is hence measurable. However, the set  $E(f > 0)$  is, in fact, the set  $P$  which is nonmeasurable. Consequently,  $f$  is not a measurable function.

**Problem 3.** Let the function  $f: [0, 1] \rightarrow \mathbb{R}$  be defined by

$$f(x) = \begin{cases} \frac{1}{x} & \text{if } 0 < x < 1 \\ 5 & \text{if } x = 0 \\ 7 & \text{if } x = 1. \end{cases}$$

Prove that  $f$  is a measurable function.

## 2 PROPERTIES OF MEASURABLE FUNCTIONS

### 2.1 Theorem

- (a) If  $f$  is a measurable function on the set  $E$  and  $E_1 \subset E$  is measurable set, then  $f$  is a measurable function on  $E_1$ .
- (b) If  $f$  is a measurable function on each of the sets in a countable collection  $\{E_i\}$  of disjoint measurable sets, then  $f$  is measurable on  $\bigcup_{n=1}^{\infty} E_i$ .

- (c) If  $f$  and  $g$  are measurable functions on a common domain  $E$ , then the set

$$A(f, g) = \{x \in E : f(x) < g(x)\}$$

is measurable.

*Proof.* (a) for each real  $\alpha$ , we have

$$E_1(f > \alpha) = E(f > \alpha) \cap E_1.$$

The result follows as the set on the right-hand side is measurable.

(b) Write  $E = \bigcup_{i=1}^{\infty} E_i$ . Clearly,  $E$  is a measurable set, cf. III-4.10. The result now follows, since for each real  $\alpha$  we have

$$E(f > \alpha) = \bigcup_{i=1}^{\infty} E_i(f > \alpha).$$

(c) Define for each rational number  $r$ ,

$$A_r = \{x \in E : f(x) < r < g(x)\}.$$

The result now follows by observing that

$$A_r = \{x \in E : f(x) < r\} \cap \{x \in E : g(x) > r\}$$

and

$$A(f, g) = \bigcup_r A_r = \{x \in E : f(x) < g(x)\}. \blacksquare$$

### Problems

- If  $f$  and  $g$  are measurable functions on a common domain  $E$ , show that each of the sets  $\{x \in E : f(x) \leq g(x)\}$  and  $\{x \in E : f(x) = g(x)\}$  are measurable. [Hint : Use Theorem 2.1(c).]
- Let  $f$  be a function defined on  $E_1 \cup E_2$ , where  $E_1$  and  $E_2$  are measurable sets. Show that the function  $f$  is measurable on  $E_1 \cup E_2$  if and only if  $f_{1E_1}$  and  $f_{1E_2}$ ; that is, its restrictions to  $E_1$  and  $E_2$  are measurable.

*Solution.* Write  $f_1 = f_{1E_1}$ ,  $f_2 = f_{1E_2}$  and  $E = E_1 \cup E_2$ . Let  $\alpha$  be any number. If  $f$  is measurable on  $E$ , then the set  $E(f > \alpha)$  is measurable for each real  $\alpha$ . Also, one can observe that

$$E_1(f_1 > \alpha) = E(f > \alpha) \cap E_1.$$

Since both the sets on the right-hand side are measurable, the set  $E(f_1 > \alpha)$  is measurable. Hence  $f_1$  is a measurable function on  $E_1$ . Similarly, one can prove that  $f_2$  is a measurable function on  $E_2$ .

Conversely, if  $f_1$  and  $f_2$  are measurable functions, then that  $f$  is so follows from the relation

$$E(f > \alpha) = E_1(f_1 > \alpha) \cup E_2(f_2 > \alpha).$$

6. Let  $f$  be a measurable function with measurable domain  $E$ . Show that  $f$  is measurable if and only if the function  $g$  defined by  $g(x) = f(x)$ , for  $x \in E$  and  $g(x) = 0$ , for  $x \notin E$  is measurable. [Hint: Take  $E_1 = E$  and  $E_2 = E^c$  in Problem 5.]

**2.2 Theorem.** *Let  $f$  be a function defined on a measurable set  $E$ . Then  $f$  is measurable if and only if, for any open set  $G$  in  $\mathbb{R}$ , the inverse image  $f^{-1}(G)$  is a measurable set.*

*Proof.* First assume that  $f$  is a measurable function. Let  $G$  be an arbitrary open set in  $\mathbb{R}$ . Then it can be written as a countable union of disjoint open intervals. Suppose

$$G = \bigcup_{n=1}^{\infty} I_n,$$

where  $I_n = ]a_n, b_n[$ . Therefore

$$\begin{aligned} f^{-1}(G) &= \bigcup_{n=1}^{\infty} \{x \in E : f(x) \in I_n\} \\ &= \bigcup_{n=1}^{\infty} [E(f > a_n) \cap E(f < b_n)]. \end{aligned}$$

The measurability of  $f$  implies that the set  $f^{-1}(G)$  is a measurable set.

Conversely, assume that the set  $f^{-1}(G)$  is measurable for any arbitrary open set  $G$  in  $\mathbb{R}$ , in particular for  $G = ]\alpha, \infty[$ ,  $\alpha$  being any real number. But, for  $G = ]\alpha, \infty[$ , we observe that

$$f^{-1}(G) = E(f > \alpha).$$

Hence  $f$  is a measurable function. ■

### Problems

- Let  $f$  be a function defined on a measurable set  $E$ . Show that  $f$  is measurable if and only if the set  $E(f > r)$  is measurable for each rational number  $r$ .
- Let  $D$  be a dense set of real numbers. Let  $f$  be an extended real-valued function on  $\mathbb{R}$  such that  $\mathbb{R}(f > \alpha)$  is measurable for each  $\alpha \in D$ . Then  $f$  is measurable.

*Solution.* Let  $\alpha$  be any real number. Since  $D$  is dense in  $\mathbf{R}$ ,  $\alpha \in D$ . This implies that  $\exists$  a decreasing sequence  $\{\alpha_n\}$  in  $D$  converging to  $\alpha$ . Thus

$$\mathbf{R}(f > \alpha) = \bigcup_{n=1}^{\infty} \mathbf{R}(f > \alpha_n).$$

Since  $\mathbf{R}(f > \alpha_n)$  is measurable for each  $n$ , the set  $\mathbf{R}(f > \alpha)$  is measurable. Hence the function  $f$  is measurable.

9. Let  $f$  be an extended real-valued function with measurable domain  $D$  and let

$$D_1 = \{x \in D : f(x) = \infty\} \quad \text{and} \quad D_2 = \{x \in D : f(x) = -\infty\}.$$

Show that  $f$  is measurable if and only if  $D_1$  and  $D_2$  are measurable sets and the restriction of  $f$  to  $D - (D_1 \cup D_2)$  is measurable.

*Solution.* Let  $f$  be a measurable function. Write

$$D_1 = \{x \in D : f(x) = \infty\} = \bigcap_{n=1}^{\infty} \{x \in D : f(x) > n\}$$

and

$$D_2 = \{x \in D : f(x) = -\infty\} = \bigcap_{n=1}^{\infty} \{x \in D : f(x) < -n\}.$$

Note that  $D_1$  and  $D_2$  are measurable sets. Hence  $D_1 \cup D_2$  is also a measurable set. In view of Theorem 2.1(a),  $f$  is measurable on  $D - (D_1 \cup D_2)$ . The other part is obvious.

10. Show that any function defined on a set of measure zero is measurable.
11. Show that any function  $f : E \rightarrow \mathbf{R}^*$  defined on a measurable set  $E$  is measurable if and only if the following conditions are satisfied:
- The sets  $f^{-1}\{\infty\}$  and  $f^{-1}\{-\infty\}$  are measurable.
  - For each Borel set  $B$ ,  $f^{-1}(B)$  is measurable.

### 3 STEP FUNCTION

**3.1 Definition.** A function  $\varphi : [a, b] \rightarrow \mathbf{R}$  is said to be a **step function** if  $\exists$  a partition

$$\{a = x_0 < x_1 < x_2 < \dots < x_n = b\}$$

of the interval  $[a, b]$  such that in every subinterval  $]x_{i-1}, x_i]$ , the function  $\varphi$  is constant, i.e.

$$\varphi(x) = c_i, \quad \forall x \in ]x_{i-1}, x_i], \quad i = 1, 2, \dots, n.$$

*Note.* A step function is one which is defined in a closed interval assuming only a finite number of values. Each of these values is assumed by the step function in an open subinterval. We can disregard the values of the function at the partitioning points  $x_i$  or assign values to the function thereat arbitrarily, since these are finite in number and as such form a set of measure zero (finite sets or more generally sets of measure zero, if omitted, do not matter in our consideration\*).

### 3.2 Examples

1. A function  $f: [a, b] \rightarrow \mathbb{R}$  defined by

$$f(x) = \begin{cases} \alpha & \text{if } a \leq x < c \\ \beta & \text{if } c \leq x \leq b, \end{cases}$$

where  $\alpha$  and  $\beta$  are constants, is a step function.

2. The Signum function  $S$  defined by

$$S(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

is a step function.

3. The greatest integer function  $f$  defined, on any open interval, by  $f(x) = [x]$  is a step function.  
 4. A function  $f: \mathbb{R} \rightarrow [0, 1[$  defined by

$$f(x) = x - [x]$$

is a step function.

As an immediate consequence of Problem 1 and Theorem 2.1(b), we have the following corollary.

**3.3 Corollary.** *A step function is a measurable function.*

## 4 OPERATIONS ON MEASURABLE FUNCTIONS

**4.1 Theorem.** *Let  $f$  and  $g$  be measurable functions on  $E$ , and  $c$  be a constant. Then each of the following functions is measurable on  $E$ :*

- (a)  $f \pm c$
- (b)  $cf$
- (c)  $f + g$

\*The Lebesgue integral of a function over a set of measure zero is zero, cf. V-4.3.

- (d)  $f - g$   
 (e)  $|f|$   
 (f)  $f^2$   
 (g)  $fg$   
 (h)  $f \div g$  ( $g$  vanishes nowhere on  $E$ ).

*Proof.* Let  $\alpha$  be an arbitrary real number.

- (a) Since  $f$  is measurable and

$$E(f \pm c > \alpha) = E(f > \alpha \pm c),$$

the function  $f \pm c$  is measurable.

(b) Assume that  $c \neq 0$  since in case  $c = 0$  the measurability of the function  $cf$  follows from Problem 1. The measurability of  $cf$  for  $c \neq 0$  follows from the obvious relation

$$E(cf > \alpha) = \begin{cases} E(f > \alpha/c) & \text{if } c > 0 \\ E(f < \alpha/c) & \text{if } c < 0. \end{cases}$$

- (c) Consider the set

$$E(f + g > \alpha) = \{x \in E : f(x) > \alpha - g(x)\}.$$

Since  $g$  is a measurable function,  $\alpha - g$  is measurable in view of (a) and (b). Hence  $f + g$  is measurable in view of Theorem 2.1(c).

- (d) It is obvious in view of (b) and (c), since

$$f - g = f + (-g).$$

- (e) Observe that

$$E(|f| > \alpha) = \begin{cases} E & \text{if } \alpha < 0 \\ E(f > \alpha) \cup E(f < -\alpha) & \text{if } \alpha \geq 0. \end{cases}$$

Both the sets on the right-hand side are measurable and hence  $|f|$  is measurable.

- (f) From the relation

$$E(f^2 > \alpha) = \begin{cases} E & \text{if } \alpha < 0 \\ E(|f| > \sqrt{\alpha}) & \text{if } \alpha \geq 0 \end{cases}$$

and using (e), it follows that  $f^2$  is a measurable function.

- (g) This follows from the identity

$$fg = \frac{1}{4} [(f+g)^2 - (f-g)^2],$$

in view of (b), (c), (d) and (f) above.

(h) In view of the identity

$$\frac{f}{g} = f \frac{1}{g}$$

and (g) above, it is sufficient to prove that  $1/g$ , where  $g \neq 0$  on  $E$ , is measurable. We note that

$$E\left(\frac{1}{g} > \alpha\right) = \begin{cases} E(g > 0) & \text{if } \alpha = 0 \\ E\left(g < \frac{1}{\alpha}\right) \cap E(g < 0) & \text{if } \alpha > 0 \\ E(g > 0) \cup E(g < 0) \cap E\left(g < \frac{1}{\alpha}\right) & \text{if } \alpha \leq 0. \end{cases}$$

This proves that in either case,  $1/g$  is a measurable function. The proof of the theorem is now complete.

*Note.* The results in Theorem 4.1 hold good for extended real-valued functions except that  $f+g$  is not defined when  $f = \infty$  and  $g = -\infty$  or vice-versa, for then

$$E(f+g > \alpha) = \bigcup_{i=1}^{\infty} [E(f > r_i) \cap E(g > \alpha - r_i)] \\ \cup [E(f = \infty) - E(g = -\infty)] \\ \cup [E(g = \infty) - E(f = -\infty)]$$

a measurable set.

*Remark.* The converse of Theorem 4.1(a) may not be true.

**4.2 Example.** Let  $P$  be a non-measurable subset of  $E = [0, 1]$ . Define a function  $f: E \rightarrow \mathbb{R}$  by

$$f(x) = \begin{cases} 1 & \text{if } x \in P \\ -1 & \text{if } x \notin P. \end{cases}$$

The function  $f$  is not measurable, since  $E(f > 0) (= P)$  is a non-measurable set. But  $|f|$  is measurable as the set

$$E(|f| > \alpha) = \begin{cases} E & \text{if } \alpha < 1 \\ \phi & \text{if } \alpha \geq 1 \end{cases}$$

is measurable.

### Problems

12. Are the results of (c) and (d) in Theorem 4.1 true for an infinite number of measurable functions? Explain.

13. If  $f$  is measurable, prove that any positive integral power of  $f$  is also measurable.

**4.3 Definition.** Let  $f_1$  and  $f_2$  be real-valued functions with common domain  $E$ . Then the functions  $f^* = \max(f_1, f_2)$  and  $f_* = \min(f_1, f_2)$  are defined to be the real-valued functions on  $E$ , where values at any point  $x \in E$  are given by

$$f^*(x) = \max(f_1(x), f_2(x))$$

and

$$f_*(x) = \min(f_1(x), f_2(x)),$$

respectively.

**4.4 Theorem.** If  $f_1$  and  $f_2$  are measurable functions, then the functions  $f^*$  and  $f_*$  are measurable.

*Proof.* Let  $\alpha$  be any real number. We note that

$$E(f^* > \alpha) = E(f_1 > \alpha) \cup E(f_2 > \alpha)$$

$$E(f_* > \alpha) = E(f_1 > \alpha) \cap E(f_2 > \alpha).$$

Since  $f_1$  and  $f_2$  are measurable functions, the sets  $E(f_1 > \alpha)$  and  $E(f_2 > \alpha)$  are measurable. Also, the union and intersection of two measurable sets are again measurable sets. This proves that  $f^*$  and  $f_*$  are measurable functions. ■

**4.5 Definition.** Let  $f$  be a real-valued function. Then its *positive part*, written  $f^+$ , and its *negative part*, written  $f^-$ , are defined to be the non-negative functions given by

$$f^+ = \max(f, 0)$$

and

$$f^- = \max(f, 0),$$

respectively.

We observe that

$$f = f^+ - f^-$$

and

$$|f| = f^+ + f^-.$$

**4.6 Theorem.** A function is measurable if and only if both its positive and negative parts are measurable functions.

**Problem 14.** Work out the details of the proof of Theorem 4.6.

Let  $\{f_i\}$  be a sequence of functions defined on a common domain  $E$ . Then the function  $\max(f_1, f_2, \dots, f_n)$  and  $\min(f_1, f_2, \dots, f_n)$ , for any finite  $n$ , can be defined similarly as in Definition 4.3. Also, in the case when  $n$  is infinite, the functions  $\sup_n f_n$  and  $\inf_n f_n$  could be composed of likewise. We shall denote by  $\sup_n f_n$ , the function whose value at  $x \in E$  is the supremum of  $\{f_1(x), f_2(x), \dots\}$ . We further denote by  $\limsup_n f_n$ , the function whose value at  $x$  is  $\limsup_n f_n(x)$ . Similar notational agreements apply to  $\inf_n f_n$  and  $\liminf_n f_n$ . We may note that

$$\limsup_n f_n = \inf_n \left( \sup_{k \geq n} f_k \right) \tag{1}$$

$$\inf_n f_n = - \sup_n (-f_n) \tag{2}$$

$$\liminf_n f_n = - \limsup_n (-f_n) = \sup_n \left( \inf_{k \geq n} f_k \right). \tag{3}$$

*Note.* We may be using frequently the notations  $\overline{\lim}$  and  $\underline{\lim}$  in place of  $\limsup$  and  $\liminf$ , respectively.

**4.7 Theorem.** Let  $\{f_n\}$  be a sequence of measurable functions (with the same domain  $E$  of definition). Then the functions  $\max\{f_1, f_2, \dots, f_n\}$ ,  $\min\{f_1, f_2, \dots, f_n\}$ ,  $\sup_n f_n$ ,  $\inf_n f_n$ ,  $\limsup_n f_n$  and  $\liminf_n f_n$  are all measurable.

*Proof.* In view of Eqs. (1)–(3), it will suffice to prove that  $\max\{f_1, f_2, \dots, f_n\}$  and  $\sup_n f_n$  are measurable functions.

Let  $h = \max\{f_1, f_2, \dots, f_n\}$ . We note that

$$E(h > \alpha) = \bigcup_{i=1}^n E(f_i > \alpha).$$

Measurability of each  $f_i$  implies that the function  $h$  is measurable.

Similarly, by writing  $g$  as  $\sup_n f_n$  and observing that

$$E(g > \alpha) = \bigcup_{i=1}^{\infty} E(f_i > \alpha),$$

the measurability of the function  $g$  follows. ■

**4.8 Corollary.** *If  $\{f_n\}$  is a sequence of measurable functions converging to  $f$  on  $E$ , then  $f$  is a measurable function.*

*Proof.* Since  $f_n \rightarrow f$ , we have

$$\overline{\lim} f_n = \underline{\lim} f_n = f.$$

In view of Theorem 4.7,  $f$  is a measurable function. ■

**4.9 Corollary.** *The set of points on which a sequence  $\{f_n\}$  of measurable functions converges is measurable.*

*Proof.* The required set

$$\{x : \overline{\lim} f_n(x) - \underline{\lim} f_n(x) = 0\}$$

is, clearly, measurable. ■

*Note.* We may conclude that all operations of analysis, including limit operations, when applied to measurable functions lead to a measurable function.

**Problem 15.** Does there exist a nonmeasurable function on  $[a, b]$ ? Justify your answer.

## 5 CHARACTERISTIC FUNCTION

**5.1 Definition.** Let  $E$  be a given set. If  $A$  is a subset of  $E$ , then the characteristic function  $\chi_A$  of  $A$  is a real-valued function defined on  $E$  by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

*Note.* The characteristic function  $\chi_A$  of the set  $A$  is also called the indicator function of  $A$ .

A few simple properties of a characteristic function are given in the following theorem, the proof of which is straightforward and hence omitted.

**5.2 Theorem.** *Let  $A$  and  $B$  be subsets of  $E$ . Then:*

- (a)  $\chi_\emptyset = 0$  and  $\chi_E = 1$ .
- (b)  $A \subset B \Rightarrow \chi_A \leq \chi_B$ .

$$(c) \chi_{A \cup B} = \chi_A + \chi_B - \chi_{A \cap B}.$$

$$(d) \chi_{A \cap B} = \chi_A \cdot \chi_B.$$

$$(e) \chi_{A^c} = 1 - \chi_A.$$

$$(f) \chi_A = \sum_{n=1}^{\infty} \chi_{A_n}, \text{ where } A = \bigcup_n A_n \text{ and the sequence } \{A_n\} \text{ consists of disjoint subsets of } E.$$

$$(g) \chi_{A-B} = \chi_A - \chi_{A \cap B}.$$

$$(h) \chi_{A \times B} = \chi_A \cdot \chi_B. \text{ (The set } B \text{ can be considered to be a subset of some other set } F \text{.)}$$

We define the measurability of a function  $f$  over a set  $E$ , provided the domain set  $E$  is measurable. If  $E$  is a nonmeasurable set, the function  $f$  is nonmeasurable. But it does not mean that if  $E$  is a measurable set, the function  $f$  is always measurable. In fact, we prove the following theorem.

**5.3 Theorem.** *Let  $E$  be a measurable set. Then the set  $A \subseteq E$  and its characteristic function  $\chi_A$  are simultaneously measurable or nonmeasurable.*

*Proof.* Let  $\alpha$  be any real number. Then

$$E(\chi_A > \alpha) = \begin{cases} \phi & \text{if } \alpha \geq 1 \\ A & \text{if } 0 \leq \alpha < 1 \\ E & \text{if } \alpha < 0. \end{cases}$$

If  $A$  is a measurable set, then the set  $E(\chi_A > \alpha)$  is measurable for each real number  $\alpha$  and hence  $\chi_A$  is a measurable function. On the other hand, if  $\chi_A$  is a measurable function, the set  $A$  turns out to be measurable by choosing an  $\alpha$  such that  $0 \leq \alpha < 1$ . ■

*Remark.* Theorem 5.3 asserts that the characteristic function of nonmeasurable sets are nonmeasurable even though the domain set is measurable.

## 6 SIMPLE FUNCTION

A very useful concept in the theory of measurable functions is that of a simple function. A function is said to be a simple function if it assumes only a finite number of values, each of them being finite. We put more precisely as follows.

**6.1 Definition.** A function  $f: E \rightarrow \mathbb{R}$  is said to be a simple function

if there is a finite disjoint class  $\{E_1, E_2, \dots, E_n\}$  of measurable sets with  $\bigcup_{i=1}^n E_i = E$  and a finite set  $\{a_1, a_2, \dots, a_n\}$  of real numbers such that

$$f(x) = a_i, x \in E_i, 1 \leq i \leq n.$$

Let  $f$  be a simple function as described above. Then, we have

$$f(x) = \sum_{i=1}^n a_i \chi_{E_i(x)},$$

where  $\chi_{E_i}$  is the characteristic function of the measurable set  $E_i$ . Clearly, the sets  $E_i = \{x \in E : f(x) = a_i\}$ ,  $i = 1, 2, \dots, n$  form a partition of  $E$ . It is easy to verify that a simple function is always measurable.

## 6.2 Examples

1. Each step function is a simple function.
2. Each characteristic function of a measurable set is a simple function.
3. Any linear combination of the form  $\sum_{i=1}^n a_i \chi_{E_i}$  represents a simple function defined on the set  $E = \bigcup_{i=1}^n E_i$ . (The sets  $E_i$ 's are measurable but not necessarily disjoint.)

*Note.* The representation of the function  $\varphi$  described in Example 3 above is not unique. On the other hand, if  $\varphi$  is a simple function defined on a set  $E$  and the set  $\{a_1, a_2, \dots, a_m\}$  its range, then

$$\varphi = \sum_{i=1}^m a_i \chi_{A_i}, \quad (4)$$

where  $A_i = \{x \in E : \varphi(x) = a_i\}$ ,  $i = 1, 2, \dots, m$ . This representation of  $\varphi$  is called the **canonical representation** and it is characterized by the fact that the  $A_i$ 's are disjoint and the  $a_i$ 's distinct and nonzero.

**Problem 16.** Prove that the product of two simple functions and any finite linear combination of simple functions are again simple functions.

## 7 CONTINUOUS FUNCTION

**7.1 Theorem.** *A continuous function defined on a measurable set is measurable.*

*Proof.* Let  $f$  be a function defined and continuous on  $E$  (measurable). Let  $\alpha$  be any arbitrary real number. Consider the set

$$A = \{x \in E : f(x) \leq \alpha\}.$$

Let  $x_0$  be a limit point of  $A$ . Then,  $\exists$  a sequence  $\{x_n\}$  of points in  $A$  with  $x_n \rightarrow x_0$ . Therefore,

$$f(x_n) \leq \alpha, \quad \forall n.$$

By the continuity of  $f$  at  $x_0$ , we have

$$f(x_0) \leq \alpha$$

and, as such,  $x_0 \in A$ . Hence the set  $A$  is closed and, therefore, measurable. This proves that  $f$  is a measurable function. ■

*Remark.* The converse of Theorem 7.1 may not be true; more precisely, a measurable function need not be continuous.

**7.2 Example.** Consider a function  $f : \mathbb{R} \rightarrow \{0, 1\}$  given by

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, the function  $f$  is measurable but not continuous. The point at  $x = 0$  is a point of discontinuity of  $f$ .

**7.3 Theorem.** *If  $\varphi$  is a measurable function on a set  $E$  (measurable) and  $f$  a function defined and continuous on the range of  $\varphi$ , then  $f \circ \varphi$  is a measurable function on  $E$ .*

*Proof.* Let  $\alpha$  be an arbitrary real number. Then

$$E(f \circ \varphi > \alpha) = \{x \in E : \varphi(x) \in G\},$$

where  $G = \{u : f(u) > \alpha\}$ . But  $G$  is an open set since  $f$  is continuous. Hence, in view of Theorem 2.2, the set  $E(f \circ \varphi > \alpha)$  is measurable. This proves that  $f \circ \varphi$  is a measurable function. ■

**Remark.** The result in Theorem 7.3 may not be true in reverse order; that is, a measurable function of a continuous function need not be measurable.

**7.4 Example.** Consider the Cantor ternary set  $C$ , and the increasing and continuous Cantor function  $f$  defined on  $[0, 1]$ . Let  $g$  be defined on  $[0, 1]$  by  $g(x) = f(x) + x$ . The function  $g$  so defined is clearly strictly increasing and continuous on  $[0, 1]$ . Hence  $g^{-1}$  exists with domain  $[0, 2]$  and range  $[0, 1]$ . Moreover,  $g^{-1}$  is continuous, and, therefore, measurable on its domain.

Let  $D$  denote the image of Cantor set  $C$  under the function  $g$ . Then  $D \subset [0, 2]$ . Since  $[0, 2] - C$  is an open set, it is a countable union of disjoint open intervals  $\{]a_i, b_i[ \}$ . On each such interval, the Cantor function  $f$  has a constant value, say,  $\alpha_i$ . Therefore

$$g(]a_i, b_i[) = ]a_i + \alpha_i, b_i + \alpha_i[$$

and

$$m(]a_i, b_i[) = m(]a_i + \alpha_i, b_i + \alpha_i[).$$

This further gives that

$$m([0, 2] - D) = m([0, 1] - C).$$

Hence,  $m(D) = 1$ . This implies that  $g$  maps a set of measure zero (Cantor set  $C$ ) onto a set having a nonzero measure.

Now, suppose  $F$  denotes a nonmeasurable subset of  $D$ . Denote by  $h$  the characteristic function of  $F$ . Clearly,  $h$  is a nonmeasurable function.

Define a function  $\varphi$  on  $[0, 1]$  by  $\varphi = h \circ g$  and having range consisting of two numbers 0 and 1. If  $\varphi(x) = 1$ , then  $g(x) \in F$  which implies  $x \in C$  since  $F \subset D$ . Therefore  $\{x : \varphi(x) = 1\} \subset C$  and hence of measure zero. This implies  $\varphi$  is equal to zero function almost everywhere. Accordingly, by Theorem 8.6,  $\varphi$  is a measurable function on  $[0, 1]$ .

Thus, the function  $h = \varphi \circ g^{-1}$  is an example of a nonmeasurable function that is a measurable function of a continuous function.

**7.5 Corollary.** *If  $f$  is a measurable function, then so are  $|f|$ ,  $|f|^p$  ( $p > 0$ ),  $\exp(cf)$ ,  $f^+$  and  $f^-$ .*

**Proof.** It follows from Theorem 7.3 as each of the functions  $\varphi(t) = |t|$ ,  $|t|^p$  ( $p > 0$ ),  $\exp(ct)$ ,  $t^+$  and  $t^-$  is a continuous function. ■

**Note.** Some of the results in Corollary 7.5 are obtained independently in § 4.

**Problem 17.** Show by an example that a measurable function of a measurable function may not be measurable.

**7.6 Theorem.** *Let  $f$  and  $g$  be real-valued functions defined and measurable on  $\mathbb{R}$ . Then the real-valued function*

$$h(x) = F(f(x), g(x)), \quad x \in \mathbb{R}$$

*defined and continuous on the Euclidean plane  $\mathbb{R}^2$  is measurable.*

*Proof.* Let  $\alpha$  be an arbitrary real number. Consider the set

$$G_\alpha = \{(u, v) : F(u, v) > \alpha\}, \quad u = f(x) \text{ and } v = g(x).$$

Obviously,  $G_\alpha$  is an open subset of  $\mathbb{R}^2$  and hence can be expressed as a countable union of open sets  $I_n$ , that is,

$$G_\alpha = \bigcup_{n=1}^{\infty} I_n,$$

$$\begin{aligned} \text{where } I_n &= \{(u, v) : a_n < u < b_n, c_n < v < d_n\} \\ &= \{(f(x), g(x)) : a_n < f(x) < b_n, c_n < g(x) < d_n\}. \end{aligned}$$

Now

$$\{x : (f(x), g(x)) \in I_n\} = \{x : a_n < f(x) < b_n\} \cap \{x : c_n < g(x) < d_n\}.$$

Since the functions  $f$  and  $g$  are measurable, the set

$$\{x : (f(x), g(x)) \in I_n\}$$

is measurable. Further, we note that

$$\begin{aligned} \{x : h(x) > \alpha\} &= \{x : (f(x), g(x)) \in G_\alpha\} \\ &= \bigcup_{n=1}^{\infty} \{x : (f(x), g(x)) \in I_n\}. \end{aligned}$$

Hence,  $h$  is a measurable function. ■

**7.7 Corollary.** *If  $f$  and  $g$  are measurable functions, then so are  $f+g$  and  $fg$ .*

*Note.* The results in Corollary 7.7 are already obtained independently in § 4.

## 8 SETS OF MEASURE ZERO

Sets of measure zero are just the sets that are negligible in the theory of Lebesgue integration.

**8.1 Definition.** A property  $\mathcal{P}$  is said to hold good almost everywhere (abbreviated a.e.) on a set  $S$  if the set of points of  $S$  where  $\mathcal{P}$  fails to hold has measure zero.

### 8.2 Examples

1. If a function  $f$  defined on a set  $E$  is discontinuous on a subset of  $E$

whose measure is zero and is continuous elsewhere on  $E$ , then we say that the function  $f$  is continuous a.e. on  $E$ .

2. Let  $f$  be a function defined on  $\mathbb{R}$  by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ 1 & \text{if } x \text{ is rational.} \end{cases}$$

Then  $f(x) = 0$  a.e. This can equivalently be written as  $f = 0$  p.p. (presque partout).

**8.3 Definition.** Two functions  $f$  and  $g$  defined on the same domain  $E$  are said to be equivalent on  $E$ , written  $f \sim g$  on  $E$ , if  $f = g$  a.e. on  $E$ ; i.e.,  $f(x) = g(x)$  for all  $x \in E - E_1$ , where  $E_1 \subset E$  with  $m(E_1) = 0$ .

**8.4 Theorem.** Let  $f$  and  $g$  be two equivalent functions defined on a set  $E$ . If  $f$  and  $g$  are continuous on  $E$ , then they coincide throughout  $E$ .

*Proof.* Suppose  $x_0 \in E$  be a point such that  $f(x_0) \neq g(x_0)$ . Then, since  $f - g$  is continuous at  $x_0$ ,  $\exists$  a neighbourhood  $N_{x_0}$  of  $x_0$  in which  $f - g$  is nonzero. However, the measure of  $N_{x_0}$  is positive and so

$$m\{x : f(x) \neq g(x)\} > 0.$$

Thus,  $f$  and  $g$  cannot be equivalent. This is in contradiction with the hypothesis. Hence the result follows. ■

*Remark.* Two continuous functions cannot be equivalent if they differ even at a single point of the domain. However, discontinuous functions can obviously be equivalent without being identical.

**8.5 Example.** The function  $f$  defined in Example 2 in 8.2 is equivalent to the function  $g \equiv 0$ .

**8.6 Theorem.** Let  $f$  and  $g$  be two functions defined on a common domain  $E$  such that  $f = g$  a.e. and  $g$  is measurable. Then  $f$  is measurable.

*Proof.* Let  $E_1 = \{x \in E : f(x) = g(x)\}$  and  $E_2 = \{x \in E : f(x) \neq g(x)\}$ . Then  $E = E_1 \cup E_2$  and  $m(E_2) = 0$ .

Let  $\alpha$  be any arbitrary real number. Consider the set

$$A = \{x \in E : f(x) > \alpha\}.$$

Since  $A \cap E_2 \subset E_2$ , it follows that  $m^*(A \cap E_2) = 0$ . Also,

$$A \cap E_1 = \{x \in E : g(x) > \alpha\} \cap E_1,$$

whence  $A \cap E_1$  is measurable.

The measurability of  $f$  follows from the relation

$$A = (A \cap E_1) \cup (A \cap E_2). \blacksquare$$

**8.7 Theorem.** *If a function  $f$  defined on  $E$  is continuous a.e., then  $f$  is measurable on  $E$ .*

*Proof.* It runs on the lines of proof of Theorem 7.1 and hence left to the reader.  $\blacksquare$

Since the behaviour of measurable functions on sets of measure zero is often unimportant, it is natural to introduce the following generalization of the ordinary notion of convergence of a sequence of functions.

**8.8 Definition.** A sequence  $\{f_n\}$  of functions defined on  $E$  is said to converge a.e. to a function  $f$  if

$$\lim_{n \rightarrow \infty} f_n(x) = f(x),$$

for all  $x \in E - E_1$ , where  $E_1 \subset E$  with  $m(E_1) = 0$ .

**8.9 Example.** The sequence  $\{f_n\}$  given by  $f_n(x) = (-1)^n x^n$ ,  $x \in [0, 1]$  converges a.e. to the function  $f \equiv 0$ , in fact, everywhere except at point  $x = 1$ .

**8.10 Theorem.** *If a sequence  $\{f_n\}$  of measurable functions converges a.e. to the function  $f$ , then  $f$  is measurable.*

*Proof.* It follows from Corollary 4.8 and Definition 8.8.  $\blacksquare$

### Problems

18. Let  $f$  be a measurable function. Then  $\inf \{\alpha : f \leq \alpha \text{ a.e.}\}$  is called the essential supremum of  $f$ , denoted by  $\text{ess sup } f$ , and  $\sup \{\alpha : f \geq \alpha \text{ a.e.}\}$  the essential infimum of  $f$ , denoted by  $\text{ess inf } f$ . Prove that  $f \leq \text{ess sup } f$  a.e. and  $\text{ess sup } f = -\text{ess inf } (-f)$ . Also prove that if  $f$  and  $g$  are any two measurable functions, then  $\text{ess sup } (f+g) \leq \text{ess sup } f + \text{ess sup } g$  and give an example of strict inequality. [Hint: For inequality, take  $f = \chi_{[-1, 0]} - \chi_{[0, 1]}$  and  $g = -f$ . Then inequality reads :  $2 > 0$ .]
19. Let  $f = g$  a.e., where  $g$  is a measurable and  $f$  a continuous function. Then show that

$$\text{ess sup } f = \text{ess sup } g = \sup f.$$

**9. BOREL MEASURABLE FUNCTION**

In analogy with Definition 1.1, we define the Borel measurable function as follows.

**9.1 Definition.** A function  $f$  defined on a set  $E$  is said to be a **Borel measurable function**, or more precisely, **Borel function** on  $E$  if the set  $E(f > \alpha)$  is a Borel set for all real numbers  $\alpha$ .

As an immediate consequence of the definition, we note that *every Borel measurable function is Lebesgue measurable*. Now the question arises: Is every Lebesgue measurable function Borel measurable? The answer is **no**.

**9.2 Example.** The characteristic function of a set which is Lebesgue measurable and non-Borel is Lebesgue measurable but not a Borel measurable function. (A Lebesgue non-Borel measurable set is given in Example III-5.6.)

*Remark.* If we replace a Lebesgue measurable set by a Borel set and Lebesgue measurable function by Borel measurable function in Theorems 1.2, 4.1, 4.4 and 4.7, the results hold good. In fact, one can prove the following results.

**9.3 Theorem.** *Let  $f$  be an extended real-valued function defined on  $E$ . Then the following statements are equivalent:*

- (a)  $E(f > \alpha)$  is a Borel set for all  $\alpha \in \mathbb{R}$ .
- (b)  $E(f \geq \alpha)$  is a Borel set for all  $\alpha \in \mathbb{R}$ .
- (c)  $E(f < \alpha)$  is a Borel set for all  $\alpha \in \mathbb{R}$ .
- (d)  $E(f \leq \alpha)$  is a Borel set for all  $\alpha \in \mathbb{R}$ .

**9.4 Theorem.** *Let  $f$  and  $g$  be Borel measurable functions on  $E$ , and let  $c$  be any constant. Then each of the following functions is Borel measurable on  $E$ :*

- (a)  $f \pm c$
- (b)  $cf$
- (c)  $f + g$
- (d)  $f - g$
- (e)  $|f|$
- (f)  $f^2$
- (g)  $fg$
- (h)  $f \div g$  ( $g$  vanishes nowhere in  $E$ ).

**9.5 Theorem.** *If  $f_1$  and  $f_2$  are Borel measurable functions, then the functions  $f^*$  and  $f_*$  are Borel measurable.*

**9.6 Theorem.** *Let  $\{f_n\}$  be a sequence of Borel measurable functions (with the same domain  $E$  of definition). Then the functions  $\max\{f_1, f_2, \dots, f_n\}$ ,  $\min\{f_1, f_2, \dots, f_n\}$ ,  $\sup_n f_n$ ,  $\inf_n f_n$ ,  $\limsup_n f_n$ , and  $\liminf_n f_n$  are all Borel measurable.*

*Remark.* Theorem 8.6 cannot be extended to Borel measurable functions.

**9.7 Example.** In Example III-5.6, we note that  $\exists$  sets of measure zero which are not Borel-measurable. Let  $E$  be one such set, i.e.,  $m(E) = 0$  and  $E$  not a Borel measurable set. Define

$$g = \chi_E \quad \text{and} \quad f \equiv 0.$$

Clearly  $f$  is a Borel measurable function while  $g$  is not even though  $f = g$  a.e.

**9.8 Theorem.** *A continuous function is Borel measurable. The converse may not be true.*

*Proof.* As it runs on the lines of proof of Theorem 7.1, it is left to the reader. ■

In Theorem 2.2, it is proved that inverse image of an open set  $G$  under a measurable function is a measurable set. If we take the set  $G$  simply to be a measurable set, then  $f^{-1}(G)$ , where  $f$  remains a measurable function, need not be a measurable set.

**9.9 Example.** Let  $\varphi$  be a Cantor function and  $P$  be a nonmeasurable set in  $[0, 1]$ . Then  $A = \varphi(P)$  lies in  $C$ , the Cantor set, and has a measure 0. So  $A$  is measurable. However,  $\varphi$  is one-one and onto, so  $\varphi^{-1}(A) = P$ , a nonmeasurable set.

However, if  $f$  is a Borel measurable function and  $G$  a Borel set, we have the following.

**9.10 Theorem.** *If  $f$  is a Borel measurable function and  $B$  a Borel set, then  $f^{-1}(B)$  is a Borel set.*

*Proof.* It is left to the reader as an exercise. ■

**9.11 Theorem.** *If  $f$  and  $g$  are Borel measurable functions, so is  $f \circ g$ .*

*Proof.* Let  $\alpha$  be an arbitrary real number. Then

$$\begin{aligned}\{x: (f \circ g)(x) > \alpha\} &= \{x: f(g(x)) > \alpha\} \\ &= \{x: g(x) \in A\} = g^{-1}(A),\end{aligned}$$

where  $A = \{u: f(u) > \alpha\}$ . The set  $A$  is, clearly, a Borel set since  $f$  is a Borel function. Therefore  $g^{-1}(A)$  is a Borel set, cf. Theorem 9.10. Hence  $f \circ g$  is a Borel function as the set  $\{x: (f \circ g)(x) > \alpha\}$  is Borel. ■

*Remark.* Compare Theorem 9.11 with Problem 17.

**9.12 Theorem.** *If  $f$  is a Borel measurable function and  $g$  a Lebesgue-measurable function then  $f \circ g$  is a Lebesgue measurable function. In other words, a Borel measurable function of a Lebesgue measurable function is a Lebesgue measurable function.*

*Proof.* It is enough to establish: “If  $f$  is a Lebesgue measurable function and  $B$  a Borel set, then  $f^{-1}(B)$  is a Lebesgue measurable set.” For this recall

$$f^{-1}\left(\bigcup_i A_i\right) = \bigcup_i f^{-1}(A_i)$$

and

$$f^{-1}(A^c) = (f^{-1}(A))^c.$$

Thus, the class of sets whose inverse images under  $f$  are measurable forms a  $\sigma$ -algebra. But this class clearly contains the intervals. Hence it must contain all the Borel sets. ■

*Remark.* The result in Theorem 9.12 is stronger than that in Theorem 7.3.

### Problems

20. Show that to every measurable function  $f$  there corresponds a Borel measurable function  $g$  such that  $g = f$  a.e.
21. If  $f$  is an increasing function on  $\mathbb{R}$ , then prove that  $f$  is a Borel measurable function.
22. Prove Theorem 9.8.
23. Prove Theorem 9.10.

## 10 SEQUENCE OF FUNCTIONS

**10.1 Theorem.** *Let  $E$  be a measurable set with  $m(E) < \infty$ , and  $\{f_n\}$  a sequence of measurable functions defined on  $E$ . Let  $f$  be a measurable (real-valued) function such that  $f_n(x) \rightarrow f(x)$  for each  $x \in E$ . Then given*

$\epsilon > 0$  and  $\delta > 0$ , there is a measurable set  $A \subset E$  with  $m(A) < \delta$  and an integer  $N$  such that

$$|f_n(x) - f(x)| < \epsilon,$$

for all  $x \in E - A$  and all  $n \geq N$ .

*Proof.* Consider the sets

$$G_n = \{x : x \in E, |f_n(x) - f(x)| \geq \epsilon\}.$$

Since the functions  $f_n$  and  $f$  are measurable, the sets  $G_n$  are measurable. Define

$$\begin{aligned} E_k &= \bigcup_{n=k}^{\infty} G_n \\ &= \{x : x \in G_n \text{ for some } n \geq k\} \\ &= \{x : x \in E, |f_n(x) - f(x)| \geq \epsilon \text{ for some } n \geq k\}. \end{aligned}$$

We observe that  $E_{k+1} \subseteq E_k$ , and for each  $x \in E$ , there must be some set  $E_k$  such that  $x \notin E_k$ . On the contrary, if we assume that  $x \in E_k$  for all  $k$ , then for any given fixed  $k$ , we must have

$$|f_n(x) - f(x)| \geq \epsilon \quad \text{for some } n \geq k.$$

This would lead to  $f_n(x) \not\rightarrow f(x)$ . Thus,  $\{E_k\}$  is a decreasing sequence of measurable sets for which  $\bigcap_{k=1}^{\infty} E_k = \phi$ . Therefore, in view of corollary III-6.3, we have

$$\lim_{n \rightarrow \infty} m(E_n) = 0.$$

Hence, given a  $\delta > 0$ ,  $\exists$  an integer  $N$  such that  $m(E_k) < \delta$ , for some  $k \geq N$ . In particular,  $m(E_N) < \delta$ , i.e.,

$$m\{x : x \in E, |f_n(x) - f(x)| \geq \epsilon \text{ for some } n \geq N\} < \delta.$$

If we write  $A = E_N$ , then  $m(A) < \delta$  and

$$E - A = \{x : x \in E, |f_n(x) - f(x)| < \epsilon \text{ for all } n \geq N\}.$$

In other words,

$$|f_n(x) - f(x)| < \epsilon, \quad \forall n \geq N \text{ and for } x \in E - A. \blacksquare$$

A trivial modification of Theorem 10.1 is the following result.

**10.2 Theorem.** Let  $E$  be a measurable set with  $m(E) < \infty$  and  $\{f_n\}$  a sequence of measurable functions converging a.e. to a real-valued function  $f$  defined on  $E$ . Then, given  $\epsilon > 0$  and  $\delta > 0$ , there is a set  $A \subset E$  with  $m(A) < \delta$  and an integer  $N$  such that

$$|f_n(x) - f(x)| < \epsilon,$$

for all  $x \in E - A$  and all  $n \geq N$ .

*Proof.* Let  $F$  be the set of points of  $E$  for which  $f_n \not\rightarrow f$ . Then  $m(F) = 0$  and  $f_n(x) \rightarrow f(x)$  for all  $x \in E - F = E_1$  (say). Applying Theorem 10.1 for the set  $E_1$ , we get a set  $A_1 \subset E_1$  with  $m(A_1) < \delta$  and an integer  $N$  such that

$$|f_n(x) - f(x)| < \epsilon, \quad (4)$$

for all  $n \geq N$  and  $x \in E_1 - A_1$ . The result now follows by taking  $A = A_1 \cup F$  since  $m(F) = 0$  and  $E - A = E_1 - A_1$ . ■

**Problem 24.** Show by an example that the condition  $m(E) < \infty$  in Theorem 10.1 cannot be relaxed.

*Solution.* Suppose  $E = \{x : x \geq 0\}$ . Then  $m(E) = \infty$ . Define a sequence  $\{f_n\}$  of functions  $f_n : E \rightarrow \mathbb{R}$  by

$$f_n(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq n \\ 1 & \text{if } x > n. \end{cases}$$

The sequence  $\{f_n\}$  is, clearly, a sequence of measurable functions converging to  $f \equiv 0$ . If we take  $\epsilon = 1$  and a  $\delta$  ( $0 < \delta < 1$ ), there may not exist any set  $A \subset E$  with  $m(A) < \delta$  and an integer  $N$  satisfying the inequality (4).

Let us recall that the concept of uniform convergence of a sequence of functions is stronger than that of point-wise convergence. Uniform convergence implies point-wise convergence but not vice versa. Consider, for instance, a sequence  $\{f_n\}$  of functions defined on  $[0, 1]$  by  $f_n(x) = x^n$  ( $0 \leq x \leq 1$ ). Then  $\{f_n\}$  converges pointwise to the zero function on  $[0, 1]$ . The convergence is not uniform. However,  $\{f_n\}$  converges to zero function uniformly on the closed intervals  $[0, 1 - \epsilon]$ , for each  $\epsilon > 0$ . In other words, to each given  $\epsilon > 0$ ,  $\exists$  a set  $A \subset [0, 1]$  with  $m(A) < \epsilon$  such that  $\{f_n\}$  converges uniformly to zero function on the set  $[0, 1] - A$ .

This motivates the introduction of the concept of almost uniform convergence.

**10.3 Definition.** A sequence  $\{f_n\}$  of measurable functions is said to converge almost uniformly to a measurable function  $f$  defined on a measurable set  $E$  if for each  $\epsilon > 0$ ,  $\exists$  a measurable set  $A \subset E$  with  $m(A) < \epsilon$  such that  $\{f_n\}$  converges to  $f$  uniformly on  $E - A$ .

*Remark.* Let  $f_n \rightarrow f$  almost uniformly. Then for each positive integer  $n$ ,  $\exists E_n$  such that  $m(E_n) < 1/n$  and  $f_n \rightarrow f$  uniformly on  $E - E_n$ . Let  $A = \bigcup_{n=1}^{\infty} (E - E_n)$ . Then  $m(E - A) = 0$  and for each  $x \in A$ ,  $f_n(x) \rightarrow f(x)$ .

Hence almost uniform convergence implies convergence almost everywhere.

Towards its converse, we prove the following result.

**10.4 Theorem (Egoroff's Theorem).** *Let  $E$  be a measurable set with  $m(E) < \infty$  and  $\{f_n\}$  a sequence of measurable functions which converge to  $f$  a.e. on  $E$ . Then, given  $\eta > 0$ , there is a set  $A \subset E$  with  $m(A) < \eta$  such that the sequence  $\{f_n\}$  converges to  $f$  uniformly on  $E - A$ .*

*Proof.* Applying Theorem 10.2 with  $\epsilon = 1$ ,  $\delta = \eta/2$ , we get a measurable set  $A_1 \subset E$  with  $m(A_1) < \delta$  and a positive integer  $N_1$ , such that

$$|f_n(x) - f(x)| < 1,$$

for all  $n \geq N_1$  and  $x \in E_1 (= E - A_1)$ . Again, taking  $\epsilon = 1/2$  and  $\delta = \eta/2^2$ , we get another measurable set  $A_2 \subset E_1$  with  $m(A_2) < \delta$  and a positive integer  $N_2$  such that

$$|f_n(x) - f(x)| < 1/2,$$

for all  $n \geq N_2$  and  $x \in E_2 (= E_1 - A_2)$ ; and so on. At the  $n^{\text{th}}$  stage, we get a measurable set  $A_n \subset E_{n-1}$  with  $m(A_n) < \delta$  and a positive integer  $N_n$  such that

$$|f_n(x) - f(x)| < \frac{1}{n},$$

for all  $n \geq N_n$  and  $x \in E_n (= E_{n-1} - A_n)$ . Setting

$$A = \bigcup_{n=1}^{\infty} A_n,$$

we note that

$$m(A) \leq \sum_{n=1}^{\infty} m(A_n) < \sum_{n=1}^{\infty} \frac{\eta}{2^n} = \eta.$$

Also,

$$\begin{aligned} E - A &= E - \bigcup_n A_n \\ &= \bigcap_n (E - A_n) = \bigcap_n (E_{n-1} - A_n) \\ &= \bigcap_n E_n. \end{aligned}$$

Let  $x \in E - A$ . Then  $x \in E_n$  for all  $n$  and so

$$|f_n(x) - f(x)| < \frac{1}{n}, \quad \forall n \geq N_n.$$

Choosing  $n$  such that  $\frac{1}{n} < \epsilon$ , we get

$$|f_n(x) - f(x)| < \epsilon,$$

for all  $x \in E - A$  and  $n \geq N_n = N$ . ■

*Remark.* Theorem 10.4 is no longer true if  $E$  is of infinite measure. A counter-example is obtained by taking  $E = \mathbb{R}$  and  $f_n = \chi_{[n, n+1]}$ , for each  $n$ .

*Note.* One should not confuse the term 'almost uniform convergence' with 'uniformly convergent almost everywhere'.

## 11 THE STRUCTURE OF MEASURABLE FUNCTIONS

While studying the properties of a complicated function, it becomes a natural temptation to represent it, if possible, exactly or approximately in terms of comparatively simpler functions of similar nature. In this connection we establish some results approximating measurable functions in terms of other known functions which are simpler in nature, viz., simple functions and continuous functions.

**11.1 Theorem.** *Let  $f$  be a measurable function defined on a set  $E$ . Then there exists a sequence  $\{f_n\}$  of simple functions which converges (pointwise) to  $f$  on  $E$ . In case  $f \geq 0$ , the sequence  $\{f_n\}$  can be chosen such that  $0 \leq f_n \leq f_{n+1}$ ,  $\forall n \in \mathbb{N}$ .*

*Proof.* Suppose first that  $f$  is nonnegative on  $E$ . Consider a finite collection of subsets of  $E$  defined by

$$E_{n,\alpha} = \left\{ x \in E : \frac{\alpha-1}{2^n} \leq f(x) < \frac{\alpha}{2^n} \right\},$$

for  $\alpha = 1, 2, 2^2, \dots, 2^{2n}$  and

$$E_{n, 1+2^{2n}} = \{x \in E : f(x) > 2^n\}.$$

Define a sequence  $\{f_n\}$  of functions as

$$f_n = \sum_{\alpha=1}^{1+2^{2n}} \left( \frac{\alpha-1}{2^n} \right) \chi_{E_{n,\alpha}}.$$

It is easy to verify that each  $f_n$  is a simple function and the sequence  $\{f_n\}$  is an increasing one which converges to  $f$  on  $E$ .

In the general case, apply the preceding construction to the non-negative measurable functions

$$g = f^+ \quad \text{and} \quad h = f^-$$

and obtain sequences  $\{g_n\}$  and  $\{h_n\}$ , respectively. Since

$$f = g - h,$$

we obtain a sequence  $\{f_n\}$  given by

$$f_n = g_n - h_n,$$

of simple functions with the required property. ■

**11.2 Theorem (Lusin Theorem).** *Let  $f$  be a measurable function defined on  $E$ . Then for each  $\epsilon > 0$ ,  $\exists$  a closed set  $F \subset E$  with  $m(E - F) < \epsilon$  such that  $f$  is continuous on  $F$ .*

*Proof.* First, suppose  $f$  to be a simple function and let  $\{a_1, a_2, \dots, a_m\}$  be the set of non-zero values of  $f$ . Then,

$$f = \sum_{i=1}^m a_i \chi_{E_i},$$

the sets  $E_i$  being measurable and disjoint. Write  $E_{m+1} = E - \bigcup_{i=1}^m E_i$  and let  $\epsilon > 0$  be given. Then, corresponding to each  $E_i$ ,  $i = 1, 2, \dots, m+1$  we can find a closed set  $F_i \subset E_i$  with  $m(E_i - F_i) < \epsilon$ , cf. III-7.1. Let

$$F = \bigcup_{i=1}^{m+1} F_i.$$

Clearly, the set  $F$  is closed,  $F \subset E$  and satisfies

$$m(E - F) = \sum_{i=1}^{m+1} m(E_i - F_i) < (m+1)\epsilon.$$

The sets  $F_i$ ,  $i = 1, 2, \dots, (m+1)$ , being disjoint and the function  $f$  having a constant value on each  $F_i$ , it follows that  $f$  is continuous on  $F$ .

Now, let  $f$  be any measurable function defined on  $E$ . Then, by Theorem 11.1, there exists a sequence  $\{f_n\}$  of simple functions converging to  $f$ . Thus, in view of the above, given an  $\epsilon > 0$  and for each  $n$ , there exists a measurable set  $A_n \subset E$  such that  $f_n$  is continuous on  $A_n$  and  $m(E - A_n) < \epsilon/2^n$ . Set

$$A = \bigcap_n A_n.$$

Then, each  $f_n$  is continuous on  $F$  and  $F$  is a measurable set satisfying

$$m(E - F) = m\left(\bigcup_n [E - A_n]\right) \leq \sum_n \frac{\epsilon}{2^n} = \epsilon.$$

Corresponding to each positive integer  $p$ , define

$$B_p = A \cap \{x: p-1 \leq |x| \leq p\}.$$

Clearly, each  $B_p$  is a measurable set and  $A = \bigcup_p B_p$ . By Egoroff's Theorem, to each  $p$  there corresponds a measurable set  $C_p \subset B_p$  such that  $m(B_p - C_p) < \epsilon/2^p$  and  $\{f_n\}$  converges uniformly on  $C_p$ . Further, in view of III-7.1, there exists closed sets  $F_p \subset C_p$  such that  $m(C_p - F_p) < \epsilon/2^p$ . This implies that  $F_p \subset B_p$  with  $m(B_p - F_p) < \epsilon/2^{p-1}$  and  $\{f_n\}$  converges to  $f$  uniformly on  $F_p$ . Hence  $f$  is continuous on each  $F_p$ .

Let  $F = \bigcup_p F_p$ . Then  $F$  is closed [For each positive integer  $n$ , if  $F_n$  is a closed set such that  $F_n \subset \{x: n-1 \leq |x| \leq n\}$ , then  $F = \bigcup_{n=1}^{\infty} F_n$  is a closed set]. Moreover,  $f$  is continuous on  $F$  because of the construction of  $F$  (it may be noted that what is true in the instance of  $\bigcup_p F_p$  is not true of unions of closed sets in general). Finally, we observe

$$\begin{aligned} m(E - F) &= m(E - A) + m(A - F) \\ &< \epsilon + m\left(\bigcup_p B_p - \bigcup_p F_p\right) \\ &< \epsilon + m\left(\bigcup_p [B_p - F_p]\right) \\ &= \epsilon + \sum_p m(B_p - F_p) \\ &< \epsilon + \sum_p \frac{\epsilon}{2^{p-1}} = 3\epsilon. \end{aligned}$$

This completes the proof of the theorem. ■

**11.3. Theorem.** *Let  $f$  be a measurable function defined on  $E$ . Then for each  $\epsilon > 0$ ,  $\exists$  a continuous function  $g$  defined on  $\mathbb{R}$  such that  $m(\{x \in E: f(x) \neq g(x)\}) < \epsilon$ . Further, if  $|f(x)| \leq M$  on  $E$ , then  $g$  may be chosen so that  $|g(x)| \leq M$  on  $\mathbb{R}$ .*

*Proof.* By the Lusin Theorem, for each  $\epsilon > 0$ , there exists a closed set  $F \subset E$  such that  $m(E - F) < \epsilon$  and  $f$  is continuous on  $F$ . Using the result: "Let  $f$  be a continuous function defined on a closed set  $F$ . Then there exists a continuous function  $g$  defined on  $\mathbb{R}$  such that  $g|_F = f$ . Moreover, if  $f$  is bounded,  $g$  can be constructed so that it attains a maximum equal to that of  $f$ " studied in real analysis, the required function  $g$  exists on  $\mathbb{R}$  which is continuous on  $\mathbb{R}$  and satisfies  $g|_F = f$ . Therefore the set

$$A = \{x \in E: f(x) \neq g(x)\} \subset E - F.$$

Clearly, the set  $A$  is measurable since  $f$  and  $g$  both are measurable functions and  $m(A) < \epsilon$ .

Finally, the boundedness of  $g$  follows directly from its construction. ■

**11.4 Theorem (Fréchet Theorem).** *Let  $f$  be a measurable function defined on  $E$ . Then there exists a sequence  $\{g_n\}$  of continuous functions on  $\mathbb{R}$  such that  $g_n \rightarrow f$  a.e. on  $E$ .*

*Proof.* By Theorem 11.3, for each  $n$  there exists a continuous function  $g_n$  such that the set  $A_n = \{x \in E: f(x) \neq g_n(x)\}$  has a measure less than  $2^{-n}$ . Hence, for each  $m$ , the set  $\bigcup_{n=m}^{\infty} A_n$  is measurable and

$$m\left(\bigcup_{n=m}^{\infty} A_n\right) \leq \sum_{n=m}^{\infty} m(A_n) < \sum_{n=m}^{\infty} 2^{-n} = 2^{1-m}.$$

Define

$$A = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} A_n.$$

Then,  $A$  is a measurable set and in view of III-6.3, its measure is given by

$$m(A) = \lim_{m \rightarrow \infty} m\left(\bigcup_{n=m}^{\infty} A_n\right) = 0.$$

Let  $x \in E \cap A^c$ . Then, since  $x \notin A$ ,

$$x \notin \bigcup_{n=p}^{\infty} A_n, \quad \text{for some } p$$

and as such  $x \notin A_n$  for all  $n \geq p$ . This verifies that

$$g_n(x) = f(x), \quad \forall n \geq p.$$

Hence

$$\lim_{n \rightarrow \infty} g_n(x) = f(x), \quad x \in E \cap A^c.$$

The proof will now be complete if

$$m(A^c \cap E) = m(E).$$

But it follows trivially since  $m(A) = 0$ . ■

## 12. CONVERGENCE IN MEASURE

The notions of pointwise convergence and uniform convergence are known from a basic course in Analysis. Pointwise convergence is more general than uniform convergence. An even more general notion than

that of pointwise convergence; namely, convergence almost everywhere is introduced in §8. But the concept of convergence almost everywhere is not very different in the sense that it remains the same concept as that of pointwise convergence except that it is now on a reduced domain of definition. Moreover, these concepts—pointwise convergence and convergence a.e.—reduce to the convergence of a numerical sequence at individual points of the domain. In the theory of measure and integration as well as in its application, other notions of convergence which no longer reduce directly to the convergence of numerical sequences are very important; for instance, the notion of the convergence in the mean (cf. VII-5) and that of convergence in measure. The notion of convergence in measure was introduced by F. Riesz and E. Fischer in 1906-1907. Sometimes, it is also called approximate convergence.

**12.1 Definition.** A sequence  $\{f_n\}$  of measurable functions is said to converge in measure\* to a measurable function  $f$  on a set  $E$ , written  $f_n \xrightarrow{m} f$  on  $E$ , if for each  $\delta > 0$  and  $\epsilon > 0$ , there exists a positive integer  $N$  such that

$$m(\{x \in E : |f_n(x) - f(x)| \geq \epsilon\}) < \delta, \quad \forall n > N.$$

In fact, the concept  $f_n \xrightarrow{m} f$  on  $E$  means that for all sufficiently large values of  $n$ , the functions  $f_n$  in the sequence  $\{f_n\}$  differ from the limit function  $f$  by a small quantity (less than  $\epsilon$ ) with the exception of a set of points whose measure is arbitrary small (less than  $\delta$ ). The set in which the function  $f_n$  differs from  $f$  by more than or equal to  $\epsilon$  depends in general on  $n$  but its measure remains small for all sufficiently large values of  $n$  ( $n > N$ ). Definition 12.1 can equivalently be formulated as follows.

**12.2 Definition.** A sequence  $\{f_n\}$  of measurable functions is said to converge in measure to a measurable function  $f$  on  $E$ , if

$$\lim_{n \rightarrow \infty} m(\{x \in E : |f_n(x) - f(x)| \geq \epsilon\}) = 0,$$

for each  $\epsilon > 0$ .

We first establish that the limit function of a sequence which converges in measure exists uniquely almost everywhere.

**12.3 Theorem.** *If a sequence  $\{f_n\}$  converges in measure to  $f$ , then it*

\*It is also called *asymptotic convergence*.

converges in measure to every function  $g$  which is equivalent to the function  $f$ .

*Proof.* For each  $\epsilon > 0$ , we observe that

$$\{x: |f_n(x) - g(x)| \geq \epsilon\} \subset \{x: f(x) \neq g(x)\} \cup \{x: |f_n(x) - f(x)| \geq \epsilon\},$$

and since  $m(\{x: f(x) \neq g(x)\}) = 0$ , we infer that

$$m(\{x: |f_n(x) - g(x)| \geq \epsilon\}) \leq m(\{x: |f_n(x) - f(x)| \geq \epsilon\}).$$

This proves the result. ■

**12.4 Theorem.** *If a sequence  $\{f_n\}$  converges in measure to  $f$ , then the limit function  $f$  is unique a.e.*

*Proof.* Let  $g$  be another function such that  $f_n \xrightarrow{m} g$ . Since

$$|f - g| \leq |f - f_n| + |f_n - g|,$$

we observe, for each  $\epsilon > 0$ , that

$$\begin{aligned} \{x: |f(x) - g(x)| \geq \epsilon\} \subset & \left\{ x: |f(x) - f_n(x)| \geq \frac{\epsilon}{2} \right\} \\ & \cup \left\{ x: |f_n(x) - g(x)| \geq \frac{\epsilon}{2} \right\}. \end{aligned}$$

Since, by a proper choice of  $\epsilon$ , the measure of both the sets on the right can be made arbitrarily small, we have

$$m(\{x: |f(x) - g(x)| \geq \epsilon\}) = 0.$$

This implies  $f \sim g$ . ■

A relation between pointwise convergence and convergence in measure is given as follows.

**12.5 Theorem.** *Let  $\{f_n\}$  be a sequence of measurable functions which converge to  $f$  a.e. on  $E$ . Then  $f_n \xrightarrow{m} f$  on  $E$ .*

*Proof.* Let us consider, for each  $n \in \mathbb{N}$  and  $\epsilon > 0$ , the sets

$$S_n(\epsilon) = \{x \in E: |f_n(x) - f(x)| \geq \epsilon\}.$$

Let  $\delta > 0$  be any arbitrary number. Then, by Theorem 10.2, there exists a measurable set  $A \subset E$  with  $m(A) < \delta$  and a positive integer  $N$  such that

$$|f_n(x) - f(x)| < \epsilon,$$

for all  $x \in E - A$  and  $n \geq N$ . Thus

$$S_n(\epsilon) \subset A, \quad \forall n \geq N$$

$$\Rightarrow m(S_n(\epsilon)) \leq m(A) < \delta, \quad \forall n \geq N.$$

This proves the result. ■

*Remark.* The converse of Theorem 12.5 need not be true. In other words, convergence in measure is more general than convergence almost everywhere. In fact, there are sequences of measurable (even continuous) functions that converge in measure but fail to converge at any point.

**12.6 Example.** Consider the interval  $[0, 1]$ . For each  $n \in \mathbb{N}$ , divide  $[0, 1]$  into  $n$  sub-intervals:

$$\left[0, \frac{1}{n}\right], \left[\frac{1}{n}, \frac{2}{n}\right], \dots, \left[\frac{n-1}{n}, 1\right].$$

Write

$$E_{nk} = \left[\frac{k-1}{n}, \frac{k}{n}\right], \quad k = 1, 2, \dots, n.$$

Enumerate all these intervals as follows:

$$\begin{aligned} & [0, 1] \\ & \left[0, \frac{1}{2}\right], \left[\frac{1}{2}, 1\right] \\ & \left[0, \frac{1}{3}\right], \left[\frac{1}{3}, \frac{2}{3}\right], \left[\frac{2}{3}, 1\right] \\ & \vdots \\ & \vdots \end{aligned}$$

i.e.,  $E_{11}, E_{21}, E_{22}, E_{31}, E_{32}, E_{33}, \dots$

Let  $\{E_n\}$  designate the above sequence of intervals. Define

$$f_n = \chi_{E_n}.$$

We clearly note that the sequence  $\{f_n\}$  converges in measure to the zero function since  $m(E_n) \rightarrow 0$  as  $n \rightarrow \infty$ .

On the other hand, for each  $\epsilon > 0$ , we observe that

$$\{x: x \in [0, 1], |f_n(x)| \geq \epsilon\} = E_n, \quad \forall n.$$

Then, for each  $x \in [0, 1]$ ,  $f_n(x) = 1$  for infinitely many values of  $n$ . Hence  $f_n(x) \not\rightarrow 0$  for any  $x \in [0, 1]$ .

Although the converse of Theorem 12.5 is not true, we, however, prove the following.

**12.7 Theorem (Riesz Theorem).** *If a sequences  $\{f_n\}$  converges in measure to  $f$  on  $E$ , then  $\exists$  a subsequence  $\{f_{n_k}\}$  of  $\{f_n\}$  which converges to  $f$  a.e. on  $E$ .*

*Proof.* Let us consider two sequences  $\{\epsilon_n\}$  and  $\{\delta_n\}$  of positive real numbers such that  $\epsilon_n \rightarrow 0$  as  $n \rightarrow \infty$  and  $\sum_{n=1}^{\infty} \delta_n < \infty$ . We now choose an increasing (strictly) sequence  $\{n_k\}$  of positive integers as follows.

Let  $n_1$  be a positive integer such that

$$m(\{x \in E, |f_{n_1}(x) - f(x)| \geq \epsilon_1\}) < \delta_1.$$

Such a number  $n_1$  must exist since, in view of  $f_n \xrightarrow{m} f$ , for a given  $\epsilon_1 > 0$  and  $\delta_1 \geq 0$ ,  $\exists$  a positive integer  $n_1$  such that

$$m(\{x \in E, |f_n(x) - f(x)| \geq \epsilon_1\}) < \delta_1,$$

for all  $n \geq n_1$ , in particular, for  $n = n_1$ . Similarly, let  $n_2$  be a positive number such that

$$m(\{x: x \in E, |f_{n_2}(x) - f(x)| \geq \epsilon_2\}) < \delta_2,$$

and that  $n_2 \geq n_1$ ; and so on. In general, we get the number  $n_k$  such that

$$m(\{x: x \in E, |f_{n_k}(x) - f(x)| \geq \epsilon_k\}) < \delta_k,$$

and that  $n_k \geq n_{k-1}$ . We shall now prove that the subsequence  $\{f_{n_k}\}$  converges to  $f$  a.e.

Define

$$A_k = \bigcup_{i=k}^{\infty} \{x: x \in E, |f_{n_i}(x) - f(x)| \geq \epsilon_i\}, \quad k \in \mathbb{N}$$

and

$$A = \bigcap_{k=1}^{\infty} A_k.$$

Clearly,  $\{A_k\}$  is a decreasing sequence of measurable sets. Therefore, in view of III-6.3, we have

$$m(A) = \lim_{k \rightarrow \infty} m(A_k).$$

But

$$m(A_k) \leq \sum_{i=k}^{\infty} \delta_i \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Hence,  $m(A) = 0$ . It remains to be verified that the sequence  $\{f_{n_k}\}$  converges to  $f$  on  $E - A$ .

Let  $x_0 \in E - A$ . Then  $x_0 \notin A_{k_0}$  for some positive integer  $k_0$ . In other words,

$$x_0 \notin \{x \in E: |f_{n_k}(x) - f(x)| \geq \epsilon_k\}, \quad k \geq k_0.$$

This gives

$$|f_{n_k}(x_0) - f(x_0)| < \epsilon_k, \quad k \geq k_0.$$

But  $\epsilon_k \rightarrow 0$  as  $k \rightarrow \infty$ . Hence

$$\lim_{k \rightarrow \infty} f_{n_k}(x_0) = f(x_0).$$

This completes the proof. ■

**12.8 Theorem.** Let  $f_n \xrightarrow{m} f$  and  $g_n \xrightarrow{m} g$  on  $E$ . Then:

- (a)  $f_n + g_n \xrightarrow{m} f + g$ .
- (b)  $\alpha f_n \xrightarrow{m} \alpha f$ ,  $\alpha$  is a real number.
- (c)  $f_n^+ \xrightarrow{m} f^+$ ,  $f_n^- \xrightarrow{m} f^-$  and  $|f_n| \xrightarrow{m} |f|$ .

Further, if  $m(E) < \infty$ , then:

- (d)  $f_n^2 \xrightarrow{m} f^2$ .
- (e)  $f_n \cdot g_n \xrightarrow{m} f \cdot g$ .

*Proof.* (a) It follows from the relation

$$\begin{aligned} & \{x \in E: |(f_n + g_n)(x) - (f + g)(x)| \geq \epsilon\} \\ & \subseteq \left\{x \in E: |f_n(x) - f(x)| \geq \frac{\epsilon}{2}\right\} \\ & \quad \cup \left\{x \in E: |g_n(x) - g(x)| \geq \frac{\epsilon}{2}\right\}. \end{aligned}$$

(b) In case  $\alpha = 0$ , it follows obviously. Let  $\alpha \neq 0$ . Then

$$\begin{aligned} & \{x \in E: |(\alpha f_n)(x) - (\alpha f)(x)| \geq \epsilon\} \\ & = \left\{x \in E: |f_n(x) - f(x)| \geq \frac{\epsilon}{|\alpha|}\right\}, \end{aligned}$$

and the result follows.

(c) This follows from the relations

$$\begin{aligned} |f_n^+ - f^+| & \leq |f_n - f|, \\ |f_n^- - f^-| & \leq |f_n - f| \end{aligned}$$

and

$$| |f_n| - |f| | \leq |f_n - f|.$$

(d) From Definition 12.1, it is obvious that the limit function is finite valued almost everywhere. Therefore, for  $\epsilon > 0$ , there exists a set  $A$  and  $k > 0$  such that  $m(A) < \epsilon/2$  and  $|f| \leq k$  on  $A^c$ . Write

$$E_\delta = \{x: |f_n(x) - f(x)| \geq \delta\}.$$

Then, on  $(A \cup E_\delta)^c$ , we have

$$\begin{aligned} |f_n^2 - f^2| &= |f_n + f| |f_n - f| \\ &\leq (\delta + 2k)\delta \\ &< \epsilon, \end{aligned}$$

for an appropriate  $\delta > 0$ . Also  $m(E_\delta) < \epsilon/2$  for all large  $n$ . So, for all large  $n$ , it follows that

$$m(\{x: |f_n^2(x) - f^2(x)| \geq \epsilon\}) < \epsilon,$$

giving the result.

(e) From the identity

$$f_n \cdot g_n = \frac{1}{4} [(f_n + g_n)^2 - (f_n - g_n)^2],$$

the result follows. ■

*Remark.* The condition  $m(E) < \infty$  in (d) and (e) in Theorem 12.8 is essential.

**12.9 Example.** Take  $E = ]0, \infty[$ . Consider, for each  $n$ ,  $f_n(x) = x$ ,  $x \in E$ . Then,  $f(x) = x$ . Let  $g_n(x) = c_n$ , where  $\{c_n\}$  is a sequence of positive real numbers such that  $c_n \rightarrow 0$  as  $n \rightarrow \infty$ .

Here  $m(E) = \infty$ , and we note that

$$\begin{aligned} m(\{x: |f_n(x)g_n(x) - f(x)g(x)| \geq \epsilon\}) \\ = m(\{x: |c_n x| \geq \epsilon\}) = \infty, \quad \forall n. \end{aligned}$$

This shows that  $f_n \cdot g_n \xrightarrow{m} f \cdot g$  on  $E$ .

**12.10 Definition.** A sequence  $\{f_n\}$  of measurable functions is said to be a **Cauchy sequence in measure** or **fundamental in measure** if given an  $\epsilon > 0$  there exists a positive integer  $N$  such that

$$m(\{x: |f_n(x) - f_p(x)| \geq \epsilon\}) < \epsilon,$$

for all  $n, p \geq N$ .

**12.11 Theorem.** *If a sequence  $\{f_n\}$  converges in measure to  $f$ , then  $\{f_n\}$  is a Cauchy sequence in measure.*

*Proof.* It follows obviously from the relation

$$\{x: |f_n(x) - f_p(x)| \geq \epsilon\} \subset \left\{x: |f_n(x) - f(x)| \geq \frac{\epsilon}{2}\right\} \\ \cup \left\{x: |f_p(x) - f(x)| \geq \frac{\epsilon}{2}\right\}. \blacksquare$$

**12.12 Theorem.** *If  $\{f_n\}$  is a sequence of measurable functions which is fundamental in measure, then there exists a measurable function  $f$  such that  $\{f_n\}$  converges in measure to  $f$ .*

*Proof.* Since  $\{f_n\}$  is fundamental in measure, for every positive integer  $k$ , we may find an integer  $n_k$  such that if  $n \geq n_k$  and  $p \geq n_k$ , then

$$m\left(\left\{x: |f_n(x) - f_p(x)| \geq \frac{1}{2^k}\right\}\right) < \frac{1}{2^k},$$

and assume that  $n_{k+1} > n_k$  for each  $k$ . This gives a sequence  $\{f_{n_k}\}$ , an infinite subsequence of  $\{f_n\}$ .

Let 
$$E_k = \left\{x: |f_{n_k}(x) - f_{n_{k+1}}(x)| \geq \frac{1}{2^k}\right\}.$$

Then for each  $x \notin \bigcup_{i=k}^{\infty} E_i$ , we have

$$|f_{n_i}(x) - f_{n_j}(x)| \leq \sum_{p=i}^{j-1} |f_{n_p}(x) - f_{n_{p+1}}(x)| < \frac{1}{2^{i-1}}.$$

for  $j \geq i \geq k$ . Hence the sequence  $\{f_{n_k}\}$  is a uniformly Cauchy sequence in  $\left(\bigcup_{i=k}^{\infty} E_i\right)^c$ . Since

$$m\left(\bigcup_{i=k}^{\infty} E_i\right) \leq \sum_{i=k}^{\infty} m(E_i) < \frac{1}{2^{k-1}},$$

the sequence  $\{f_{n_k}\}$  is an almost uniformly Cauchy sequence and hence a Cauchy sequence a.e.

Let us write

$$f(x) = \lim_{k \rightarrow \infty} f_{n_k}(x),$$

for every  $x$  for which the limit exists. We observe, for every  $\epsilon > 0$ , that

$$\{x: |f_n(x) - f(x)| \geq \epsilon\} \subset \left\{x: |f_n(x) - f_{n_k}(x)| \geq \frac{\epsilon}{2}\right\}$$

$$\cup \left\{ x: |f_{n_k}(x) - f(x)| \geq \frac{\epsilon}{2} \right\}.$$

The measure of the first term on the right is by hypothesis arbitrarily small if  $n$  and  $n_k$  are sufficiently large, and the measure of the second term also tends to 0 as  $k \rightarrow \infty$ , in view of the fact that almost uniform convergence implies convergence in measure; see Problem 27. ■

### Problems

25. Let the function  $f$  be finite valued and measurable on a set  $E$ . Then prove that:

(a)  $|f|^\alpha$  ( $\alpha \geq 0$ ) is measurable.

(b)  $|f|^\alpha$  ( $\alpha < 0$ ) is measurable if  $f(x) \neq 0$  on  $E$ .

[Hint: The function  $\varphi(z) = |z|^\alpha$  ( $\alpha \geq 0$ ) is continuous for all  $z$ , and so  $|f(x)|^\alpha$  is defined for all  $x \in E$  and is measurable. Further, if  $\alpha < 0$ , the function  $|f|^\alpha$  is defined and continuous on the open set  $z \neq 0$ ; so if  $f(x) \neq 0$ , the power  $|f(x)|^\alpha$  is defined and measurable on  $E$ .]

26. Show that, for every  $a \in \mathbb{R}^*$  and measurable functions  $f, g: E \rightarrow \mathbb{R}^*$ , the functions  $h, k: E \rightarrow \mathbb{R}^*$  defined by

$$h(x) = \begin{cases} a & \text{if } x \in D \\ f(x) + g(x) & \text{if } x \in E - D \end{cases}$$

and

$$k(x) = \begin{cases} a & \text{if } x \in D \\ f(x)g(x) & \text{if } x \in E - D, \end{cases}$$

where  $D = \{f^{-1}(\infty) \cap g^{-1}(-\infty)\} \cup \{f^{-1}(-\infty) \cap g^{-1}(\infty)\}$ ,

are measurable.

27. Prove that almost uniform convergence implies convergence in measure.

28. Let  $F'(x)$  exist for every  $x$  in  $[a, b]$  and

$$f(x) = F'(x) \quad (a \leq x \leq b).$$

Prove that  $f$  is a measurable function. [Hint: Define  $F(x) = F(b)$  for  $x > b$ . Let  $f_n(x) = n \left[ F \left( x + \frac{1}{n} \right) - F(x) \right]$  ( $a \leq x \leq b, n \in \mathbb{N}$ ). Then each  $f_n$  is measurable and  $f_n \rightarrow f$  on  $[a, b]$ .]

29. Show that an increasing function is a Borel function.

30. Prove that an increasing function of a measurable function is a measurable function.
31. Show that  $\lim_{p \rightarrow \infty} \lim_{q \rightarrow \infty} (\cos p! \pi x)^{2q}$  defines a measurable function.
32. Prove that every subsequence of a Cauchy sequence in measure is again a Cauchy sequence in measure.
33. Is the supremum of an arbitrary family of measurable functions measurable? Justify.

## V

# Lebesgue Integral

The theory of Riemann integration though very useful and adequate for solving many mathematical problems, both pure and applied, is not free from defects. It does not meet the needs of a number of important branches of mathematics and physics of comparatively recent development. First of all, the Riemann integral of a function is defined on a closed interval and cannot be defined on an arbitrary set. Investigations in probability theory, partial differential equations, hydromechanics and quantum mechanics often pose problems which require integration over sets. Second and more important is the fact that the Riemann integrability depends upon the continuity of the function. Of course, there are functions which are discontinuous and yet Riemann-integrable, but these functions are continuous almost everywhere, cf. Theorem 1.1. Again, given a sequence of Riemann-integrable functions converging to some function in a domain, the limit of the sequence of integrated functions may not be the Riemann integral of the limit function. In fact, the Riemann integral of the limit function may not even exist. This is a major drawback of the Riemann theory of integration, apart from the fact that even relatively simple functions are not integrable in the sense of Riemann integration. H. Lebesgue in his classical work, introduced the concept of an integral, known after his name the Lebesgue integral, based on the measure theory that generalizes the Riemann integral. It has the advantage that it takes care of both bounded and unbounded functions and simultaneously allows their domains to be more general sets and thereby enlarges the class of functions for which the Lebesgue integral is defined. Also, it gives more powerful and useful convergence theorems relating to the interchange of the limit and integral valid under less restrictive conditions required for the Riemann integral, cf. Theorems 4.1, 5.5, and 6.6.

**1 RIEMANN INTEGRAL**

The classical definition of an integral, given first by Cauchy and later developed by Riemann, runs as follows.

Let  $f$  be a bounded real-valued function defined on the interval  $[a, b]$  and let

$$P = \{a = x_0 < x_1 < x_2 \dots < x_n = b\}$$

be a partition (or subdivision) of  $[a, b]$ . For each partition  $P$  of  $[a, b]$ , consider the sums

$$S(P) = \sum_{i=1}^n (x_i - x_{i-1})M_i$$

and

$$s(P) = \sum_{i=1}^n (x_i - x_{i-1})m_i,$$

where  $M_i = \sup \{f(x) : x \in ]x_{i-1}, x_i]\}$  and  $m_i = \inf \{f(x) : x \in ]x_{i-1}, x_i]\}$ , for  $i = 1, 2, \dots, n$ .

The upper Riemann integral of  $f$  over  $[a, b]$  is defined by

$$\mathcal{R} \int_a^b f(x) dx = \inf S(P)$$

and the lower Riemann integral of  $f$  over  $[a, b]$  is defined by

$$\mathcal{R} \int_a^b f(x) dx = \sup s(P),$$

where the supremum and infimum are taken over all possible partitions  $P$  of  $[a, b]$ . If

$$\mathcal{R} \int_a^b f(x) dx = \mathcal{R} \int_a^b f(x) dx,$$

we say that the Riemann integral of  $f$  over  $[a, b]$  exists and denote it by  $\mathcal{R} \int_a^b f(x) dx$ . It may be noted that in order that the function  $f$  be Riemann integrable, it is necessary for it to be bounded.

Let us try to look at the definition of the Riemann integral of a bounded function through a slightly different angle; namely, through step functions, as below.

Let  $\psi$  be a step function (cf. IV, § 4) defined on the closed interval  $[a, b]$ . Then

$$\psi(x) = c_i, \quad x_{i-1} < x < x_i \quad (i = 1, 2, \dots, n)$$

where  $\{a = x_0 < x_1 < x_2 \dots < x_n = b\}$  is a partition of  $[a, b]$ . Define the elementary integral of  $\psi$  over  $[a, b]$  as

$$\int_a^b \psi(x) dx = \sum_{i=1}^n c_i(x_i - x_{i-1}).$$

The elementary integral of  $\psi$  is defined since the representation of  $\psi$  is unique and the sum is finite. Keeping this in mind we observe that

$$\mathcal{R} \int_a^b \psi(x) dx = \inf \int_a^b \psi(x) dx,$$

for all step functions  $\psi \geq f$ . Similarly

$$\mathcal{R} \int_a^b f(x) dx = \sup \int_a^b \psi(x) dx$$

for all step functions  $\psi \leq f$ .

*Note.* While defining the elementary integral of a step function  $\psi$ , there was nothing in mind in regard to 'in what sense' it was defined. Rather, it was simply the name given to the sum  $\sum_i c_i(x_i - x_{i-1})$ . However, now one can easily verify that the elementary integral of a step function is nothing but the Riemann integral of the function.

It may be observed that the definition of the Riemann integral involves the approximation of the integrand by step functions over the partitions of the domain of the integrand. If the bounded function  $f$  defined over  $[a, b]$  is not too discontinuous, the Riemann integral of  $f$  is the limit of the integrals of step functions which approximate  $f$ . However, when the oscillations of  $f$  cannot be kept sufficiently small on the subdivisions, the Riemann integral fails to exist. Consider, for instance, the function  $f : [0, 1] \rightarrow \mathbf{R}$  given by

$$f(x) = \begin{cases} 3 & \text{if } x \text{ is rational} \\ 4 & \text{if } x \text{ is irrational.} \end{cases}$$

For such a function, the oscillations in any partition of  $[0, 1]$  cannot be made less than 1, and the lower and the upper Riemann integrals take the values 3 and 4, respectively. Thus,  $f$  fails to be Riemann integrable. In fact, Lebesgue gave the following characterization of the Riemann integrable functions in terms of their discontinuities.

**1.1 Theorem.** *A bounded function is Riemann integrable if and only if it is continuous almost everywhere.*

*Proof.* For the proof, refer to Appendix I.

**2 LEBESGUE INTEGRAL OF A BOUNDED FUNCTION**

We have seen that given a class of functions—the step functions defined on  $[a, b]$ —for which the integral (elementary integral) has been defined, the integral then can be extended to those functions on  $[a, b]$  which admit arbitrary close upper and lower approximations by functions of the class in question. The resulting Riemann-integrable functions have all the properties we may reasonably expect. The same sort of thing may be done, starting from any class of functions in which an ‘elementary integral’ is defined. If the class with which we start is relatively complicated, the class of integrable functions so obtained is relatively large and adequate for a reasonable theory of integration. If on the other hand, the initial class is simple, the resulting “integrable” functions may be inadequate, and the process must be repeated. Thus, in order to obtain a class of integrable functions wider than that of Riemann integrable functions, we replace the class of step functions by a larger class of functions known as simple functions (cf. IV § 6).

Let  $\varphi$  be a simple function (measurable) with its canonical representation given by

$$\varphi = \sum_{i=1}^n a_i \chi_{A_i},$$

where the sets  $A_i = \{x : \varphi(x) = a_i\}$  are disjoint and measurable, and numbers  $a_i$  ( $i = 1, 2, \dots, n$ ) are distinct and non-zero. Assume that  $\varphi$  vanishes outside a set of finite measure. Then, we define the “elementary integral” of  $\varphi$  by

$$\int \varphi(x) dx = \sum_{i=1}^n a_i m(A_i).$$

We sometimes abbreviate the expression for the elementary integral of  $\varphi$  by  $\int \varphi$ . If  $E$  is a measurable set, we define the integral of  $\varphi$  over  $E$  by

$$\int_E \varphi = \int \varphi \cdot \chi_E.$$

The elementary integral of a simple function has been defined through its canonical representation. However, we prove that the elementary integral is independent of the choice of the representation of the simple function.

**2.1 Lemma.** *Let  $\varphi = \sum_{i=1}^n a_i \chi_{E_i}$ , where each  $E_i$  is a measurable set of finite measure and  $E_i \cap E_j = \emptyset$  for  $i \neq j$ . Then*

$$\int \varphi = \sum_{i=1}^n a_i m(E_i).$$

*Proof.* Clearly,  $\varphi$  is a simple function defined on  $\bigcup_{i=1}^n E_i$ . Let  $a$  be an element in the range of  $\varphi$ . Then the canonical representation of  $\varphi$  is given by

$$\varphi = \sum_a a \chi_{A_a},$$

where  $a$  varies over the range of  $\varphi$  and the set  $A_a$  is given by

$$A_a = \{x : \varphi(x) = a\} = \bigcup_{a_i = a} E_i.$$

Observe, in view of the additivity of  $m$ , that

$$am(A_a) = \sum_{a_i = a} a_i m(E_i).$$

Hence

$$\begin{aligned} \int \varphi &= \sum_a am(A_a) \\ &= \sum_a \sum_{a_i = a} a_i m(E_i) \\ &= \sum_{i=1}^n a_i m(E_i) \blacksquare \end{aligned}$$

*Note.* In view of Lemma 2.1, we can write the elementary integral for a simple function even when its representation is not canonical.

**2.2 Theorem.** *Let  $\varphi$  and  $\psi$  be simple functions which vanish outside a set of finite measure. Then:*

- (a)  $\int a\varphi + b\psi = a \int \varphi + b \int \psi$ , for all reals  $a$  and  $b$ .
- (b) If  $\varphi \geq \psi$  a.e., then  $\int \varphi \geq \int \psi$ .

*Proof.* (a) Let  $\{A_i\}$  and  $\{B_j\}$  be the sets occurring in the canonical representations of  $\varphi$  and  $\psi$ . Let  $A_0$  and  $B_0$  be the sets where  $\varphi$  and  $\psi$ , respectively, vanish. Then

$$\begin{aligned} a\varphi + b\psi &= a \sum_{i=1}^m \alpha_i \chi_{A_i} + b \sum_{j=1}^n \beta_j \chi_{B_j} \\ &= \sum_{i=0}^m \sum_{j=0}^n (a\alpha_i + b\beta_j) \chi_{A_i \cap B_j}, \end{aligned}$$

Since the collection of sets obtained by taking all the intersection  $A_i \cap B_j$  ( $i=0, 1, 2, \dots, m; j=0, 1, 2, \dots, n$ ) forms a finite disjoint collection of measurable sets, by Lemma 2.1, we get

$$\begin{aligned} \int a\varphi + b\psi &= \sum_{i=0}^m \sum_{j=0}^n (a\alpha_i + b\beta_j) m(A_i \cap B_j) \\ &= \sum_{i=0}^m \sum_{j=0}^n a\alpha_i m(A_i \cap B_j) + \sum_{i=0}^m \sum_{j=0}^n b\beta_j m(A_i \cap B_j) \\ &= \sum_{i=0}^m a\alpha_i m\left(A_i \cap \left[\bigcup_{j=0}^n B_j\right]\right) + \sum_{j=0}^n b\beta_j m\left(\left[\bigcup_{i=0}^m A_i\right] \cap B_j\right) \\ &= a \sum_{i=1}^m \alpha_i m(A_i) + b \sum_{j=1}^n \beta_j m(B_j) \\ &= a \int \varphi + b \int \psi, \end{aligned}$$

in view of the fact that

$$\begin{cases} A_i \cap \left[\bigcup_{j=0}^n B_j\right] = A_i, & i=0, 1, 2, \dots, m \\ \left[\bigcup_{i=0}^m A_i\right] \cap B_j = B_j, & j=0, 1, 2, \dots, n; \end{cases}$$

and that  $\alpha_0 = 0 = \beta_0$ .

(b) Taking  $a = 1$  and  $b = -1$  in (a), we get

$$\int \varphi - \int \psi = \int (\varphi - \psi)$$

Since  $\varphi - \psi \geq 0$  a.e. is a simple function, by the definition of the elementary integral, we have

$$\int (\varphi - \psi) \geq 0,$$

and the result follows. ■

Let  $f: E \rightarrow \mathbb{R}$  be a bounded function and  $E$  a measurable set with  $m(E) < \infty$ . In analogy with the Riemann integral, we consider the numbers

$$\inf_{\psi \geq f} \int_E \psi \quad \text{and} \quad \sup_{\varphi \leq f} \int_E \varphi,$$

where  $\psi$  and  $\varphi$  range over the set of all simple functions defined on the set  $E$ . These two numbers do exist and are, respectively, called the upper Lebesgue integral, written  $\mathcal{L} \int_E f(x) dx$ , and the lower Lebesgue integral, written  $\underline{\mathcal{L}} \int_E f(x) dx$ .

*Justification.* Since  $f$  is a bounded function on  $E$ , there exist real numbers  $\alpha$  and  $\beta$  such that

$$\begin{cases} \alpha = \inf \{f(x) : x \in E\} \\ \beta = \sup \{f(x) : x \in E\}. \end{cases}$$

Then the constants  $\alpha$  and  $\beta$ , regarded as constant functions, are simple functions on  $E$  satisfying  $\alpha \leq f \leq \beta$ .

Consider the set

$$L(f) = \{\varphi : \varphi \text{ is a simple function defined on } E \text{ and } \varphi \leq f\}.$$

For any  $\varphi \in L(f)$ , we have  $\varphi \leq f \leq \beta$ . This gives

$$\int_E \varphi \leq \int_E \beta = \beta m(E).$$

Thus, the set  $\left\{ \int_E \varphi : \varphi \in L(f) \right\}$  is a non empty subset of real numbers since at least  $\alpha$  is a member of it, and it has an upper bound,  $\beta m(E)$ , in  $\mathbf{R}$ . Hence, it has the supremum,  $\sup_{\varphi \in L(f)} \int_E \varphi$ . Similarly, by considering the set

$$U(f) = \{\psi : \psi \text{ is a simple function defined on } E \text{ and } \psi \geq f\},$$

we obtain that the set  $\left\{ \int_E \psi : \psi \in U(f) \right\}$  is a nonempty subset of real numbers and has the infimum,  $\inf_{\psi \in U(f)} \int_E \psi$ .

It is quite clear that every bounded function defined on a set of finite measure possesses both lower and upper Lebesgue integrals. It is natural to enquire whether the upper and lower Lebesgue integrals of  $f$  are equal. The answer is certainly in the affirmative when  $f$  is a simple function.

**2.3 Theorem.** *Let  $f : E \rightarrow \mathbf{R}$  be a simple function. Then*

$$\underline{\mathcal{L}} \int_E f(x) dx = \int_E f = \overline{\mathcal{L}} \int_E f(x) dx.$$

*Proof.*  $f$  being a simple function,  $f \in L(f)$  and  $f \in U(f)$ . Hence

$$\int_E f \leq \underline{\mathcal{L}} \int_E f(x) dx \leq \overline{\mathcal{L}} \int_E f(x) dx \leq \int_E f,$$

and the result follows. ■

**2.4 Definition.** A bounded function  $f$  defined on a set  $E$  of finite measure is said to be **Lebesgue integrable** over  $E$ , if

$$\overline{\mathcal{L}} \int_E f(x) dx = \underline{\mathcal{L}} \int_E f(x) dx.$$

The common value is called the **Lebesgue integral** of  $f$  over  $E$ , written  $\mathcal{L} \int_E f(x) dx$ .

*Remark.* Every simple function is Lebesgue integrable and its Lebesgue integral is nothing but the same as the elementary integral of  $f$ .

*Note.* Since we deal mostly with Lebesgue integrals, we shall write  $\int_E f(x) dx$  for  $\mathcal{L} \int_E f(x) dx$ . We sometimes write the integral simply as  $\int_E f$ . In the special case, when  $E = [a, b]$ , the integral  $\int_{[a,b]} f$  is traditionally denoted by  $\int_a^b f$ . If  $f$  is a bounded measurable function which vanishes outside a set  $E$  of finite measure, we write  $\int f$  for  $\int_E f$ . Observe that  $\int_E f = \int f \cdot \chi_E$ . However, for Riemann integral, we use the notation  $\mathcal{R} \int_a^b f(x) dx$  in order to distinguish it from the Lebesgue integral.

Next, we show the important role played by measurable functions. We shall prove that every bounded measurable function defined on a set whose measure is finite is Lebesgue integrable.

**2.5 Theorem.** *A bounded function  $f$  defined on a measurable set  $E$  of finite measure is Lebesgue integrable if and only if  $f$  is measurable.*

*Proof.* Let  $f$  be Lebesgue integrable over  $E$ . Then

$$\inf_{\psi \geq f} \int_E \psi(x) dx = \sup_{\varphi \leq f} \int_E \varphi(x) dx = I \text{ (say),}$$

for all simple functions  $\varphi$  and  $\psi$ .

Given an integer  $n$ ,  $\exists$  simple functions  $\varphi_n$  and  $\psi_n$  such that  $\varphi_n(x) \leq f(x) \leq \psi_n(x)$  satisfying

$$\int_E \psi_n(x) dx < I + \frac{1}{2n}$$

and 
$$\int_E \varphi_n(x) dx > I - \frac{1}{2n}.$$

This gives

$$\int_E \psi_n(x) dx - \int_E \varphi_n(x) dx < \frac{1}{n}. \quad (1)$$

Define the functions

$$\psi^* = \inf \psi_n \quad \text{and} \quad \varphi^* = \sup \varphi_n.$$

Since for each  $n$ ,  $\varphi_n$  and  $\psi_n$  are measurable functions, the functions  $\varphi^*$  and  $\psi^*$  are measurable (cf. IV-4.7) and  $\varphi^*(x) \leq f(x) \leq \psi^*(x)$ . Now, consider the sets

$$\begin{aligned} \Delta &= \{x : \varphi^*(x) < \psi^*(x)\}, \\ \Delta_v &= \left\{x : \varphi^*(x) < \psi^*(x) - \frac{1}{v}\right\}, \\ \Delta_{v,n} &= \left\{x : \varphi_n(x) < \psi_n(x) - \frac{1}{v}\right\}. \end{aligned}$$

We note that:

(a)  $\Delta = \bigcup_{v=1}^{\infty} \Delta_v.$

(b)  $\Delta_v \subset \Delta_{v,n}, \forall n.$

(c)  $m(\Delta_{v,n}) < v/n$ ; for if  $m(\Delta_{v,n}) \geq v/n$ , then

$$\begin{aligned} \int_{\Delta_{v,n}} \psi_n(x) dx - \int_{\Delta_{v,n}} \varphi_n(x) dx &= \int_{\Delta_{v,n}} \left\{ \psi_n(x) - \varphi_n(x) \right\} dx \\ &> \frac{1}{v} m(\Delta_{v,n}) \\ &\geq \frac{1}{n}, \end{aligned}$$

which contradicts (1).

Since  $n$  is arbitrary, we, in view of (b) and (c) above, have  $m(\Delta_v) = 0$  and hence  $m(\Delta) = 0$ . This proves that  $\varphi^* \geq \psi^*$  a.e. But  $\varphi^* \leq \psi^*$ . Hence  $\varphi^* = \psi^* = f$  a.e., and since each of the functions  $\varphi^*$  and  $\psi^*$  is measurable, the function  $f$  is measurable.

On the other hand, assume that  $f$  is a measurable function on  $E$ . Suppose  $f$  is bounded by  $M$ . Then

$$-M \leq f(x) \leq M, \quad \forall x \in E.$$

Divide the interval  $[-M, M]$  into  $2n$  equal parts and consider the sets

$$E_k = \left\{x \in E : \frac{M}{n} k \geq f(x) > \frac{M}{n} (k-1)\right\}, \quad -n \leq k \leq n.$$

Clearly,  $\{E_k : -n \leq k \leq n\}$  is a countable collection of pairwise disjoint measurable sets such that  $E = \bigcup_{k=-n}^n E_k$ . Therefore,

$$m(E) = \sum_{k=-n}^n m(E_k).$$

For each  $n$ , if we define simple functions  $\psi_n$  and  $\varphi_n$  as

$$\psi_n(x) = \frac{M}{n} \sum_{k=-n}^n k \chi_{E_k}(x)$$

and

$$\varphi_n(x) = \frac{M}{n} \sum_{k=-n}^n (k-1) \chi_{E_k}(x),$$

then they satisfy  $\varphi_n(x) \leq f(x) \leq \psi_n(x)$ . Thus,

$$\begin{cases} \int \inf_{\psi \geq f} \int_E \psi(x) dx \leq \int_E \psi_n(x) dx = \frac{M}{n} \sum_{k=-n}^n km(E_k) \\ \int \sup_{\varphi \leq f} \int_E \varphi(x) dx \geq \int_E \varphi_n(x) dx = \frac{M}{n} \sum_{k=-n}^n (k-1)m(E_k) \end{cases}$$

$$\Rightarrow \inf_{\psi \geq f} \int_E \psi(x) dx - \sup_{\varphi \leq f} \int_E \varphi(x) dx \leq \frac{M}{n} \sum_{k=-n}^n m(E_k) = \frac{M}{n} m(E).$$

Since  $n$  is arbitrary, we have

$$0 \leq \inf_{\psi \geq f} \int_E \psi(x) dx - \sup_{\varphi \leq f} \int_E \varphi(x) dx \leq 0.$$

Hence  $f$  is Lebesgue integrable on  $E$ . ■

*Remark.* The connection between integrability and measurability provides the major justification for introducing the ideas of measurable sets and measurable functions in the preceding chapters.

### 3 COMPARISON OF RIEMANN INTEGRAL AND LEBESGUE INTEGRAL

In this section, we undertake to establish that the Lebesgue integral is a generalization of the Riemann integral.

**3.1 Theorem.** *Let  $f$  be a bounded function defined on  $[a, b]$ . If  $f$  is Riemann integrable over  $[a, b]$ , then it is Lebesgue integrable and*

$$\mathcal{R} \int_a^b f(x) dx = \int_a^b f(x) dx.$$

*Proof.* Since  $f$  is Riemann integrable over  $[a, b]$ , we have

$$\inf_{\psi_1 \geq f} \int_a^b \psi_1(x) dx = \sup_{\varphi_1 \leq f} \int_a^b \varphi_1(x) dx = \mathcal{R} \int_a^b f(x) dx,$$

where  $\varphi_1$  and  $\psi_1$  vary over all step functions defined on  $[a, b]$ .

But every step function is a simple function. Therefore,

$$\begin{aligned} \sup_{\varphi_1 \leq f} \int_a^b \varphi_1(x) \, dx &\leq \sup_{\varphi \leq f} \int_a^b \varphi(x) \, dx \\ \inf_{\psi_1 \geq f} \int_a^b \psi_1(x) \, dx &\geq \inf_{\psi \geq f} \int_a^b \psi(x) \, dx, \end{aligned}$$

where  $\varphi$  and  $\psi$  vary over all the simple functions defined on  $[a, b]$ . Thus, in view of the above relations, we have

$$\begin{aligned} \mathcal{R} \int_a^b f(x) \, dx &\leq \sup_{\varphi \leq f} \int_a^b \varphi(x) \, dx \leq \inf_{\psi \geq f} \int_a^b \psi(x) \, dx \leq \mathcal{R} \int_a^b f(x) \, dx \\ \Rightarrow \sup_{\varphi \leq f} \int_a^b \varphi(x) \, dx &= \inf_{\psi \geq f} \int_a^b \psi(x) \, dx = \mathcal{R} \int_a^b f(x) \, dx \\ \Rightarrow \int_a^b f(x) \, dx &= \mathcal{R} \int_a^b f(x) \, dx. \blacksquare \end{aligned}$$

*Remark.* Though all bounded measurable functions are Lebesgue integrable, there are functions of this kind which are not Riemann integrable.

**3.2 Example.** Consider the Dirichlet function  $f : [0, 1] \rightarrow \mathbb{R}$  defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

The function  $f$  is clearly bounded and measurable on  $[0, 1]$  and hence Lebesgue integrable. Also  $\int_0^1 f(x) \, dx = 0$ . However,  $f$  is not Riemann integrable since

$$\mathcal{R} \int_0^1 f(x) \, dx = 1 \quad \text{and} \quad \mathcal{R} \int_0^1 f(x) \, dx = 0.$$

*Note.* We have observed that a bounded function defined on an interval may be Lebesgue integrable without being Riemann integrable (cf. Example 3.2). On the other hand, if a function is Riemann integrable, then it is Lebesgue integrable too (cf. Theorem 3.1) and the two integrals are equal. Thus the class of Lebesgue integrable functions is wider than that of Riemann integrable functions. However, the situation is entirely different in the case of improper Riemann integrals. There do exist improper Riemann integrable functions which are not Lebesgue integrable (see § 7).

**Problem 1.** Prove that a Riemann integrable function defined on  $[a, b]$  is measurable.

#### 4 PROPERTIES OF THE LEBESGUE INTEGRAL FOR BOUNDED MEASURABLE FUNCTIONS

**4.1 Theorem.** *Let  $f$  and  $g$  be bounded measurable functions defined on a set  $E$  of finite measure. Then:*

(a)  $\int_E af = a \int_E f$ , for all real numbers  $a$ .

(b)  $\int_E (f+g) = \int_E f + \int_E g$ .

(c) If  $f = g$  a.e., then

$$\int_E f = \int_E g.$$

(d) If  $f \leq g$  a.e., then

$$\int_E f \leq \int_E g.$$

Hence

$$\left| \int_E f \right| \leq \int_E |f|.$$

(e) If  $\alpha \leq f(x) \leq \beta$ , then

$$\alpha m(E) \leq \int_E f(x) dx \leq \beta m(E).$$

(f) If  $E_1$  and  $E_2$  are disjoint measurable subsets of  $E$ , then

$$\int_{E_1 \cup E_2} f = \int_{E_1} f + \int_{E_2} f.$$

*Proof.* (a) The result is trivially true when  $a = 0$ . Assume that  $a \neq 0$ . If  $\psi$  is a simple function then so is  $a\psi$ ; and conversely. Hence for  $a > 0$

$$\int_E af = \inf_{a\psi \geq af} \int_E a\psi = a \inf_{\psi \geq f} \int_E \psi = a \int_E f;$$

and for  $a < 0$

$$\int_E af = \inf_{a\psi \geq af} \int_E a\psi = a \sup_{\psi \leq f} \int_E \psi = a \int_E f.$$

(b) If  $\psi_1$  and  $\psi_2$  are simple functions such that  $\psi_1 \geq f$  and  $\psi_2 \geq g$ , then  $\psi_1 + \psi_2$  is a simple function and  $\psi_1 + \psi_2 \geq f + g$ . Hence

$$\int_E (f+g) \leq \int_E (\psi_1 + \psi_2) = \int_E \psi_1 + \int_E \psi_2$$

$$\Rightarrow \int_E (f+g) \leq \inf_{\psi_1 \geq f} \int_E \psi_1 + \inf_{\psi_2 \geq g} \int_E \psi_2 = \int_E f + \int_E g.$$

Similarly, if  $\varphi_1 \leq f$  and  $\varphi_2 \leq g$  are simple functions, then  $\varphi_1 + \varphi_2$  is a simple function with  $\varphi_1 + \varphi_2 \leq f + g$ , and so

$$\int_E (f+g) \geq \int_E (\varphi_1 + \varphi_2) = \int_E \varphi_1 + \int_E \varphi_2,$$

which gives

$$\int_E (f+g) \geq \int_E f + \int_E g.$$

Hence the result follows.

(c) It is sufficient to show that

$$\int_E (f-g) = 0.$$

Since  $f-g=0$  a.e., it follows that if  $\psi \geq f-g$ , then  $\psi \geq 0$  a.e. Therefore, in view of Theorem 2.2 (b), we have

$$\int_E \psi \geq 0$$

$$\Rightarrow \int_E (f-g) \geq 0.$$

Similarly, one can prove that

$$\int_E (f-g) \leq 0.$$

This proves the result.

(d) It is similar to that of (c).

(e) It follows from (d) and the fact that

$$\int_E 1 = m(E).$$

(f) It is an immediate consequence of (b) and the fact that  $\chi_{E_1 \cup E_2} = \chi_{E_1} + \chi_{E_2}$  since  $E_1$  and  $E_2$  are disjoint sets. ■

We derive certain simple results from Theorem 4.1 in the form of corollaries.

**4.2 Corollary.** If  $f(x) \geq 0$  on  $E$ , then  $\int_E f(x) dx \geq 0$ ; and if  $f(x) \leq 0$  on  $E$ , then  $\int_E f(x) dx \leq 0$ .

**4.3 Corollary.** If  $m(E) = 0$ , then  $\int_E f = 0$ .

**4.4 Corollary.** If  $f(x) = k$  a.e. on  $E$ , then  $\int_E f = km(E)$ . In particular, if  $f = 0$  a.e. on  $E$ , then  $\int_E f = 0$ ; and if  $f = 1$  a.e. on  $E$ , then  $\int_E f = m(E)$ .

*Remarks.* 1. The converse of Theorem 4.1 (c) and Corollary 4.4 need not be true (cf. Problems 2 and 3).

2. The result in Theorem 4.1 (c) indicates one of the great advantages of the Lebesgue integral over the Riemann integral. If  $f$  is a bounded measurable function on  $[a, b]$  (and hence Lebesgue integrable over  $[a, b]$ ), then changing the values of the function  $f$  on a set of measure zero has no effect either on the Lebesgue integrability of or on the value of its integral over  $[a, b]$ . On the other hand, changing the values of a Riemann integrable function on a set of measure zero may even destroy the Riemann integrability of the function.

**4.5 Example.** Consider the function  $f$  defined by  $f(x) = 1$ , ( $0 \leq x \leq 1$ ). Then the characteristic function  $\chi_A$  of the set  $A$ , where  $A$  is the set of all irrationals in  $[0, 1]$ , can be regarded as a function obtained by changing the values of  $f$  on a set of measure zero in  $[0, 1]$ ; namely, on the rational points where it takes value zero. Now, one can easily observe that  $f$  is Riemann integrable in  $[0, 1]$  while  $\chi_A$  is not.

**Problem 2.** Show by an example that the converse of Theorem 4.1(c) need not be true.

*Solution.* Let the functions  $f : [-1, 1] \rightarrow \mathbb{R}$  and  $g : [-1, 1] \rightarrow \mathbb{R}$  be defined by

$$f(x) = \begin{cases} 2 & \text{if } x \leq 0 \\ 0 & \text{if } x > 0 \end{cases}$$

and

$$g(x) = 1, \quad \forall x.$$

Then

$$\int_{-1}^1 f(x) dx = 2 = \int_{-1}^1 g(x) dx.$$

But  $f \neq g$  a.e. In other words, they are not equal even for a single point in  $[-1, 1]$ .

**Problem 3.** Show by an example that the converse of Corollary 4.4 need not be true.

*Solution.* Consider the function  $f : [-1, 1] \rightarrow \mathbb{R}$  defined by

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0. \end{cases}$$

Then

$$\int_{-1}^1 f(x) dx = \int_{-1}^0 f(x) dx + \int_0^1 f(x) dx = 0.$$

But  $f \neq 0$  a.e. on  $[-1, 1]$ .

In fact, towards the converse of Corollary 4.4, we prove the following.

**4.6 Theorem.** If  $\int_E f = 0$  and  $f(x) \geq 0$  on  $E$ , then  $f = 0$  a.e.

*Proof.* Suppose  $A \subset E$  is the set on which  $f(x) > 0$ . Then

$$\begin{aligned} A &= \{x \in E : f(x) > 0\} \\ &= \bigcup_{n=1}^{\infty} \left\{ x \in E : f(x) > \frac{1}{n} \right\}. \end{aligned}$$

Now it is enough to prove that the measure of each of the sets in the union is zero. Let, if possible, there be a natural number  $N$  such that

$$m\left(\left\{x \in E : f(x) > \frac{1}{N}\right\}\right) = \lambda (\neq 0).$$

Write

$$\begin{aligned} E_1 &= \left\{x \in E : f(x) > \frac{1}{N}\right\}, \\ E_2 &= \left\{x \in E : f(x) \leq \frac{1}{N}\right\}. \end{aligned}$$

Thus  $E_1$  and  $E_2$  are two measurable disjoint sets such that  $E = E_1 \cup E_2$ . Therefore

$$\int_E f = \int_{E_1} f + \int_{E_2} f.$$

But

$$\int_{E_1} f > \frac{1}{N}m(E_1) \geq \frac{\lambda}{N} > 0.$$

Consequently,  $\int_E f > 0$  which contradicts the hypothesis that

$$\int_E f = 0.$$

This completes the proof of the theorem. ■

We now proceed to prove a result concerning the interchange of the limit and integral operations of a convergent sequence of bounded measurable functions.

**4.7 Theorem (Bounded Convergence Theorem).** *Let  $\{f_n\}$  be a sequence of measurable functions defined on a set  $E$  of finite measure. Suppose there is a real number  $M$  such that  $|f_n(x)| \leq M$ , for all  $x$  and all  $n$ . If  $f(x) = \lim_{n \rightarrow \infty} f_n(x)$  for each  $x \in E$ , then*

$$\int_E f = \lim_{n \rightarrow \infty} \int_E f_n.$$

*Proof.* Since the function  $f$  is the limit of a sequence of measurable functions, it is measurable (cf. IV-4.8) and hence Lebesgue integrable. Further, by Theorem IV-10.1, for a given  $\epsilon > 0$ , there is a measurable set  $A \subset E$  with  $m(A) < \epsilon/4M$  and a positive integer  $N$  such that

$$|f_n(x) - f(x)| < \frac{\epsilon}{2m(E)},$$

for all  $n \geq N$  and  $x \in E - A$ . Also,

$$|f_n(x)| \leq M, \quad \forall n \in \mathbb{N} \text{ and } x \in E$$

$$\Rightarrow |f(x)| \leq M, \quad x \in E$$

$$\Rightarrow |f_n(x) - f(x)| \leq 2M, \quad x \in E \text{ and, in particular, for } x \in A.$$

Therefore

$$\begin{aligned} \left| \int_E f_n - \int_E f \right| &= \left| \int_E (f_n - f) \right| \\ &\leq \int_E |f_n - f| \\ &= \int_{E-A} |f_n - f| + \int_A |f_n - f| \\ &< \frac{\epsilon}{2m(E)}m(E-A) + 2Mm(A) \\ &< \epsilon, \quad \forall n \geq N. \end{aligned}$$

Hence  $\lim_{n \rightarrow \infty} \int_E f_n = \int_E f$ . ■

A slightly improved form of Theorem 4.7 is the following.

**4.8 Theorem.** Let  $\{f_n\}$  be a sequence of measurable functions defined on a set  $E$  of finite measure, and suppose that  $|f_n(x)| \leq M$  for all  $n$  and all  $x$  in  $E$ . If  $f_n \rightarrow f$  a.e. on  $E$ , then  $f$  is integrable and

$$\int_E f = \lim_{n \rightarrow \infty} \int_E f_n.$$

*Remark.* The results in Theorems 4.7 and 4.8 need not be true in case of Riemann integrals.

**4.9 Example.** Let  $\{r_i\}$  be an enumeration of all rational numbers in  $[a, b]$ . Write  $S_n = \{r_i : i = 1, 2, 3, \dots, n\}$ ,  $n \in \mathbb{N}$ . Define, for each  $n \in \mathbb{N}$ , the function  $f_n : [a, b] \rightarrow \mathbb{R}$  by

$$f_n(x) = \begin{cases} 1 & \text{if } x \in S_n \\ 0 & \text{if } x \notin S_n. \end{cases}$$

We note that  $f_n$  is discontinuous only at  $n$  points in  $[a, b]$ ; namely, the points of  $S_n$ . Therefore  $f_n$  is Riemann integrable over  $[a, b]$ , and we have

$$\mathcal{R} \int_a^b f_n(x) dx = 0, \quad \forall n \in \mathbb{N}.$$

Further, we observe that  $f_n \rightarrow f$  on  $[a, b]$ , where the limit function  $f$  is given by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational,} \end{cases}$$

which is not Riemann integrable although  $f$  is Lebesgue integrable, cf. Example 3.2.

We find from Example 4.9 that the result in Theorem 4.8 does not hold good for Riemann integral simply because the limit function of a convergent sequence for Riemann integrable functions can be a function which is not Riemann integrable. However, we prove the following.

**4.10 Theorem.** Let  $\{f_n\}$  be a sequence of Riemann integrable functions defined on  $[a, b]$  such that  $|f_n(x)| \leq M$ , for all  $n$  and  $x \in [a, b]$ .

If  $\{f_n\}$  converges a.e. to a Riemann integrable function  $f$  defined on  $[a, b]$ , then

$$\mathcal{R} \int_a^b f(x) dx = \lim_{n \rightarrow \infty} \mathcal{R} \int_a^b f_n(x) dx.$$

*Proof.* It follows in view of Theorems 3.1 and 4.8.  $\blacksquare$

## 5 INTEGRAL OF NONNEGATIVE MEASURABLE FUNCTIONS

In this section, we generalize the definition of the Lebesgue integral to include unbounded functions, particularly nonnegative functions (measurable) defined on sets of finite or infinite measure.

**5.1 Definition.** If  $f$  is a nonnegative measurable function defined on a set  $E$  (measurable), we define

$$\int_E f = \sup_{h \leq f} \int_E h,$$

where  $h$  is a bounded measurable function such that  $m(\{x \in E : h(x) \neq 0\})$  is finite.

**5.2 Theorem.** Let  $f$  and  $g$  be nonnegative measurable functions defined on a set  $E$ . Then:

- (a)  $\int_E cf = c \int_E f$ ,  $c > 0$ .
- (b)  $\int_E (f+g) = \int_E f + \int_E g$ .
- (c) If  $f \leq g$  a.e., then  $\int_E f \leq \int_E g$ .

*Proof.* (a) By Definition 5.1, we have

$$\int_E cf = \sup_{ch \leq cf} \int_E ch = c \sup_{h \leq f} \int_E h = c \int_E f.$$

(b) Let  $h$  and  $k$  be bounded and measurable functions such that  $h \leq f$ ,  $k \leq g$  and vanish outside sets of finite measure, i.e.

$$\begin{cases} m(\{x \in E : h(x) \neq 0\}) < \infty \\ m(\{x \in E : k(x) \neq 0\}) < \infty. \end{cases}$$

Then, we have  $h+k \leq f+g$ , and so

$$\int_E h + \int_E k = \int_E (h+k) \leq \int_E (f+g)$$

$$\Rightarrow \sup_{h \leq f} \int_E h + \sup_{k \leq g} \int_E k \leq \int_E (f+g).$$

This gives

$$\int_E f + \int_E g \leq \int_E (f+g).$$

To prove the reverse inequality, let  $\psi$  be a bounded measurable function which vanishes outside a set of finite measure and satisfies  $\psi \leq f+g$ . Define functions  $h$  and  $k$  by setting

$$\begin{cases} h(x) = \min(f(x), \psi(x)) \\ k(x) = \psi(x) - h(x). \end{cases}$$

Now, we claim that  $h(x) \leq f(x)$  and  $k(x) \leq g(x)$ . The first assertion is obvious from the definition. In order to establish the second relation we note that

$$h(x) + k(x) = \psi(x) \leq f(x) + g(x).$$

If  $h(x) = \psi(x)$ , then  $k(x) = 0$  and hence  $k(x) \leq g(x)$  since  $g$  is nonnegative; if  $h(x) = f(x)$ , then obviously  $k(x) \leq g(x)$ .

Since  $h$  and  $k$  are bounded by the bounds of  $\psi$ , and also vanish where  $\psi$  vanishes, we have

$$\begin{aligned} \int_E \psi &= \int_E (h+k) = \int_E h + \int_E k \\ &\leq \int_E f + \int_E g \end{aligned}$$

$$\Rightarrow \sup_{\psi \leq f+g} \int_E \psi \leq \int_E f + \int_E g.$$

Hence

$$\int_E f + \int_E g \geq \int_E (f+g).$$

This completes the proof of (b).

(c) It follows directly in view of Definition 5.1 and Theorem 4.1 (d). ■

**Problem 4.** Let  $f$  be a nonnegative measurable function. Show that  $f=0$  a.e. on  $E$  if and only if  $\int_E f=0$ .

**5.3 Theorem (Fatou's Lemma).** Let  $\{f_n\}$  be a sequence of nonnegative measurable functions and  $f_n \rightarrow f$  a.e. on  $E$ . Then

$$\int_E f \leq \liminf_{n \rightarrow \infty} \int_E f_n.$$

*Proof.* We may assume, without any loss of generality, that the sequence  $\{f_n\}$  converges to  $f$  everywhere on  $E$  since the integrals over sets of measure zero are zero.

Let  $h$  be a bounded measurable function such that  $h \leq f$  and vanishes outside a set of finite measure, viz.

$$m(\{x \in E : h(x) \neq 0\}) < \infty.$$

Let us denote this set by  $E'$ . Define a sequence  $\{h_n\}$  of functions by setting

$$h_n(x) = \min \{h(x), f_n(x)\}.$$

Then each  $h_n$  is clearly bounded by the bounds of  $h$  and vanishes outside  $E'$ . Moreover, we note that

$$\begin{aligned} \lim_{n \rightarrow \infty} h_n(x) &= \lim_{n \rightarrow \infty} \min \{h(x), f_n(x)\} \\ &= \min \{h(x), f(x)\} \\ &= h(x), \quad x \in E'. \end{aligned}$$

Thus  $\{h_n\}$  is a uniformly bounded sequence of measurable functions such that  $h_n \rightarrow h$  on  $E'$ . Therefore, by the Bounded Convergence Theorem (cf. Theorem 4.7), we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{E'} h_n &= \int_{E'} h \\ \Rightarrow \int_E h &= \int_{E'} h = \lim_{n \rightarrow \infty} \int_{E'} h_n \leq \liminf_{n \rightarrow \infty} \int_E f_n. \end{aligned}$$

Hence, taking the supremum over all  $h \leq f$ , we get

$$\int_E f \leq \liminf_{n \rightarrow \infty} \int_E f_n. \blacksquare$$

*Remark.* Theorem 5.3 need not hold good, even in the presence of uniform convergence, unless the functions  $f_n$  are nonnegative,

**5.4 Example.** Let the functions  $f_n : [0, 1] \rightarrow \mathbb{R}$  be given by

$$f_n(x) = \begin{cases} -n & \text{if } \frac{1}{n} \leq x \leq \frac{2}{n} \\ 0 & \text{if otherwise.} \end{cases}$$

Then  $f(x) = \lim_{n \rightarrow \infty} f_n(x) = 0$  a.e. on  $[0, 1]$ , and

$$\lim_{n \rightarrow \infty} \int_0^1 f_n(x) dx = -1,$$

while  $\int_0^1 f(x) dx = 0$ .

**Problem 5.** Show that the strict inequality exists in Theorem 5.3.

*Solution.* Consider a sequence  $\{f_n\}$  of functions defined on  $\mathbf{R}$ , where

$$f_n(x) = \begin{cases} 1 & \text{if } n \leq x < n+1 \\ 0 & \text{if otherwise.} \end{cases}$$

We note that

$$\lim_{n \rightarrow \infty} f_n(x) = 0,$$

and so  $f = 0$ . Therefore  $\int_{\mathbf{R}} f = 0$ .

If we write  $E_1 = [n, n+1[$  and  $E_2 = \mathbf{R} - [n, n+1[$ , then

$$\int_{\mathbf{R}} f_n = \int_{E_1} f_n + \int_{E_2} f_n = 1.$$

Hence

$$\int_{\mathbf{R}} f < \lim_{n \rightarrow \infty} \int_{\mathbf{R}} f_n.$$

*Note* For another example which ascertains that the strict inequality exists in Theorem 5.3, see Problem 19.

The conclusion of Theorem 5.3 is weak as it only asserts that  $\int f \leq \lim_{n \rightarrow \infty} \int f_n$ . In order to get equality between  $\int f$  and  $\lim_{n \rightarrow \infty} \int f_n$ , we should pose some more conditions on the sequence  $\{f_n\}$ . In fact, we prove the following.

**5.5 Theorem (Monotone Convergence Theorem).** Let  $\{f_n\}$  be an increasing sequence of nonnegative measurable functions, and let  $f = \lim_{n \rightarrow \infty} f_n$ . Then,

$$\int f = \lim_{n \rightarrow \infty} \int f_n.$$

*Proof* By Theorem 5.3, we have

$$\int f \leq \varliminf_{n \rightarrow \infty} \int f_n.$$

Since for each  $n$ ,  $f_n \leq f$  and so  $\int f_n \leq \int f$ , c.f. Theorem 5.2 (c). This implies

$$\varliminf_{n \rightarrow \infty} \int f_n \leq \int f,$$

and hence the result follows. ■

**5.6 Corollary.** *Let  $\{u_n\}$  be a sequence of nonnegative measurable functions, and let  $f = \sum_{n=1}^{\infty} u_n$ . Then*

$$\int f = \sum_{n=1}^{\infty} \int u_n.$$

*Proof.* Setting  $f_n = \sum_{i=1}^n u_i$ , we get a sequence  $\{f_n\}$  satisfying the hypothesis of Theorem 5.5. Since  $\lim_{n \rightarrow \infty} f_n = \sum_{n=1}^{\infty} u_n = f$ , we have

$$\begin{aligned} \int f &= \lim_{n \rightarrow \infty} \int f_n \\ &= \lim_{n \rightarrow \infty} \int \sum_{i=1}^n u_i. \end{aligned}$$

In view of Theorem 5.2 (b), we may write

$$\int \sum_{i=1}^n u_i = \sum_{i=1}^n \int u_i.$$

Hence

$$\int f = \sum_{n=1}^{\infty} \int u_n. \quad \blacksquare$$

**5.7 Corollary.** *Let  $\{E_i\}$  be a sequence of disjoint measurable sets and  $E = \bigcup_i E_i$ . If  $f$  is a nonnegative measurable function defined on  $E$ , then,*

$$\int_E f = \sum_i \int_{E_i} f.$$

*Proof.* Set  $f_i = f \cdot \chi_{E_i}$ . Clearly, each  $f_i$  is a nonnegative measurable function. Also

$$\begin{aligned} f \cdot \chi_E &= f \cdot \chi_{\bigcup_i E_i} \\ &= f(\chi_{E_1} + \chi_{E_2} + \cdots + \chi_{E_n} + \cdots) \end{aligned}$$

$$\begin{aligned}
 &= \sum_I f \cdot \chi_{E_i} \\
 &= \sum_I f_i.
 \end{aligned}$$

Therefore, using Corollary 5.6, we obtain

$$\begin{aligned}
 \int f \cdot \chi_E &= \sum_{i=1}^{\infty} \int f_i = \sum_I \int f \cdot \chi_{E_i} \\
 \Rightarrow \int_E f &= \sum_I \int_{E_i} f. \blacksquare
 \end{aligned}$$

**Problem 6.** Show that Theorem 5.5 need not hold good for a decreasing sequence of functions.

*Solution.* Consider a sequence  $\{f_n\}$  of functions each defined on  $\mathbb{R}$ , where

$$f_n(x) = \begin{cases} 0 & \text{if } x < n \\ 1 & \text{if } x \geq n. \end{cases}$$

Clearly,  $\{f_n\}$  is a decreasing sequence of nonnegative measurable functions and converging to zero function,  $f$ . Thus

$$\int_{-\infty}^{\infty} f(x) dx = 0.$$

But

$$\begin{aligned}
 \int_{-\infty}^{\infty} f_n(x) dx &= \int_{-\infty}^n f(x) dx + \int_n^{\infty} f(x) dx \\
 \Rightarrow \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} f_n(x) dx &= \infty.
 \end{aligned}$$

**Problem 7.** Prove that if  $\{f_n\}$  is a sequence of nonnegative measurable functions defined on  $E$ , then

$$\int_E \lim_{n \rightarrow \infty} f_n \leq \lim_{n \rightarrow \infty} \int_E f_n.$$

*Solution.* Corresponding to each positive integer  $n$ , define  $h_n = \inf \{f_v : v \geq n\}$ . Each  $h_n$  is measurable. Since  $h_n \leq f_v$ , for  $v \geq n$  we have

$$\begin{aligned}
 \int_E h_n &\leq \int_E f_v, \quad \text{for } v \geq n \\
 \Rightarrow \int_E h_n &\leq \lim_{v \rightarrow \infty} \int_E f_v, \quad \forall n \in \mathbb{N}.
 \end{aligned}$$

One may verify that  $\{h_n\}$  is an increasing sequence which converges to the limit function  $h(= \lim_{n \rightarrow \infty} f_n)$ . Therefore, in view of Theorem 5.5, we have

$$\int_E \lim_{n \rightarrow \infty} f_n = \int_E h = \lim_{n \rightarrow \infty} \int_E h_n \leq \lim_{n \rightarrow \infty} \int_E f_n.$$

*Remark.* The result in Problem 7 is a generalization of Fatou's Lemma (cf. Theorem 5.3).

Unlike the convention in the case of the Riemann integral, we now acknowledge  $\infty$  as a possible value for a Lebesgue integral. It is necessary to distinguish Lebesgue integrals having finite and infinite values.

**5.8 Definition.** A nonnegative measurable function  $f$  defined on a set  $E$  (measurable) is said to be **integrable** (or **summable**) if

$$\int_E f < \infty.$$

*Note.* Every nonnegative measurable function has an integral (Lebesgue) but only functions having finite integrals are called integrable (or summable).

**5.9 Theorem.** Let  $f$  and  $g$  be nonnegative measurable functions. If  $f > g$  on  $E$  and  $f$  is integrable over  $E$ , then  $g$  is also integrable over  $E$  and

$$\int_E (f - g) = \int_E f - \int_E g.$$

*Proof.* By Theorem 5.2 (b), we have

$$\int_E f = \int_E (f - g + g) = \int_E (f - g) + \int_E g, \quad (2)$$

since the functions  $f - g$  and  $g$  are nonnegative and measurable. Further,  $f$  being integrable over  $E$ ,  $\int_E f < \infty$ . Therefore, each integral on the right of (2) is finite. In particular,

$$\int_E g < \infty.$$

This verifies that  $g$  is an integrable function over  $E$ . The second part of the theorem is now obvious. ■

**5.10 Theorem.** Let  $f$  be a nonnegative function which is integrable over a set  $E$ . Then given  $\epsilon > 0$  there is a  $\delta > 0$  such that for every set  $A \subset E$  with  $m(A) < \delta$ , we have

$$\int_A f < \epsilon.$$

*Proof.* The result is trivially true if  $f$  is a bounded function on  $E$  since in that case  $|f(x)| \leq M, \forall x \in E$ , for some number  $M$  and thus given  $\epsilon > 0$ , one can determine a  $\delta (= \epsilon/M) > 0$  for which

$$\int_A f \leq Mm(A) < \epsilon.$$

Assume that  $f$  is an unbounded function on  $E$ . Consider now a sequence  $\{f_n\}$  of functions defined on  $E$ , where

$$f_n(x) = \begin{cases} f(x) & \text{if } f(x) \leq n \\ n & \text{if } f(x) > n. \end{cases}$$

Observe that  $\{f_n\}$  is an increasing sequence of bounded functions such that  $f_n \rightarrow f$  on  $E$ . Then, by the Monotone Convergence Theorem, given  $\epsilon > 0$ , there is an integer  $N$  such that

$$\int_E f_N > \int_E f - \frac{\epsilon}{2}.$$

Therefore, by Theorem 5.9, we have

$$\int_E (f - f_N) = \int_E f - \int_E f_N < \frac{\epsilon}{2},$$

since  $f$  is an integrable function.

Choose  $\delta < \frac{\epsilon}{2N}$ . If  $m(A) < \delta$ , then

$$\begin{aligned} \int_A f &= \int_A (f - f_N) + \int_A f_N \\ &< \int_E (f - f_N) + N \cdot m(A) \\ &< \frac{\epsilon}{2} + N \cdot \frac{\epsilon}{2N} = \epsilon. \blacksquare \end{aligned}$$

*Note.* The technique used in the proof of Theorem 5.10 helps us to evaluate the Lebesgue integral of nonnegative functions, bounded or unbounded, see Problems 8 and 9.

**Problem 8.** Evaluate the integral (Lebesgue) of the function  $f: [0, 1] \rightarrow \mathbf{R}$  defined by

$$f(x) = \begin{cases} \frac{1}{\sqrt[3]{x}} & \text{if } 0 < x \leq 1 \\ 0 & \text{if } x = 0. \end{cases}$$

*Solution.* Here  $f$  is an unbounded nonnegative measurable function defined on  $[0, 1]$ . Define a sequence  $\{f_n\}$  of functions on  $[0, 1]$ , where

$$f_n(x) = \begin{cases} f(x) & \text{if } f(x) \leq n \\ n & \text{if } f(x) > n. \end{cases}$$

In other words,

$$f_n(x) = \begin{cases} \frac{1}{\sqrt[3]{x}} & \text{if } x \geq \frac{1}{n^3} \\ n & \text{if } x < \frac{1}{n^3}. \end{cases}$$

Clearly, as in proof of Theorem 5.10, we observe that  $\{f_n\}$  is an increasing sequence of nonnegative measurable function such that  $f_n \rightarrow f$  on  $[0, 1]$ . Therefore, by the Monotone Convergence Theorem, we get

$$\begin{aligned} \int_0^1 f &= \lim_{n \rightarrow \infty} \int_0^1 f_n = \lim_{n \rightarrow \infty} \left[ \int_0^{1/n^3} n \, dx + \int_{1/n^3}^1 x^{-1/3} \, dx \right] \\ &= \lim_{n \rightarrow \infty} \left[ \frac{1}{n^2} + \frac{3}{2} - \frac{3}{2n^2} \right] \\ &= \frac{3}{2}. \end{aligned}$$

Hence, by Definition 5.8,  $f$  is integrable over  $[0, 1]$ , and the value of its integral is  $3/2$ .

*Remark.* We note that the integral

$$\int_0^1 \frac{1}{\sqrt[3]{x}} \, dx$$

regarded as a Riemann integral is an improper integral, whereas, regarded as a Lebesgue integral, it is perfectly proper, even though the integrand is unbounded. Also, observe that the value of the integral in each case is the same, i.e.  $3/2$ .

**Problem 9.** Evaluate the integral (Lebesgue) of the function:  $f: [0, 1] \rightarrow \mathbb{R}$  defined by

$$f(x) = \begin{cases} \frac{1}{x} & \text{if } 0 < x \leq 1 \\ 0 & \text{if } x = 0, \end{cases}$$

and show that  $f$  is not Lebesgue integrable on  $[0, 1]$ .

*Solution.* The solution runs on the lines of that of Problem 8. In this case  $\int_0^1 f = \infty$ . Hence  $f$  is not integrable (Lebesgue) over  $[0, 1]$ .

*Note.* Problem 9 provides an example of an unbounded nonnegative measurable function which is not integrable (Lebesgue) over  $[0, 1]$ , whereas, the function considered in Problem 8, although an unbounded nonnegative measurable function, is integrable over  $[0, 1]$ .

**Problem 10.** Let  $f$  be a nonnegative integrable function. Show that the function  $F$  defined by

$$F(x) = \int_{-\infty}^x f(t) dt$$

is continuous on  $\mathbb{R}$ .

*Solution.* In view of Theorem 5.10, given an  $\epsilon > 0$  there is a  $\delta > 0$  such that for every set  $A \subset \mathbb{R}$  with  $m(A) < \delta$ , we have

$$\left| \int_A f \right| < \epsilon.$$

If  $x_0 \in \mathbb{R}$ , then for every  $x \in \mathbb{R}$  with  $|x - x_0| < \delta$ , we have

$$\begin{aligned} & \left| \int_{x_0}^x f(t) dt \right| < \epsilon \\ \Rightarrow & \left| \int_{-\infty}^x f(t) dt - \int_{-\infty}^{x_0} f(t) dt \right| < \epsilon \\ \Rightarrow & |F(x) - F(x_0)| < \epsilon. \end{aligned}$$

Hence  $F$  is continuous at  $x_0$ . Since  $x_0 \in \mathbb{R}$  is arbitrary,  $F$  is continuous on  $\mathbb{R}$ .

**Problem 11.** Let  $\{f_n\}$  be a sequence of nonnegative measurable functions which converge to  $f$ , and suppose  $f_n \leq f$  for each  $n$ . Show that

$$\int f = \lim_{n \rightarrow \infty} \int f_n.$$

*Solution.* Since  $f_n \leq f$ , in view of Theorem 5.2(c), we have

$$\int f_n \leq \int f, \quad \forall n \in \mathbb{N}$$

and hence

$$\overline{\lim}_{n \rightarrow \infty} \int f_n \leq \int f.$$

The result follows by using Fatou's Lemma (cf. Theorem 5.3).

## 6 GENERAL LEBESGUE INTEGRAL

We now undertake to extend the definition of the Lebesgue integral to include the most general possible functions (measurable) that take both positive and negative values. To tackle such functions, we show that a measurable function can be written as the difference of two nonnegative measurable functions.

Let us recall that if  $f$  is a real-valued function defined on a set  $E$ , its positive and negative parts are defined as

$$f^+ = \max(f, 0)$$

and

$$f^- = \max(-f, 0),$$

and that  $f$  is measurable if and only if both  $f^+$  and  $f^-$  are measurable, cf. IV-4.6.

One can easily verify that

$$f = f^+ - f^-$$

and

$$|f| = f^+ + f^-.$$

**6.1 Definition.** A measurable function  $f$  is said to be **integrable** over  $E$  if  $f^+$  and  $f^-$  both are integrable over  $E$ . In this case, we define

$$\int_E f = \int_E f^+ - \int_E f^-,$$

**6.2 Theorem.** A function  $f$  is integrable over  $E$  if and only if  $|f|$  is integrable over  $E$ .

*Proof.* If  $f$  is integrable over  $E$ , it follows that  $f^+$  and  $f^-$  both are integrable over  $E$ . By Theorem 5.2 (b), we have

$$\int_E |f| = \int_E f^+ + \int_E f^-.$$

Hence the function  $|f|$  is integrable over  $E$ . Conversely, if  $\int_F |f|$  is finite, then so are  $\int_F f^+$  and  $\int_E f^-$ , and, hence, the result follows. ■

*Remark.* Lebesgue integration is absolute integration in the sense that  $f$  is Lebesgue integrable if and only if  $|f|$  is so. In other words, the Lebesgue integral integrates only those functions whose absolute value functions are also integrable. However, this is not always true in the case of Riemann integrals.

**6.3 Example.** Consider a function  $f: [0, 1] \rightarrow \mathbb{R}$  given by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ -1 & \text{if } x \text{ is irrational.} \end{cases}$$

Observe that  $|f| = 1$  on  $[0, 1]$  and hence Riemann integrable while  $f$  is not.

**Problem 12.** Show that the function  $f: [0, \infty[ \rightarrow \mathbb{R}$  defined by

$$f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is not Lebesgue integrable over  $[0, \infty[$ .

*Solution.* In view of Theorem 6.2, it is sufficient to prove that

$$\int_0^\infty \left| \frac{\sin x}{x} \right| = \infty.$$

For this, consider the integral

$$\begin{aligned} \int_0^{n\pi} \frac{|\sin x|}{x} dx &= \sum_{k=1}^n \int_{(k-1)\pi}^{k\pi} \frac{|\sin x|}{x} dx \\ &= \sum_{k=1}^n \int_0^\pi \frac{|\sin \{z + (k-1)\pi\}|}{z + (k-1)\pi} dz \\ &\geq \sum_{k=1}^n \int_0^\pi \frac{|\sin \{z + (k-1)\pi\}|}{k\pi} dz \\ &= \frac{1}{\pi} \sum_{k=1}^n \frac{1}{k} \int_0^\pi |\sin z| dz \\ &= \sum_{k=1}^n \frac{2}{k\pi}. \end{aligned}$$

Thus

$$\lim_{n \rightarrow \infty} \int_0^\pi \frac{|\sin x|}{x} dx \geq \sum_{k=1}^{\infty} \frac{2}{k\pi} = \infty$$

$$\Rightarrow \int_0^\infty \frac{|\sin x|}{x} dx = \infty.$$

**6.4 Theorem.** *Let  $f$  be a measurable function over a set  $E$ . If there is an integrable function  $g$  such that  $|f| \leq g$ , then  $f$  is integrable over  $E$ .*

*Proof.* From  $f^+ \leq g$ , it follows that  $\int_E f^+ \leq \int_E g$ , and so  $f^+$  is integrable on  $E$ . Similarly  $f^- \leq g$  implies the integrability of  $f^-$ . Hence  $f$  is integrable over  $E$ . ■

**6.5 Theorem.** *Let  $f$  and  $g$  be integrable functions over  $E$ . Then:*

(a) *The function of  $cf$  ( $c$  is finite) is integrable over  $E$ , and*

$$\int_E cf = c \int_E f.$$

(b) *The function  $f+g$  is integrable over  $E$ , and*

$$\int_E (f+g) = \int_E f + \int_E g.$$

(c) *If  $f \leq g$  a.e., then  $\int_E f \leq \int_E g$ .*

(d) *If  $E_1$  and  $E_2$  are disjoint measurable sets in  $E$ , then*

$$\int_{E_1 \cup E_2} f = \int_{E_1} f + \int_{E_2} f.$$

*Proof.* (a) If  $c \geq 0$ , then

$$(cf)^+ = cf^+, (cf)^- = cf^-;$$

and if  $c < 0$ , then

$$(cf)^+ = (-c) \cdot f^-, (cf)^- = (-c)f^+.$$

The result now follows since the integrability of  $f$  implies that of  $f^+$  and  $f^-$ ; and conversely.

(b) We first prove that if  $f_1$  and  $f_2$  are nonnegative functions with  $f = f_1 - f_2$ , then

$$\int_E f = \int_E f_1 - \int_E f_2. \quad (3)$$

Since  $f = f^+ - f^-$ , it follows that

$$f^+ - f^- = f_1 - f_2$$

$$\Rightarrow f^+ + f_2 = f_1 + f^-.$$

This, by Theorem 5.2(b), gives

$$\int_E f^+ + \int_E f_2 = \int_E f_1 + \int_E f^-.$$

Hence, by Theorem 5.9, we have

$$\int_E f = \int_E f^+ - \int_E f^- = \int_E f_1 - \int_E f_2$$

This establishes (3) above.

Now, if  $f$  and  $g$  are integrable, then so are  $f^+ + g^+$  and  $f^- + g^-$ , and

$$f + g = (f^+ + g^+) - (f^- + g^-).$$

Therefore

$$\begin{aligned} \int_E (f + g) &= \int_E f^+ + \int_E g^+ - \int_E f^- - \int_E g^- \\ &= \int_E f + \int_E g. \end{aligned}$$

*Alternative Proof.* (b) Since  $f$  and  $g$  are integrable,  $|f|$  and  $|g|$  are integrable (cf. Theorem 6.2), and so  $|f| + |g|$  is integrable over  $E$  (cf. Theorem 5.2 (b) and Definition 5.8). Hence, from the fact that  $|f| + |g| \geq |f + g|$ , we conclude that  $f + g$  is integrable (cf. Theorems 5.9 and 6.2). Further, we have

$$\begin{aligned} \int_E (f + g) &= \int_E (f^+ - f^- + g^+ - g^-) \\ &= \int_E (f^+ + g^+) - \int_E (f^- + g^-) \\ &= \int_E f^+ + \int_E g^+ - \int_E f^- - \int_E g^- \\ &= \int_E f + \int_E g. \end{aligned}$$

(c) We note that  $f \leq g$  a.e. implies  $g - f \geq 0$  a.e. Therefore

$$\int_E (g - f) \geq 0.$$

Hence,

$$\begin{aligned}\int_E g &= \int_E (g-f) + \int_E f \\ &\geq \int_E f,\end{aligned}$$

by using part (b).

$$\begin{aligned}\text{(d)} \quad \int_{E_1 \cup E_2} f &= \int f \cdot \chi_{E_1 \cup E_2} \\ &= \int f \cdot \chi_{E_1} + \int f \cdot \chi_{E_2} \\ &= \int_{E_1} f + \int_{E_2} f.\end{aligned}$$

*Note.* Observe that  $f+g$  is not defined at points where  $f=\infty$  and  $g=-\infty$  or where  $f=-\infty$  and  $g=\infty$ . However, the set of such points must have measure zero, since  $f$  and  $g$  are integrable functions. As such the integrability and the value of  $f+g$  are independent of the choice of the values in these ambiguous cases.

**Problem 13.** If  $f$  is an integrable function such that  $f=0$  a.e., then show that  $\int f=0$ . [Hint: Apply Problem 4 to functions  $f^+$  and  $f^-$ .]

**Problem 14.** If  $f$  is an integrable function, then prove that

$$\left| \int f \right| \leq \int |f|.$$

When does equality occur?

*Solution.* Since  $|f| - f \geq 0$  and  $|f| + f \geq 0$ , we have

$$\int |f| \geq \int f \quad \text{and} \quad \int |f| \geq - \int f.$$

Thus,

$$\int |f| \geq \left| \int f \right|.$$

Further, if  $\int f \geq 0$ , then  $\int |f| = \int f$  giving  $\int (|f| - f) = 0$  so that, in view of Problem 4,  $|f| = f$  a.e.; if  $\int f \leq 0$ , then  $\int |f| = \int (-f)$  which, again in view of Problem 4, gives  $|f| = -f$  a.e. Hence equality occurs when either  $f \geq 0$  a.e. or  $f \leq 0$  a.e.

**Problem 15.** Show that if  $f$  and  $g$  are measurable functions such that  $|f| \leq |g|$  a.e. and  $g$  is integrable, then  $f$  is integrable.

*Solution.* Without any loss of generality, we may assume  $|f| \leq |g|$  as we can redefine  $f$ , if necessary, on a set of measure zero. This gives  $f^+ \leq |g|$  verifying  $\int f^+ \leq \int |g| < \infty$ . Hence  $f^+$  is integrable. Similarly,  $f^-$  is also integrable.

**Problem 16.** If  $f$  is an integrable function, then show that  $f$  is finite-valued a.e.

*Solution.* Let  $E$  be a measurable set of  $\mathbb{R}$ . The integrability of  $f$  on  $E$  implies integrability of  $|f|$  on  $E$ . Hence  $\int_E |f|$  is finite.

Let, if possible,  $|f| = \infty$  on a set  $A \subset E$  with  $m(A) > 0$ . Then

$$\int_E |f| > \int_A |f| > nm(A), \quad \forall n \in \mathbb{N}$$

since  $|f| > n$  on  $A$  for all  $n$ . However,  $n$  being arbitrary we have

$$\int_E |f| = \infty.$$

This contradicts the fact that  $\int_E |f|$  is finite. Thus  $m(A) = 0$ . Hence  $f$  is finite-valued a.e.

**Problem 17.** Let  $f$  be a measurable function and  $g$  an integrable function, and  $\alpha, \beta$  real numbers such that  $\alpha \leq f \leq \beta$  a.e. Then,  $\exists$  a real number  $\gamma$  with  $\alpha \leq \gamma \leq \beta$  such that

$$\int f |g| = \gamma \int |g|.$$

*Solution.* Since  $|fg| \leq |\beta| |g| \leq (|\alpha| + |\beta|) |g|$  a.e. and  $g$  is integrable, the function  $fg$  is integrable (cf. Problem 15). Also

$$\alpha |g| \leq f |g| \leq \beta |g| \text{ a.e.}$$

and so

$$\alpha \int |g| \leq \int f |g| \leq \beta \int |g|.$$

If  $\int |g| = 0$ , then  $g = 0$  a.e. and the result is trivial. If  $\int |g| \neq 0$ , by taking  $\gamma$  to be a real number given by

$$\gamma = \left( \int f |g| \right) / \left( \int |g| \right)$$

we get the result.

It is the restriction to nonnegative functions in Fatou's Lemma and Monotone Convergence Theorem that ensures the integrability of  $\lim f_n$ . If it is desired to relax the condition of nonnegativeness of the functions, some other conditions are to be imposed. One such concerns the notion of dominant function of the sequence.

**6.6 Theorem (Lebesgue Dominated Convergence Theorem).** *Let  $g$  be an integrable function on  $E$  and let  $\{f_n\}$  be a sequence of measurable functions such that  $|f_n| \leq g$  on  $E$  and  $\lim_{n \rightarrow \infty} f_n = f$  a.e. on  $E$ . Then*

$$\int_E f = \lim_{n \rightarrow \infty} \int_E f_n. \quad (4)$$

*Proof.* Clearly each  $f_n$  is integrable over  $E$  (cf. Theorem 6.4). Further, it follows from  $\lim_{n \rightarrow \infty} f_n = f$  a.e. on  $E$  and  $|f_n| \leq g$  a.e. on  $E$  that

$|f| \leq g$  a.e. on  $E$ . Hence  $f$  is integrable over  $E$ . Consider a sequence  $\{h_n\}$  of functions defined by  $h_n = f_n + g$ . Clearly,  $h_n$  is a nonnegative and integrable function for each  $n$ . Therefore, by Fatou's Lemma, we get

$$\begin{aligned} \int_E (f+g) &\leq \liminf_{n \rightarrow \infty} \int_E (f_n+g) \\ \Rightarrow \int_E f &\leq \liminf_{n \rightarrow \infty} \int_E f_n. \end{aligned}$$

Similarly, consider a sequence  $\{k_n\}$  defined by  $k_n = g - f_n$ , and observe that  $k_n$  is a nonnegative and integrable function for each  $n$ . So, again by Fatou's Lemma, we have

$$\begin{aligned} \int_E (g-f) &\leq \liminf_{n \rightarrow \infty} \int_E (g-f_n) \\ \Rightarrow \int_E f &\geq \overline{\lim}_{n \rightarrow \infty} \int_E f_n. \end{aligned}$$

Hence the theorem follows. ■

**6.7 Corollary.** *Let  $\{u_n\}$  be a sequence of integrable functions on  $E$  such that  $\sum_{n=1}^{\infty} u_n$  converges a.e. on  $E$ . Let  $g$  be a function which is*

integrable on  $E$  and satisfy  $|\sum_{i=1}^n u_i| \leq g$  a.e. on  $E$  for each  $n$ . Then  $\sum_{n=1}^{\infty} u_n$  is integrable on  $E$  and

$$\int_E \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \int_E u_n.$$

*Proof.* Let  $\sum_{i=1}^n u_i = f_n$ . Applying Theorem 6.6 for the sequence  $\{f_n\}$ , we get

$$\int_E \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \int_E u_n. \blacksquare$$

**6.8 Corollary.** If  $f$  is integrable over  $E$  and  $\{E_i\}$  is a sequence of disjoint measurable sets such that  $\bigcup_{i=1}^{\infty} E_i = E$ , then

$$\int_E f = \sum_{i=1}^{\infty} \int_E f_i.$$

*Proof.* Since  $\{E_i\}$  is a sequence of disjoint measurable sets, we may write

$$f = \sum_{i=1}^{\infty} f \cdot \chi_{E_i}.$$

The function  $f \cdot \chi_{E_i}$  is integrable over  $E$  since  $|f \chi_{E_i}| \leq |f|$  and  $|f|$  is integrable over  $E$ . Moreover,

$$\left| \sum_{i=1}^n f \cdot \chi_{E_i} \right| \leq |f|, \quad \forall n \in \mathbb{N}.$$

Thus the conditions of Corollary 6.7 are satisfied, and, hence,

$$\begin{aligned} \int_E f &= \int_E \sum_{i=1}^{\infty} f \cdot \chi_{E_i} \\ &= \sum_{i=1}^{\infty} \int_E f \cdot \chi_{E_i} = \sum_{i=1}^{\infty} \int_{E_i} f. \blacksquare \end{aligned}$$

*Note.* The result in Corollary 6.8 concerns a sequence of integrals, rather than a sequence of functions, based on a decomposition of the domain of an integrable function. This, in fact, extends the result in Theorem 6.5 (d) to any countable collection of disjoint measurable sets.

*Remark.* The condition of integrability of  $f$  on  $E$  in Corollary 6.8 cannot be relaxed.

**6.9 Example.** Let  $E_n = [n - 1/n, n]$  and  $E = \bigcup_{i=1}^{\infty} E_i$ . Define a function  $f: E \rightarrow \mathbf{R}$  given by

$$f(x) = (-1)^n n, \quad x \in E_n, \quad \forall n \in \mathbf{N}.$$

Clearly,

$$\begin{aligned} \int_{E_n} f &= \int f \cdot \chi_{E_n} = (-1)^n n m(E_n) \\ &= (-1)^n, \quad \forall n \in \mathbf{N} \end{aligned}$$

$$\Rightarrow \sum_{n=1}^{\infty} \int_{E_n} f = \sum_{n=1}^{\infty} (-1)^n.$$

On the other hand

$$\int_E |f| = \infty$$

$\Rightarrow f$  is not integrable on  $E$ .

If we replace the function  $g$  in Theorem 6.6 by certain appropriate  $g_n$ 's, we get the following generalization of the theorem.

**6.10 Theorem.** Let  $\{g_n\}$  be a sequence of integrable functions which converge a.e. to an integrable function  $g$ . Let  $\{f_n\}$  be a sequence of measurable functions such that  $|f_n| \leq g_n$  and  $\{f_n\}$  converge to  $f$  a.e. If

$$\int g = \lim_{n \rightarrow \infty} \int g_n,$$

then

$$\int f = \lim_{n \rightarrow \infty} \int f_n.$$

*Proof.* It follows on the lines of the proof of Theorem 6.6 by using Fatou's Lemma for the sequences  $\{g_n - f_n\}$  and  $\{g_n + f_n\}$ . ■

**6.11 Corollary.** Let  $\{f_n\}$  be a sequence of integrable functions such that  $f_n \rightarrow f$  a.e. with  $f$  integrable. Then  $\int_E |f - f_n| \rightarrow 0$  if and only if

$$\int_E |f_n| \rightarrow \int_E |f|.$$

*Proof.* Suppose first that

$$\int_E |f_n| \rightarrow \int_E |f|.$$

Then, since  $|f_n - f| \leq |f_n| + |f|$  and

$$\lim_{n \rightarrow \infty} \int (|f_n| + |f|) = 2 \int |f| < \infty,$$

we have

$$\int |f - f_n| \rightarrow 0,$$

in view of Theorem 6.10. The other part follows trivially. ■

*Remark.* Theorem 6.6 is false without the assumption that an integrable function  $g$  exists. First, a sequence of integrable functions, although convergent everywhere, can have a nonintegrable limit function and, secondly, even if the limit function is integrable, relation (4) may be false. In this direction, we now work out some examples.

### 6.12 Examples

1. Let  $f_n(x) = nx^n$ ,  $0 \leq x \leq 1$ . Then  $\lim_{n \rightarrow \infty} f_n = 0$  a.e. and

$$\lim_{n \rightarrow \infty} \int_0^1 f_n(x) dx = \lim_{n \rightarrow \infty} \int_0^1 nx^n dx = \lim_{n \rightarrow \infty} \frac{n}{n+1} = 1$$

$$\Rightarrow \lim_{n \rightarrow \infty} \int_0^1 f_n \neq \int_0^1 \lim_{n \rightarrow \infty} f_n.$$

2. Let  $f_n(x) = n^2x^n$ ,  $0 \leq x \leq 1$ . Then  $\lim_{n \rightarrow \infty} f_n = 0$  a.e. and

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_0^1 f_n(x) dx &= \lim_{n \rightarrow \infty} \int_0^1 n^2x^n dx \\ &= \lim_{n \rightarrow \infty} \frac{n^2}{n+1} = \infty \end{aligned}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \int_0^1 f_n = \infty \neq 0 = \int_0^1 \lim_{n \rightarrow \infty} f_n.$$

3. Let

$$f_n(x) = \begin{cases} n \sin nx & \text{if } 0 \leq x \leq \frac{\pi}{n} \\ 0 & \text{if } \frac{\pi}{n} \leq x \leq \pi. \end{cases}$$

Then  $\lim_{n \rightarrow \infty} f_n(x) = 0$ ,  $\forall x \in [0, \pi]$  and

$$\lim_{n \rightarrow \infty} \int_0^\pi f_n(x) dx = \lim_{n \rightarrow \infty} n \int_0^\pi \sin nx dx = 2$$

$$\Rightarrow \lim_{n \rightarrow \infty} \int_0^\pi f_n(x) dx \neq \int_0^\pi \lim_{n \rightarrow \infty} f_n(x) dx.$$

*Note.* In Examples 1 and 3 in 6.12, none of the sequences is monotone or bounded almost everywhere.

*Remark.* Theorem 6.6 shows that the interchange of limit and integral operations in the case of Lebesgue integrals is possible under much less restrictive conditions than in that of Riemann integrals. The functions in the sequence need not be bounded. Moreover, uniform convergence is not required. Consider, for instance, the example which follows. This feature, among others, tend to make the Lebesgue integral a more useful concept than Riemann integral.

**6.13 Example.** Consider a sequence  $\{f_n\}$  of functions, each defined on  $[0, 2]$ , where

$$f_n(x) = \begin{cases} \sqrt{\frac{x}{n}} & \text{if } \frac{1}{n} \leq x \leq \frac{2}{n} \\ 0 & \text{if otherwise.} \end{cases}$$

Then

$$f(x) = \lim_{n \rightarrow \infty} f_n(x) = 0, \quad \forall x \in [0, 2].$$

Also, we note that

$$|f_n(x)| \leq g(x), \quad \forall x \in [0, 2]$$

where

$$g(x) = \begin{cases} \sqrt{\frac{2}{x}} & \text{if } 0 < x < 2 \\ 0 & \text{if } x = 0. \end{cases}$$

Since  $g$  is integrable over  $[0, 2]$ , the hypothesis of Theorem 6.6 is satisfied. However, one may observe that the sequence  $\{f_n\}$  does not converge uniformly on  $[0, 2]$ .

*Remark.* The existence of an integrable 'dominant function'  $g$  in Theorem 6.6 is sufficient but not necessary for the interchange of the limit and integral operations.

**6.14 Example.** Let

$$f_n(x) = \begin{cases} \frac{1}{x} & \text{if } n - \frac{1}{2} < x < n + \frac{1}{2} \\ 0 & \text{if otherwise.} \end{cases}$$

Then  $f_n(x) \rightarrow 0$  as  $n \rightarrow \infty$  for all  $x$ , and

$$\int_0^{\infty} f_n(x) dx = \log \left( \frac{2n+1}{2n-1} \right) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Thus the result of Theorem 6.6 is true. But, if  $|f_n(x)| \leq g(x), \forall x$ , then  $g(x) \geq 1/x$ , and hence no integrable dominant function exists.

### 7 IMPROPER INTEGRALS

We have seen in § 3 that if  $f$  is a Riemann integrable function over a finite interval, then it is so in the Lebesgue sense too, and the two integrals coincide. However, this is not the case with improper Riemann integrals\* and the possible corresponding results for “improper” Lebesgue integrals. As an illustration, we give below a few examples.

#### 7.1 Examples

1. Let  $E = [0, \infty[$ . Define  $f: E \rightarrow \mathbb{R}$  by

$$f(x) = \frac{(-1)^{k+1}}{k}, \quad x \in [k-1, k],$$

for  $k = 1, 2, \dots$ . One may easily verify, by using Leibnitz’s test for alternating series, that

$$\lim_{k \rightarrow \infty} \mathcal{R} \int_0^k f$$

is finite. But, on the other hand,  $\int_E f$  does not exist, since  $\int_E |f|$  fails to exist on account of

$$\int_E |f| = \sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

2. Let  $f: [0, \infty[ \rightarrow \mathbb{R}$  be given by

$$f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0. \\ 0 & \text{if } x = 0. \end{cases}$$

Then

$$\mathcal{R} \int_0^{\infty} f = \frac{\pi}{2}.$$

\*A Riemann integral may be improper because either the function to be integrated is unbounded on a point (or points) in the interval of integration or the interval of integration itself is unbounded.

On the other hand,  $f$  is not Lebesgue integrable on  $[0, \infty[$  since

$$\int_0^{\infty} |f| = \infty.$$

3. Suppose that  $-1 < a < 0$  and let  $f: [0, 1] \rightarrow \mathbf{R}$  be defined by

$$f(x) = \begin{cases} x^a & \text{if } 0 < x \leq 1 \\ 0 & \text{if } x = 0. \end{cases}$$

For each  $n$ , define  $f_n: [0, 1] \rightarrow \mathbf{R}$  by

$$f_n(x) = \begin{cases} x^a & \text{if } \frac{1}{n} \leq x \leq 1 \\ 0 & \text{if } 0 \leq x < \frac{1}{n}. \end{cases}$$

Observe that  $\{f_n\}$  is an increasing sequence of functions such that  $f_n \rightarrow f$ , and

$$\int_0^1 f_n = \frac{1}{a+1} \left(1 - \frac{1}{n^{a+1}}\right), \quad \forall n \in \mathbf{N}$$

$$\Rightarrow \int_0^1 f_n \leq \frac{1}{a+1}, \quad \forall n \in \mathbf{N}.$$

Therefore, it follows, by the Monotone Convergence Theorem, that

$\int_0^1 f$  exists and

$$\int_0^1 f = \lim_{n \rightarrow \infty} \int_0^1 f_n = \frac{1}{a+1}.$$

On the other hand, since  $f$  is unbounded, it is not Riemann integrable over  $[0, 1]$  although the improper integral  $\mathcal{R} \int_0^1 f$  does exist and can easily be seen to be equal to  $\frac{1}{a+1}$ .

4. Let  $E = [0, 1]$ . Define  $f: E \rightarrow \mathbf{R}^*$  by

$$f(x) = \begin{cases} \infty & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational.} \end{cases}$$

One can easily observe that

$$\int_0^1 f = 1,$$

while the improper Riemann integral  $\mathcal{R} \int_0^1 f$  does not exist.

5. Let  $f: [0, \infty[ \rightarrow \mathbb{R}$  be defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is irrational} \\ 0 & \text{if } x \text{ is rational.} \end{cases}$$

Then  $f$  is neither Lebesgue integrable nor does its improper Riemann integral exist.

However, using the Monotone Convergence Theorem, we can extend Theorem 3.1 to the Riemann improper integrals of nonnegative functions.

**Problem 18.** What happens in Example 3 in 7.1 when  $a < -1$ ?

**7.2 Theorem.** *Let  $f$  be a nonnegative function defined on  $[a, b]$  such that  $f$  is Riemann integrable over every subinterval  $[a + \epsilon, b]$ ,  $\epsilon > 0$ . If the improper Riemann integral*

$$\lim_{\epsilon \rightarrow 0} \mathcal{R} \int_{a+\epsilon}^b f$$

*exists and is finite, then  $f$  is Lebesgue integrable over  $[a, b]$ , and*

$$\int_a^b f = \lim_{\epsilon \rightarrow 0} \mathcal{R} \int_{a+\epsilon}^b f.$$

*Proof.* In view of Theorem 3.1, we note that

$$\int_{a+\epsilon}^b f = \mathcal{R} \int_{a+\epsilon}^b f, \quad \epsilon > 0.$$

The result then follows by using the Monotone Convergence Theorem on letting  $\epsilon \rightarrow 0$ . ■

*Remark.* The result in Theorem 7.2 can be extended to an unbounded interval (see Problem 24).

**7.3 Corollary.** *If  $f$  is a function defined on an interval (bounded or unbounded) such that the improper Riemann integral of  $|f|$  exists, the improper Riemann integral and the Lebesgue integral of  $f$  both exist and the two integrals are equal.*

## Problems

19. Prove the Monotone Convergence Theorem and Lebesgue Dominated Convergence Theorem replacing 'almost everywhere convergence' by 'convergence in measure'.

20. Considering the sequence
- $\{f_n\}$
- defined by

$$f_n(x) = \begin{cases} n^2x & \text{if } 0 < x \leq \frac{1}{n} \\ 0 & \text{if } x > \frac{1}{n}, \end{cases}$$

show that strict inequality holds in Theorem 5.3.

21. Let
- $\{f_n\}$
- be a sequence of measurable functions such that
- $|f_n| \leq g$
- a.e. on
- $E$
- , where
- $g$
- is an integrable function on
- $E$
- . Prove that

$$\int_E \lim_{n \rightarrow \infty} f_n \leq \lim_{n \rightarrow \infty} \int_E f_n \leq \overline{\lim}_{n \rightarrow \infty} \int_E f_n \leq \int_E \overline{\lim}_{n \rightarrow \infty} f_n.$$

22. Let
- $\{f_n\}$
- be a sequence of nonnegative functions on
- $\mathbb{R}$
- such that
- $f_n \rightarrow f$
- a.e. and suppose
- $\int f_n \rightarrow \int f < \infty$
- . Then, for each measurable set
- $E$
- , prove that

$$\int_E f_n \rightarrow \int_E f.$$

23. Prove Theorem 6.10.

24. If
- $f$
- is nonnegative and improper Riemann integrable on
- $[0, \infty[$
- , prove that
- $f$
- is Lebesgue integrable.

25. Let
- $f$
- be a nonnegative measurable function. Show that

$$\int f = \sup \int \varphi, \text{ where supremum is taken over all simple functions } \varphi, 0 \leq \varphi \leq f. \text{ [Hint: Use Theorem IV-11.1.]}$$

26. Under the hypothesis of Theorem 6.6, show that

$$\lim_{n \rightarrow \infty} \int |f_n| = 0.$$

[Hint:  $|f_n - f| \leq 2g$ , for each  $n$  and Theorem 6.6 applied to  $(f_n - f)$  gives the result.]

27. Prove for each of the functions
- $f: [0, 1] \rightarrow \mathbb{R}$
- defined as below, that the improper Riemann integral exists while the Lebesgue integral does not:

$$(a) f(x) = \begin{cases} \frac{1}{x} \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

$$(b) f(x) = \begin{cases} (-1)^{[nx]} & \text{if } \frac{1}{n+1} < x \leq \frac{1}{n} \\ 0 & \text{if } x = 0. \end{cases}$$

## VI

# Differentiation and Integration

In Riemann theory of integration it is known that *differentiation and integration are inverse operations of each other* in the following sense:

1. If  $f$  is a Riemann integrable function over  $[a, b]$ , then its indefinite integral

$$F(x) = \mathcal{R} \int_a^x f(t) dt$$

defines a continuous function on  $[a, b]$ . Furthermore, if  $f$  is continuous at a point  $x_0 \in [a, b]$ , then  $F$  is differentiable there at, and  $F'(x_0) = f(x_0)$ .

2. If  $f$  is Riemann integrable over  $[a, b]$  and if there is a differentiable function  $F$  on  $[a, b]$  such that  $F'(x) = f(x)$  for  $x \in [a, b]$ , then

$$\mathcal{R} \int_a^x f(t) dt = F(x) - F(a), \quad \forall x \in [a, b].$$

(This result is usually referred to as the *Fundamental Theorem of Calculus*.)

The present chapter deals with similar types of interrelation between differentiation and Lebesgue integration.

### 1 DINI DERIVATIVES

In discussing the differentiation of an indefinite Lebesgue integral we shall have to take explicit care of the fact that a continuous function does not necessarily possess a unique derivative everywhere. By various ingenious methods, it is possible to construct nowhere differentiable continuous functions (cf. Appendix II).

For a function  $f$  which fails to possess a derivative in the ordinary sense, i.e., when the increment ratio

$$\frac{f(x+h)-f(x)}{h}$$

does not tend to a unique limit as  $|h| \rightarrow 0$ , we make use of the following four quantities known as **Dini derivatives**:

$$D^+f(x) = \overline{\lim}_{h \rightarrow 0^+} \frac{f(x+h)-f(x)}{h} \quad (\text{upper right derivative})$$

$$D_+f(x) = \lim_{h \rightarrow 0^+} \frac{f(x+h)-f(x)}{h} \quad (\text{lower right derivative})$$

$$D^-f(x) = \overline{\lim}_{h \rightarrow 0^-} \frac{f(x+h)-f(x)}{h} \quad (\text{upper left derivative})$$

$$D_-f(x) = \lim_{h \rightarrow 0^-} \frac{f(x+h)-f(x)}{h} \quad (\text{lower left derivative}).$$

We may easily prove, by using the definition, that:

1. The Dini derivatives always exist (finite or infinite) for any function  $f$ , and satisfy

$$D^+f(x) \geq D_+f(x), \quad D^-f(x) \geq D_-f(x)$$

2. If  $D^-f(x) = D_-f(x)$ , then the common value, written  $\underline{D}f(x)$ , is just the left-hand derivative of  $f$  at  $x$ ; and similarly, if  $D_+f(x) = D^+f(x)$ , then the common value, written  $\overline{D}f(x)$ , is just the right-hand derivative of  $f$  at  $x$ . In general,  $\underline{D}f(x) \leq \overline{D}f(x)$  and if  $f$  has the derivative at  $x$ , then  $\underline{D}f(x) = \overline{D}f(x)$ ; and conversely. Thus, the function  $f$  is differentiable at a point if and only if all four Dini derivatives thereat are identical and different from  $\pm \infty$ .

$$3. \quad \underline{D}f(x) = -\overline{D}(-f(x)).$$

$$4. \quad \begin{cases} \overline{D}(f+g)(x) \leq \overline{D}f(x) + \overline{D}g(x) \\ \underline{D}(f+g)(x) \geq \underline{D}f(x) + \underline{D}g(x). \end{cases}$$

Now, we give examples of functions along with their Dini derivatives.

### 1.1 Examples

1. Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $f(x) = |x|$ . Then  $D^+f(0) = D_+f(0) = 1$  and  $D^-f(0) = D_-f(0) = -1$ . As such  $\underline{D}f(0) = -1$  and  $\overline{D}f(0) = 1$ . Consequently,  $f$  is not differentiable at  $x = 0$ .

2. Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be defined by

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ x \sin\left(\frac{1}{x}\right) + 4x & \text{if } x < 0. \end{cases}$$

Then  $D^+f(0) = 1$ ,  $D_+f(0) = -1$ ,  $D^-f(0) = 5$  and  $D_-f(0) = 3$ .

3. Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be defined by

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Then  $D^+f(0) = D^-f(0) = 1$  and  $D_+f(0) = D_-f(0) = -1$ .

4. Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be defined by

$$f(x) = \begin{cases} x \{1 + \sin(\log x)\} & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ x + \sqrt{-x \sin^2(\log x)} & \text{if } x < 0. \end{cases}$$

Then  $D^+f(0) = 2$ ,  $D_+f(0) = 0$ ,  $D^-f(0) = 1$  and  $D_-f(0) = -\infty$ .

5. Let  $f: [0, 1] \rightarrow \mathbb{R}$  be defined by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational.} \end{cases}$$

Then  $D^+f(x) = \infty$ ,  $D_-f(x) = -\infty$  and  $D_+f(x) = 0 = D^-f(x)$ , if  $x$  is rational;  $D^+f(x) = 0 = D_-f(x)$ ,  $D^-f(x) = \infty$  and  $D_+f(x) = -\infty$ , if  $x$  is irrational.

Next, we establish the measurability of lower and upper derivatives of a function.

**1.2 Theorem.** *If  $f$  is any function on an interval  $I$ , then  $\overline{D}f$  and  $\underline{D}f$  are measurable functions.*

*Proof.* It is sufficient to prove that the set  $I(\overline{D}f > \alpha)$  is measurable for any real number  $\alpha$ . Consider, for a real number  $\alpha$ , the set

$$A = \{x \in I, \overline{D}f(x) > \alpha\}.$$

Corresponding to each pair of positive integers  $m$  and  $n$ , we denote by  $A_{mn}$  the union of all intervals  $[x_1, x_2]$  which are subsets of  $I$  and have the properties

$$|x_2 - x_1| < \frac{1}{m} \quad \text{and} \quad \frac{f(x_2) - f(x_1)}{x_2 - x_1} > \alpha + \frac{1}{n}.$$

Clearly, the set  $A_{mn}$  is measurable and so is the set  $\bigcup_{n=1}^{\infty} \bigcap_{m=1}^{\infty} A_{mn}$ .

Let  $c \in A$ . Then  $\exists$  an integer  $n_0$  such that  $\bar{D}f(c) > \alpha + (1/n_0)$ . Therefore, corresponding to each value of  $m$ , there exists a point  $x^*$  of  $I$  satisfying

$$0 < |x^* - c| < \frac{1}{m} \quad \text{and} \quad \frac{f(x^*) - f(c)}{x^* - c} > \alpha + \frac{1}{n_0}.$$

Then  $c \in \bigcup_{n=1}^{\infty} \bigcap_{m=1}^{\infty} A_{mn}$ , as the closed interval  $[c, x^*]$  is, contained in  $A_{mn_0}$ , whence  $c \in A_{mn_0}$  for each  $m$ . This further concludes that

$$A \subset \bigcup_{n=1}^{\infty} \bigcap_{m=1}^{\infty} A_{mn}.$$

To get the reverse inequality, let  $c \in \bigcup_{n=1}^{\infty} \bigcap_{m=1}^{\infty} A_{mn}$ . Then there is an  $n_0$  such that  $c \in \bigcap_{m=1}^{\infty} A_{mn_0}$ , whence  $c \in A_{mn_0}$  for all  $m$ . This gives, corresponding to each  $m$ , an interval  $[x_1, x_2]$  such that  $x_1 \leq c \leq x_2$ ,

$$|x_2 - x_1| < \frac{1}{m} \quad \text{and} \quad \frac{f(x_2) - f(x_1)}{x_2 - x_1} > \alpha + \frac{1}{n_0}. \quad (1)$$

Suppose  $x_1 < c < x_2$ . Then, in view of (1), at least one of the following two must be true:

$$(a) \quad \frac{f(x_1) - f(c)}{x_1 - c} > \alpha + \frac{1}{n_0}.$$

$$(b) \quad \frac{f(x_2) - f(c)}{x_2 - c} > \alpha + \frac{1}{n_0}.$$

Further, by (1) we see that if  $x_2 = c$ , then (a) is true; and if  $x_1 = c$ , then (b) is true. In either case, there exists a number  $x$  in  $I$  for which

$$0 < |x - c| < \frac{1}{m} \quad \text{and} \quad \frac{f(x) - f(c)}{x - c} > \alpha + \frac{1}{n_0},$$

so that  $\bar{D}f(c) \geq \alpha + \frac{1}{n_0} > \alpha$ . Thus  $c \in A$ . Hence

$$\bigcup_{n=1}^{\infty} \bigcap_{m=1}^{\infty} A_{mn} \subset A,$$

and so  $A$  is a measurable set. This proves that  $\bar{D}f$  is a measurable function. Similarly, it can be proved that  $\underline{D}f$  is also a measurable function. ■

**1.3 Corollary.** *If  $f$  is a differentiable function on an interval  $I$ , then  $f'$  is measurable on  $I$ .*

**Problems**

1. Let  $f : ]a, b[ \rightarrow \mathbb{R}$  be a bounded nondecreasing function. Then show that the set  $\{x \in ]a, b[ : D^+f(x) = \infty\}$  is of measure zero.
2. Let  $f : ]a, b[ \rightarrow \mathbb{R}$  be a bounded nondecreasing function and let  $0 < \alpha < \beta$ . If  $A_\alpha = \{x : D_-f(x) < \alpha\}$  and  $B_\beta = \{x : D^+f(x) > \beta\}$ , then show that  $m(A_\alpha \cap B_\beta) = 0$ .
3. Evaluate the Dini derivatives at  $x = 0$  of the function given by

$$f(x) = \begin{cases} ax \sin^2\left(\frac{1}{x}\right) + bx \cos^2\left(\frac{1}{x}\right) & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ a'x \sin^2\left(\frac{1}{x}\right) + b'x \cos^2\left(\frac{1}{x}\right) & \text{if } x < 0, \end{cases}$$

where  $a < b$  and  $a' < b'$ .

4. Give an example of the functions  $f$  and  $g$  for which

$$D^+(f+g) \neq D^+f + D^+g.$$

[Hint: Consider  $f(0) = 0$ ,  $f(x) = x$  for  $x \in \mathbb{Q}$ ,  $f(x) = -x$  for  $x \notin \mathbb{Q}$ ; and  $g = -f$ .]

**2 DIFFERENTIATION OF MONOTONE FUNCTIONS**

**2.1 Definition.** Let  $E \subset \mathbb{R}$ . A collection  $\mathcal{J}$  of intervals is said to be a **Vitali cover** of the set  $E$  if for each  $x \in E$  and each  $\epsilon > 0$ , there exists an interval  $I \in \mathcal{J}$  with  $x \in I$  and  $l(I) < \epsilon$ .

**2.2 Example.** Let  $\{r_n\}$  be the enumeration of the rationals in  $[a, b]$ . Then the collection  $\{I_{n,i}\}$  where  $I_{n,i} = \left[r_n - \frac{1}{i}, r_n + \frac{1}{i}\right]$ ,  $n, i \in \mathbb{N}$  forms a Vitali cover of  $[a, b]$ .

**2.3 Lemma (Vitali's Covering Theorem).** *Let  $E$  be a set of finite outer measure and  $\mathcal{J}$  a collection of intervals which cover  $E$  in the sense of Vitali. Then given  $\epsilon > 0$ , there is a finite disjoint collection  $\{I_1, I_2, \dots, I_N\}$  of intervals in  $\mathcal{J}$  such that*

$$m^*\left(E - \bigcup_{i=1}^N I_i\right) < \epsilon.$$

*Proof.* We may assume, without any loss of generality, that each interval in  $\mathcal{J}$  is closed; for otherwise we replace each interval by its closure and observe that the set of the endpoints of  $I_1, I_2, \dots, I_N$  has measure zero.

Let  $O$  be an open set such that  $O \supset E$  and  $m^*(O) < m^*(E) + 1 < \infty$  (cf. III-2.14). Since  $\mathcal{J}$  is a Vitali cover of  $E$ , we may assume that each interval in  $\mathcal{J}$  is contained in  $O$ ; otherwise we could discard from  $\mathcal{J}$  all those intervals which extend beyond  $O$  and the remaining intervals in  $\mathcal{J}$  still form a Vitali cover of  $E$ ; for if  $x \in E$ , then for some  $\epsilon > 0$  there is a neighbourhood  $N(x, 2\epsilon) \subset O$  and there is an interval  $I \in \mathcal{J}$  such that  $x \in I$ ,  $l(I) < \epsilon$ , and hence  $I \subset O$ .

We now choose, by the method of induction, a sequence  $\{I_n\}$  of disjoint intervals from  $\mathcal{J}$  as follows. Let  $I_1$  be an arbitrary interval of the family  $\mathcal{J}$  and let  $\eta_1$  be the least upper bound of the lengths of the intervals in  $\mathcal{J}$  which do not have any point in common with  $I_1$ . Note that  $\eta_1 < \infty$  since  $\eta_1 \leq m(O) < \infty$ . We then choose  $I_2$ , an interval from  $\mathcal{J}$  which is disjoint from  $I_1$ , such that  $l(I_2) > \frac{1}{2}\eta_1$ . Let  $\eta_2$  be the least upper bound of the lengths of the intervals in  $\mathcal{J}$  which do not have any point in common with either  $I_1$  or  $I_2$ , and note that  $\eta_2 < \infty$ . Choose  $I_3$ , an interval from  $\mathcal{J}$  which is disjoint from  $I_1 \cup I_2$ , such that  $l(I_3) > \eta_2/2$ . In general, having already chosen  $n$  disjoint intervals  $I_1, I_2, \dots, I_n$ , we denote by  $\eta_n (< \infty)$  the least upper bound of the lengths of the intervals in  $\mathcal{J}$  which do not have any point in common with  $\bigcup_{i=1}^n I_i$ , and choose  $I_{n+1}$  to be an interval from  $\mathcal{J}$  such that it is disjoint from the preceding intervals and  $l(I_{n+1}) > \eta_n/2$ . If for some  $n$ , the set  $\bigcup_{i=1}^n I_i$  contains almost every point of the set  $E$ , the lemma is proved; otherwise we get an infinite sequence  $\{I_n\}$  of intervals from  $\mathcal{J}$  such that

$$\begin{cases} I_i \cap I_j = \phi, & i \neq j \\ l(I_{n+1}) > \frac{\eta_n}{2}, & \eta_n < \infty, n = 1, 2, 3, \dots \end{cases}$$

Observe that  $\{\eta_n\}$  is a monotonic decreasing sequence of nonnegative real numbers.

Since  $\bigcup_{n=1}^{\infty} I_n \subset O$ , we have

$$\sum_{n=1}^{\infty} l(I_n) \leq m(O) < \infty.$$

Thus, for a given  $\epsilon > 0$ , there is an integer  $N$  such that

$$\sum_{n=N+1}^{\infty} l(I_n) < \frac{\epsilon}{5}.$$

Write

$$J = E - \bigcup_{n=1}^N I_n.$$

The lemma will be proved if we establish that  $m^*(J) < \epsilon$ . Let  $x \in J$ .

Then  $x \notin \bigcup_{n=1}^N I_n$ . Since  $\bigcup_{n=1}^N I_n$  is a closed set not containing  $x$ , we can find an interval  $I$  in  $\mathcal{G}$  such that  $x \in I$  and  $l(I)$  so small that  $I$  does not meet any of the intervals  $I_1, I_2, \dots, I_N$ , i.e.

$$I \cap I_i = \phi, \quad i = 1, 2, \dots, N.$$

Thus, we may have

$$l(I) \leq \eta_N < 2l(I_{N+1}).$$

If the interval  $I$  does not meet with  $I_{N+1}$  also, we should have

$$l(I) \leq \eta_{N+1}.$$

If  $I$  did not have any point in common with any of the intervals in the sequence  $\{I_n\}$ , then we should have

$$l(I) \leq \eta_n, \quad \forall n$$

which is impossible since

$$\eta_n < 2l(I_{n+1}) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

As such,  $I$  must meet at least one of the intervals in the sequence  $\{I_n\}$ . Let  $n_0$  be the smallest integer such that  $I$  meets  $I_{n_0}$ . Clearly,  $n_0 > N$  and  $l(I) \leq \eta_{n_0-1} < 2l(I_{n_0})$ . Since  $x \in I$  and  $I$  has a point in common with  $I_{n_0}$ , it follows that the distance of  $x$  from the midpoint of  $I_{n_0}$  is at most  $l(I) + \frac{1}{2}l(I_{n_0}) < \frac{5}{2}l(I_{n_0})$ . Thus, if  $J_{n_0}$  is an interval concentric with  $I_{n_0}$  such that  $l(J_{n_0}) = 5l(I_{n_0})$ , we find that  $x \in J_{n_0}$ . In other words, for every  $x \in J$ ,  $\exists$  an  $n \geq N+1$  such that  $x \in J_n$  and  $l(J_n) = 5l(I_n)$ , and as such

$$\begin{aligned} J &\subset \bigcup_{n=N+1}^{\infty} J_n \\ \Rightarrow m^*(J) &\leq \sum_{n=N+1}^{\infty} l(J_n) \\ &= 5 \sum_{n=N+1}^{\infty} l(I_n) < \epsilon. \end{aligned}$$

This completes the proof of the lemma. ■

As an application of Theorem 2.3, we have the following significant result which generalizes the result: 'The union of a countable collection of intervals is a measurable set'.

**2.4 Corollary.** *The union of any collection of intervals is a measurable set.*

*Proof.* Let  $\{I_\lambda : \lambda \in \Lambda\}$  be a collection of intervals and let

$$E = \bigcup_{\lambda \in \Lambda} I_\lambda.$$

Denote by  $\mathcal{J}$  the collection of all closed intervals  $J$  such that  $J \subset I_\lambda$  for at least one  $\lambda \in \Lambda$ . Then  $\mathcal{J}$  is a Vitali cover of  $E$ , and it follows that for each  $\epsilon > 0$ , there exists a finite collection  $\{J_1, J_2, \dots, J_n\}$  of disjoint intervals in  $\mathcal{J}$  such that

$$\begin{aligned} m^* \left( E - \bigcup_{i=1}^n J_i \right) &< \epsilon \\ \Rightarrow m^* \left( E - \bigcup_{i=1}^{\infty} J_i \right) &= 0. \end{aligned}$$

But  $J_i \subset E$ , for all  $i$ . Therefore

$$E = \left[ \bigcup_{i=1}^{\infty} J_i \right] \cup \left[ E - \bigcup_{i=1}^{\infty} J_i \right].$$

Hence the measurability of  $E$  follows from that of  $\bigcup_{i=1}^{\infty} J_i$  and

$$E - \bigcup_{i=1}^{\infty} J_i. \blacksquare$$

**2.5 Theorem (Lebesgue's Theorem).** *Let  $f$  be an increasing real-valued function defined on  $[a, b]$ . Then  $f$  is differentiable a.e. and the derivative  $f'$  is measurable. Furthermore,*

$$\int_a^b f'(x) dx \leq f(b) - f(a). \quad (2)$$

*Proof.* Let us first prove that the points of the open interval  $]a, b[$  at which not all the four Dini derivatives are equal form a set of measure zero. Since

$$D_+ f(x) \leq D^+ f(x) \quad \text{and} \quad D_- f(x) \leq D^- f(x),$$

it is sufficient to show that each of the following four sets of points of  $]a, b[$  is of measure zero:

$$A = \{x \in ]a, b[, D_- f(x) < D^+ f(x)\}$$

$$B = \{x \in ]a, b[, D_+ f(x) < D^- f(x)\}$$

$$C = \{x \in ]a, b[, D_-f(x) < D^-f(x)\}$$

$$D = \{x \in ]a, b[, D_+f(x) < D^+f(x)\}.$$

We establish the result only for the set  $A$ , and the result in respect of the sets  $B$ ,  $C$  and  $D$  can be proved similarly.

Define, for given rationals,  $u$  and  $v$  with  $u < v$ , the set

$$A_{u,v} = \{x \in ]a, b[: D_-f(x) < u < v < D^+f(x)\}.$$

Clearly,  $A$  is the union of a countable family  $\{A_{u,v}\}$  of sets, where  $u$  and  $v$  vary over all rationals. Thus, in order to show  $m^*(A) = 0$ , it is sufficient to establish  $m^*(A_{u,v}) = 0$ .

Let  $m^*(A_{u,v}) = \alpha$ . Then given an  $\epsilon > 0$ ,  $\exists$  an open set  $O \supset A_{u,v}$  such that

$$m(O) < \alpha + \epsilon.$$

Let  $x$  be any arbitrary point of  $A_{u,v}$ , since  $D_-f(x) < u$ , there is an arbitrary small interval (intervals)  $[x-h, x] \subset O$  such that

$$f(x) - f(x-h) < uh.$$

This way we obtain a Vitali cover  $\mathcal{G}$  of  $A_{u,v}$ . By Lemma 2.3, there is a finite subcollection  $\{I_1, I_2, \dots, I_N\}$  of disjoint intervals in  $\mathcal{G}$  such that

$$m^*\left(A_{u,v} - \bigcup_{i=1}^N I_i\right) < \epsilon.$$

Write  $I_i = [x_i - h_i, x_i]$ ,  $i = 1, 2, \dots, N$ . Then summing over these  $N$  intervals, we have

$$\begin{aligned} \sum_{i=1}^N [f(x_i) - f(x_i - h_i)] &< u \sum_{i=1}^N h_i \\ &< m(O) \\ &< u(\alpha + \epsilon). \end{aligned}$$

Let  $y$  be an arbitrary point of the set

$$E = A_{u,v} \cap \left[ \bigcup_{i=1}^N \text{Int}(I_i) \right].$$

Since  $D^+f(y) > v$ , there is a small interval, and hence intervals of the form  $[y, y+k] \subset I_i$ , for some  $i = 1, 2, \dots, N$  such that

$$f(y+k) - f(y) > vk.$$

This way we obtain a Vitali cover  $\mathcal{G}^*$  of  $E$ . Again, by Lemma 2.3, there is a finite subcollection  $\{J_1, J_2, \dots, J_M\}$  of disjoint intervals in  $\mathcal{G}^*$  such that

$$m^*\left(E - \bigcup_{j=1}^M J_j\right) < \epsilon.$$

Since one can easily verify that  $m^*(E) > \alpha - \epsilon$ , it follows that

$$m^* \left( E \cap \left[ \bigcup_{j=1}^M J_j \right] \right) > \alpha - 2\epsilon.$$

If we write  $J_j = [y_j, y_j + k_j]$ ,  $j = 1, 2, \dots, M$ , then summing over these  $M$  intervals, we obtain

$$\begin{aligned} \sum_{j=1}^M [f(y_j + k_j) - f(y_j)] &> v \sum_{j=1}^M k_j \\ &> v(\alpha - 2\epsilon). \end{aligned}$$

But each  $J_j$  is contained in some  $I_i$ . Thus, for a fixed  $i$ , if we sum over those  $j$  for which  $J_j \subset I_i$ , we get

$$\sum [f(y_j + k_j) - f(y_j)] \leq f(x_i) - f(x_i - h_i),$$

since  $f$  is an increasing function. Therefore, summing the right-hand side of the above over  $i = 1, 2, \dots, N$ , we obtain

$$\sum_{i=1}^N [f(x_i) - f(x_i - h_i)] \geq \sum_{j=1}^M [f(y_j + k_j) - f(y_j)]$$

$$\Rightarrow u(\alpha + \epsilon) > v(\alpha - 2\epsilon).$$

Since this is true for each  $\epsilon > 0$ , we must have  $u\alpha \geq v\alpha$ . But  $u < v$ . This implies that  $\alpha = 0$ , and so  $m^*(A) = 0$ .

We have thus established that

$$g(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

is defined a.e. in  $[a, b]$ , and that  $f$  is differentiable wherever  $g$  is finite. Write

$$g_n(x) = \frac{f\left(x + \frac{1}{n}\right) - f(x)}{\frac{1}{n}},$$

and set  $f(x) = f(b)$  for  $x \geq b$ . Then we note that  $\{g_n\}$  is a sequence of nonnegative functions, since  $f$  is increasing, such that  $g_n \rightarrow g$  a.e. Moreover,  $g$  is measurable. Hence, by Fatou's Lemma, we have

$$\begin{aligned} \int_a^b g &\leq \liminf_{n \rightarrow \infty} \int_a^b g_n \\ &= \lim_{n \rightarrow \infty} n \int_a^b \left[ f\left(x + \frac{1}{n}\right) - f(x) \right] dx \\ &= \lim_{n \rightarrow \infty} \left[ n \int_b^{b+1/n} f - n \int_a^{a+1/n} f \right] \end{aligned}$$

$$\begin{aligned}
 &= \lim_{n \rightarrow \infty} \left[ f(b) - n \int_a^{a+1/n} f \right] \\
 &\leq f(b) - f(a) \\
 &< \infty.
 \end{aligned}$$

This verifies that  $g$  is integrable and hence finite a.e. in  $[a, b]$ . Consequently  $f$  is differentiable a.e. and  $g = f'$  a.e. ■

*Remark.* Strict inequality holds in Theorem 2.5.

## 2.6 Examples

1. Consider the function  $f : [0, 1] \rightarrow \mathbb{R}$  given by

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} < x \leq 1. \end{cases}$$

Then  $f'(x) = 0$  a.e. on  $[0, 1]$ ; and

$$\int_0^1 f' = 0 < 1 = f(1) - f(0).$$

2. The Cantor function  $\psi$  satisfies the hypothesis on  $[0, 1]$  and in addition  $\psi'(0) = 0$  a.e. on  $[0, 1]$ . Thus

$$\int_0^1 \psi' = 0 < 1 = \psi(1) - \psi(0).$$

We know that equality, in the case of the Riemann integral, holds in (2) if  $f'$  is continuous in  $[a, b]$ . However, we shall obtain equality in the case of the Lebesgue integral under conditions less restrictive than that a continuity of  $f'$ , but of course stronger than the hypothesis of this theorem, cf. Theorem 4.5.

**Problem 5.** Show by an example that the condition ' $f$  is an increasing function in Theorem 2.5' is necessary.

## 3 FUNCTIONS OF BOUNDED VARIATION

In this section, we introduce functions of bounded variation. The reason for introducing such functions here is that they are related to the derivatives of Lebesgue integrals. These functions are essentially the differences of monotone functions (cf. Theorem 3.2); and we, in fact, extend the Lebesgue Theorem on differentiability of monotone functions (cf. Theorem 2.5) to the class of functions of bounded variation in § 4 which is larger than that of monotone functions.

**3.1 Definition.** A function  $f$  defined on an interval  $[a, b]$  is said to be of bounded variation if there is a constant  $K > 0$  such that for every partition

$$P = \{a = x_0 < x_1 < \dots < x_i < \dots < x_n = b\}$$

of  $[a, b]$ , we have

$$\sum_{i=1}^n |f(x_i) - f(x_{i-1})| \leq K.$$

In fact, the variation of a function  $f$  defined on  $[a, b]$  with respect to a partition  $P$  of  $[a, b]$  is given by

$$V_a^b(f, P) = \sum_{i=1}^n |f(x_i) - f(x_{i-1})|,$$

and the total variation of  $f$  on  $[a, b]$  by

$$T_a^b(f) = \sup_P V_a^b(f, P).$$

The total variation  $T_a^b(f)$  may be finite or infinite. It follows from Definition 3.1 that  $f$  is of bounded variation on  $[a, b]$  if and only if  $T_a^b(f) < \infty$ .

Further, let  $f$  be defined on  $\mathbf{R}$ . Define

$$T_{-\infty}^{\infty}(f) = \lim_{a \rightarrow \infty} T_{-a}^a(f).$$

If  $T_{-\infty}^{\infty}(f) < \infty$ , then  $f$  is said to be of bounded variation on  $\mathbf{R}$ .

We now give a few simple observations with regard to the functions of bounded variation.

(a) *A bounded monotone function is a function of bounded variation and*

$$T_a^b(f) = |f(b) - f(a)|.$$

(b) *A function of bounded variation is necessarily bounded, but not conversely.*

Let  $x \in [a, b]$  be arbitrary. Then

$$|f(x) - f(a)| + |f(b) - f(x)| \leq T_a^b(f)$$

$$\Rightarrow |f(x)| \leq f(a) + T_a^b(f).$$

For the converse, consider a function  $f: [0, 1] \rightarrow \mathbf{R}$  given by

$$f(x) = \begin{cases} x \sin\left(\frac{\pi}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases} \quad (3)$$

The function  $f$  is clearly bounded. To show that  $f$  is not of bounded variation, consider a partition  $\left\{1, \frac{2}{3}, \frac{2}{5}, \frac{2}{7}, \dots, \frac{2}{2n+1}, 0\right\}$  of  $[0, 1]$ ,  $n$  being a positive integer. For this partition, we have

$$\begin{aligned} |f(1) - f\left(\frac{2}{3}\right)| + |f\left(\frac{2}{3}\right) - f\left(\frac{2}{5}\right)| + \dots + |f\left(\frac{2}{2n+1}\right) - f(0)| \\ = \frac{2}{3} + \left(\frac{2}{3} + \frac{2}{5}\right) + \dots + \left(\frac{2}{2n+1}\right) \\ = 4 \left(\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots + \frac{1}{2n+1}\right). \end{aligned}$$

Since  $\sum_n \frac{1}{2n+1}$  is divergent, its partial sum  $\left(\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots + \frac{1}{2n+1}\right)$  is not bounded above. Hence  $T_0^1(f) = \infty$ .

(c) *A function of bounded variation need not be continuous.*

Consider the function  $f: [0, 2] \rightarrow \mathbb{R}$  defined by  $f(x) = [x]$ , where  $[x]$  denotes the greatest integer not greater than  $x$ . Here  $f$  is a function of bounded variation but not continuous in  $[0, 2]$ .

(d) *A continuous function need not be of bounded variation.*

The function  $f$  defined in (3) is continuous on  $[0, 1]$  but not of bounded variation.

(e) *If  $f$  and  $g$  are two functions of bounded variation on  $[a, b]$  and  $\lambda$  is a constant, then  $\lambda f$  and  $f + g$  are functions of bounded variation on  $[a, b]$ . Furthermore,*

$$T_a^b(f+g) \leq T_a^b(f) + T_a^b(g)$$

and

$$T_a^b(\lambda f) = |\lambda| T_a^b(f).$$

Let  $P = \{a = x_0 < x_1 < \dots < x_n = b\}$  be a partition of  $[a, b]$ . Then

$$\begin{aligned} \sum_{i=1}^n |f(x_i) + g(x_i) - f(x_{i-1}) - g(x_{i-1})| \\ \leq \sum_{i=1}^n |f(x_i) - f(x_{i-1})| + \sum_{i=1}^n |g(x_i) - g(x_{i-1})|. \end{aligned}$$

Taking suprema on both the sides over all partitions  $P$  of  $[a, b]$ , and noting that

$$\sup \{x + y : x \in A \text{ and } y \in B\} \leq \sup \{x : x \in A\} + \sup \{y : y \in B\},$$

we immediately get  $T_a^b(f+g) \leq T_a^b(f) + T_a^b(g)$ . This also verifies that  $f+g$  is a function of bounded variation.

The other part is an immediate consequence of the definition of  $T_a^b(f)$ .

*Remark.* The set of all functions of bounded variation on a given interval  $[a, b]$ , denoted by  $BV[a, b]$ , forms a linear space unlike the set of all monotone functions. In fact, the sum of two monotone functions need not be monotone, e.g.  $f(x) = \sin x + 2x$  and  $g(x) = \sin x - 2x$  defined on  $[-\pi, \pi]$ .

(f) If  $f$  is a function of bounded variation on  $[a, b]$  and  $c$  is a point of  $[a, b]$ , then  $f$  is of bounded variation on  $[a, c]$  and  $[c, b]$  both, and conversely. Furthermore,

$$T_a^b(f) = T_a^c(f) + T_c^b(f).$$

(g)  $f$  is a function of bounded variation on  $[a, b]$ , then the function  $v_f: [a, b] \rightarrow \mathbb{R}$  defined by  $v_f(x) = T_a^x(f)$  is known as the variation function. It may easily be verified that  $v_f$  is an increasing function.

**3.2 Theorem (Jordan Decomposition Theorem).** A function  $f$  defined on  $[a, b]$  is of bounded variation if and only if it can be expressed as a difference of two monotone increasing functions (real-valued) on  $[a, b]$ .

*Proof.* Let  $f$  be of bounded variation on  $[a, b]$ . Define  $g$  and  $h$  by

$$\begin{cases} g = \frac{1}{2} (v_f + f) \\ h = \frac{1}{2} (v_f - f), \end{cases}$$

so that  $f = g - h$ .

Now, if  $x_1, x_2$  is any pair of points in  $[a, b]$  with  $x_2 > x_1$ , then

$$\begin{cases} g(x_2) - g(x_1) = \frac{1}{2} [\{v_f(x_2) - v_f(x_1)\} + \{f(x_2) - f(x_1)\}] \\ h(x_2) - h(x_1) = \frac{1}{2} [\{v_f(x_2) - v_f(x_1)\} - \{f(x_2) - f(x_1)\}]. \end{cases}$$

But,  $f$  being of bounded variation on  $[a, b]$ , in particular on  $[x_1, x_2]$ , we have

$$|f(x_2) - f(x_1)| \leq T_{x_1}^{x_2}(f) = v_f(x_2) - v_f(x_1).$$

Hence  $g(x_2) > g(x_1)$  and  $h(x_2) > h(x_1)$  verifying that  $g$  and  $h$  both are monotone increasing (real-valued) functions.

Conversely, if  $f = g - h$ , where  $g$  and  $h$  are monotone increasing functions, then for any partition  $P$  of  $[a, b]$ , we have

$$\begin{aligned} \sum_{i=1}^n |f(x_i) - f(x_{i-1})| &\leq \sum_{i=1}^n [g(x_i) - g(x_{i-1})] + \sum_{i=1}^n [h(x_i) - h(x_{i-1})] \\ &= g(b) - g(a) + h(b) - h(a) \end{aligned}$$

$$\Rightarrow T_a^b(f) < \infty.$$

Hence  $f$  is a function of bounded variation. ■

**3.3 Corollary.** *If  $f$  is a function of bounded variation on  $[a, b]$ , then  $f'$  exists a.e. on  $[a, b]$ .*

*Proof.* It follows from Theorems 2.5 and 3.2. ■

**3.4 Corollary.** *If  $f$  is a function of bounded variation, then it is measurable.*

*Proof.* By Corollary 3.3,  $f$  is continuous a.e. in  $[a, b]$  and the result follows by Theorem IV-8.7. ■

*Note.* The functions  $g$  and  $h$  in the decomposition of  $f$  are called, respectively, the positive and the negative variation functions of  $f$  on  $[a, b]$ . Also, observe that

$$v_f(x) = g(x) + h(x), \quad x \in [a, b].$$

The positive and the negative variation functions of  $f$  are, respectively, denoted by  $P_a^x(f)$  and  $N_a^x(f)$ . Thus, the positive and the negative variations of  $f$  on  $[a, b]$  are, respectively, given by  $P_a^b(f)$  and  $N_a^b(f)$ . Also

$$T_a^b(f) = P_a^b(f) + N_a^b(f).$$

**3.5 Theorem.** *Let  $f$  be a function of bounded variation on  $[a, b]$ . Then  $f$  is continuous at a point in  $[a, b]$  if and only if its variation function  $v_f$  is so.*

*Proof.* Let  $f$  be continuous at a point  $x^*$  in  $[a, b]$ . Then for each  $\epsilon > 0$ , there exists a  $\delta > 0$  such that

$$|f(x) - f(x^*)| < \frac{\epsilon}{2}, \quad \text{for } |x - x^*| < \delta.$$

By the definition of  $T_{x^*}^b(f)$ ,  $\exists$  a partition

$$P = \{x^* = x_0 < x_1 < x_2 < \dots < x_n = b\}$$

of  $[x^*, b]$  such that

$$\sum_{i=1}^n |f(x_i) - f(x_{i-1})| > T_{x^*}^b(f) - \frac{\epsilon}{2}.$$

Assume that the first interval, i.e.  $[x_0, x_1]$  in  $P$  is of length less than  $\delta$ , otherwise it could be made so by introducing an additional point between  $x_0$  and  $x_1$ , which of course would not decrease the sum on the right of the above and, as such, the above inequality would still remain valid. Therefore,

$$\begin{aligned} T_{x^*}^b(f) - \frac{\epsilon}{2} &< |f(x_1) - f(x_0)| + \sum_{i=2}^n |f(x_i) - f(x_{i-1})| \\ &< \frac{\epsilon}{2} + T_{x_1}^b(f) \end{aligned}$$

$$\Rightarrow T_{x^*}^b(f) - T_{x_1}^b(f) < \epsilon$$

$$\Rightarrow T_{x^*}^b(f) < \epsilon.$$

In other words,

$$v_f(x_1) - v_f(x^*) < \epsilon, \quad 0 < x_1 - x^* < \delta$$

$$\Rightarrow \lim_{x \rightarrow x^* + 0} v_f(x) = v_f(x^*).$$

Similarly

$$\lim_{x \rightarrow x^* - 0} v_f(x) = v_f(x^*).$$

Hence,  $v_f$  is continuous at  $x^*$ .

Conversely, if the variation function  $v_f$  is continuous at a point  $x^*$  in  $[a, b]$ , to each  $\epsilon > 0$ ,  $\exists$  a  $\delta > 0$  such that

$$|v_f(x) - v_f(x^*)| < \epsilon, \quad \text{for } |x - x^*| < \delta.$$

The result now follows from the fact that

$$|f(x) - f(x^*)| \leq |v_f(x) - v_f(x^*)|. \blacksquare$$

**3.6 Corollary.** *A continuous function is of bounded variation if and only if it can be expressed as a difference of two continuous monotone increasing functions.*

*Proof.* It follows from Theorems 3.2 and 3.5.  $\blacksquare$

*Remark.* If  $f$  is a continuous function of bounded variation, it may

also be expressed as the difference of two monotone increasing functions which have discontinuities (even countably many), e.g.

$$f = (g + k) - (h + k),$$

where  $k$  is an increasing function having discontinuities.

### Problems

6. Show that the function  $f: [0, 1] \rightarrow \mathbb{R}$  defined by

$$f(x) = \begin{cases} x^\alpha \sin\left(\frac{1}{x^\beta}\right) & \text{if } 0 < x \leq 1 \\ 0 & \text{if } x = 0. \end{cases}$$

is of bounded variation if  $\alpha > \beta$  but not if  $\alpha \leq \beta$ .

7. Let  $f$  be defined by  $f(0) = 0$  and  $f(x) = x^2 \sin\left(\frac{1}{x^2}\right)$  for  $x \neq 0$ . Is  $f$  of bounded variation on  $[-1, 1]$ ?
8. Show that the function  $f: [0, 1] \rightarrow \mathbb{R}$  defined by

$$f(x) = \begin{cases} x \cos\left(\frac{\pi x}{2}\right) & \text{if } 0 < x \leq 1 \\ 0 & \text{if } x = 0 \end{cases}$$

is continuous but not of bounded variation on  $[0, 1]$ .

9. Show that a step function defined on  $[a, b]$  is of bounded variation on  $[a, b]$ .
10. Let  $\{f_n\}$  be a sequence of functions defined on  $[a, b]$ . If  $f_n \rightarrow f$  on  $[a, b]$ , then  $T_a^b(f) \leq \liminf_{n \rightarrow \infty} T_a^b(f_n)$ .
11. Let  $f$  be a function of bounded variation. Prove that the set of points of discontinuity of  $f$  is countable.

## 4 DIFFERENTIATION OF AN INTEGRAL

If  $f$  is an integrable function on an interval  $[a, b]$ , then  $f$  is integrable on any interval  $[a, x] \subset [a, b]$ . The function  $F$  given by

$$F(x) = \int_a^x f(t) dt + c,$$

where  $c$  is a constant, called the **indefinite integral** of  $f$ .

**4.1 Theorem.** *Let  $f$  be an integrable function on  $[a, b]$ . Then the indefinite integral of  $f$  is a continuous function of bounded variation on  $[a, b]$ .*

*Proof.* Let  $x_0$  be any point in  $[a, b]$ . Then

$$\begin{aligned} |F(x) - F(x_0)| &= \left| \int_{x_0}^x f(t) dt \right| \\ &\leq \int_{x_0}^x |f(t)| dt. \end{aligned}$$

But  $f$  being integrable, the function  $|f|$  is integrable over  $[a, b]$ , (cf. V-6.2). Therefore, given  $\epsilon > 0$ , there is a  $\delta > 0$  such that for every measurable set  $A \subset [a, b]$  with  $m(A) < \delta$ , we have (cf. V-5.10)

$$\int_A |f| < \epsilon.$$

In particular

$$\left| \int_{x_0}^x |f(t)| dt \right| < \epsilon, \text{ for } |x - x_0| < \delta.$$

Hence  $|F(x) - F(x_0)| < \epsilon$ , whenever  $|x - x_0| < \delta$ . This proves the continuity of  $F$  at  $x_0$ , and hence in  $[a, b]$ .

In order to establish that  $F$  is a function of bounded variation, let

$$P = \{a = x_0 < x_1 < x_2 \dots < x_n = b\}$$

be a partition of  $[a, b]$ . Then

$$\begin{aligned} \sum_{i=1}^n |F(x_i) - F(x_{i-1})| &= \sum_{i=1}^n \left| \int_{x_{i-1}}^{x_i} f(t) dt \right| \\ &\leq \sum_{i=1}^n \int_{x_{i-1}}^{x_i} |f(t)| dt \\ &= \int_a^b |f(t)| dt \end{aligned}$$

$$\Rightarrow T_a^b(F) = \int_a^b |f(t)| dt < \infty.$$

Hence the result follows. ■

**4.2 Theorem.** *Let  $f$  be an integrable function on  $[a, b]$ . If*

$$\int_a^x f(t) dt = 0,$$

*for all  $x \in [a, b]$ , then  $f = 0$  a.e. in  $[a, b]$ .*

*Proof.* Let if possible,  $f \neq 0$  a.e. in  $[a, b]$ . Suppose  $f(t) > 0$  on a set  $E$  of positive measure. Then there exists a closed set  $F \subset E$  with  $m(F) > 0$ . Put  $A = ]a, b[ - F$ . Then  $A$  is an open set and we have

$$\begin{aligned}
 0 &= \int_a^b f(t) dt = \int_{A \cup F} f(t) dt = \int_A f(t) dt + \int_F f(t) dt \\
 \Rightarrow \int_A f(t) dt &= - \int_F f(t) dt.
 \end{aligned}$$

But  $f(t) > 0$  on  $F$  with  $m(F) > 0$  implies

$$\int_F f(t) dt \neq 0$$

Therefore

$$\int_A f(t) dt \neq 0.$$

Now,  $A$  being an open set, it can be expressed as a union of countable collection  $\{]a_n, b_n[ \}$  of disjoint open intervals. Thus

$$\begin{aligned}
 0 &\neq \int_A f(t) dt = \sum_n \int_{a_n}^{b_n} f(t) dt \\
 \Rightarrow \int_{a_n}^{b_n} f(t) dt &\neq 0 \quad \text{for some } n \\
 \Rightarrow \text{either } \int_a^{a_n} f(t) dt &\neq 0 \quad \text{or} \quad \int_a^{b_n} f(t) dt \neq 0.
 \end{aligned}$$

In either case, we see that if  $f$  is positive on a set of positive measure, then for some  $x \in [a, b]$  we have

$$\int_a^x f(t) dt \neq 0.$$

Similar assertion is obtained if  $f$  is negative on a set of positive measure. Hence the result follows by contradiction. ■

**4.3 Theorem.** *Let  $f$  be a bounded and measurable function defined on  $[a, b]$ . If*

$$F(x) = \int_a^x f(t) dt + F(a),$$

*then  $F'(x) = f(x)$  a.e. in  $[a, b]$ .*

*Proof.* Since  $f$  is bounded and measurable, it is integrable (cf. IV-2.1). Therefore, in view of Theorem 4.1,  $F$  is a continuous function of bounded variation on  $[a, b]$  and hence  $F'$  exists a.e. in  $[a, b]$ , cf. Corollary 3.3. Let  $|f| \leq K$ . Set

$$f_n(x) = \frac{F(x+h) - F(x)}{h},$$

with  $h = \frac{1}{n}$ . Then

$$f_n(x) = \frac{1}{h} \int_x^{x+h} f(t) dt$$

$$\Rightarrow |f_n| \leq K.$$

Also,  $f_n \rightarrow F'$  a.e. If  $c \in [a, b]$  is arbitrary, then the Bounded Convergence Theorem (cf. V-4.7) implies that

$$\begin{aligned} \int_a^c F'(x) dx &= \lim_{n \rightarrow \infty} \int_a^c f_n(x) dx \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \int_a^c [F(x+h) - F(x)] dx \\ &= \lim_{h \rightarrow 0} \left[ \frac{1}{h} \int_c^{c+h} F(x) dx - \frac{1}{h} \int_a^{a+h} F(x) dx \right]. \end{aligned}$$

But  $F$  being a continuous function, we note that

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{1}{h} \int_c^{c+h} F(x) dx &= \lim_{h \rightarrow 0} \frac{1}{h} \mathcal{R} \int_c^{c+h} F(x) dx \\ &= \lim_{h \rightarrow 0} \frac{1}{h} F(c + \theta h), \quad 0 < \theta < 1 \\ &= F(c); \end{aligned}$$

and similarly

$$\lim_{h \rightarrow 0} \frac{1}{h} \int_a^{a+h} F(x) dx = F(a).$$

Thus

$$\int_a^c F'(x) dx = F(c) - F(a) = \int_a^c f(x) dx$$

$$\Rightarrow \int_a^c [F'(x) - f(x)] dx = 0.$$

This is true for all  $c$  in  $[a, b]$ . Hence, by Theorem 4.2, we have  $F' = f$  a.e. ■

Note that in Theorem 4.3, the function considered is bounded and measurable. We extend it, in the following theorem, to any measurable function which of course is integrable.

**4.4 Theorem.** *Let  $f$  be an integrable function on  $[a, b]$ , and suppose*

$$F(x) = \int_a^x f(t) dt + F(a).$$

*Then  $F'(x) = f(x)$  a.e. in  $[a, b]$ .*

*Proof.* We may assume, without any loss of generality, that  $f \geq 0$ . Define a sequence  $\{f_n\}$  of functions  $f_n: [a, b] \rightarrow \mathbf{R}$ , where

$$f_n(x) = \begin{cases} f(x) & \text{if } f(x) \leq n, \\ n & \text{if } f(x) > n. \end{cases}$$

Clearly, each  $f_n$  is a bounded and measurable function and so, by Theorem 4.3, we have

$$\frac{d}{dx} \int_a^x f_n = f_n(x) \text{ a.e.}$$

Also,  $f - f_n \geq 0$  for all  $n$ , and, therefore, the function  $G_n$  defined by

$$G_n(x) = \int_a^x (f - f_n)$$

is an increasing function of  $x$ , which must have a derivative almost everywhere, cf. Theorem 2.5; and this derivative would, clearly, be nonnegative. Thus, from the relation

$$F(x) = \int_a^x f(t) dt + F(a) = G_n(x) + \int_a^x f_n(t) dt + F(a),$$

it follows that

$$\begin{aligned} F'(x) &= G'_n(x) + f_n(x) \text{ a.e.} \\ &\geq f_n(x) \text{ a.e.,} \quad \forall n. \end{aligned}$$

Since  $n$  is arbitrary, we have

$$F'(x) \geq f(x) \text{ a.e.}$$

$$\Rightarrow \int_a^b F'(x) dx \geq \int_a^b f(x) dx = F(b) - F(a).$$

Consequently, in view of Theorem 2.5, we get

$$\begin{aligned} \int_a^b F'(x) dx &= F(b) - F(a) \\ &= \int_a^b f(x) dx \end{aligned}$$

$$\Rightarrow \int_a^b \{F'(x) - f(x)\} dx = 0.$$

Since  $F'(x) - f(x) \geq 0$ , this gives that  $F'(x) - f(x) = 0$  a.e., and so  $F'(x) = f(x)$  a.e. ■

### Remarks

1. An indefinite integral need not be differentiable everywhere, and even if it is differentiable, it need not follow that  $F' = f$  everywhere, cf. Examples 1 and 2 in 4.5.

2. The condition  $F' = f$  a.e. is not sufficient to ensure that  $F$  is an indefinite integral of  $f$ . Consider for instance examples in 2.6.

#### 4.5 Examples

1. Consider the function  $f: [0, 2] \rightarrow \mathbb{R}$  given by

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ 2 & \text{if } 1 < x \leq 2. \end{cases}$$

Then

$$F(x) = \int_0^x f(t) dt = \begin{cases} x & \text{if } 0 \leq x \leq 1, \\ 2x - 1 & \text{if } 1 < x \leq 2. \end{cases}$$

Here  $F$  defines a function which is continuous but not differentiable in  $[0, 2]$ . In fact,  $F$  is not differentiable at  $x = 1$ .

2. Consider the function  $f: [0, 1] \rightarrow \mathbb{R}$  given by

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = p/q \\ 0 & \text{if otherwise.} \end{cases}$$

Then

$$F(x) = \int_0^x f(t) dt = 0.$$

Here  $F$  defines a function differentiable in  $[0, 1]$ . However,  $F(x) \neq f(x)$  for  $x = p/q$  in  $[0, 1]$ .

#### Problems

12. If  $f$  is the greatest integer function and  $F(x) = \int_0^x f(t) dt$ , determine  $F$  on  $[0, 5]$  and verify that  $F'(x) = f(x)$  a.e. in  $[0, 5]$ .

13. Let

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

If  $F$  is as in Problem 12, then verify that  $F'(x) = f(x)$  a.e. in  $[0, 1]$ .

#### 5 LEBESGUE SETS

In this section, we obtain interesting properties of integrable function which of course will not be used in the book but are of importance in applications.

**5.1 Definition.** Let  $f$  be an integrable function on  $[a, b]$ . A point  $x$  in  $[a, b]$  is said to be a **Lebesgue point** of  $f$  if

$$\lim_{h \rightarrow 0} \frac{1}{h} \int_x^{x+h} |f(t) - f(x)| dt = 0.$$

**5.2 Theorem.** Let  $x$  be a Lebesgue point of the function  $f$ . Then the indefinite integral

$$F(x) = \int_a^x f(t) dt + F(a)$$

is differentiable at the point  $x$ , and  $F'(x) = f(x)$ .

*Proof.* Since  $x$  is a Lebesgue point of  $f$ , by Definition 5.1, we have

$$\lim_{h \rightarrow 0} \int_x^{x+h} |f(t) - f(x)| dt = 0.$$

It is easy to show that

$$\begin{aligned} \frac{F(x+h) - F(x)}{h} - f(x) &= \frac{1}{h} \int_x^{x+h} \{f(t) - f(x)\} dt \\ \Rightarrow \left| \frac{F(x+h) - F(x)}{h} - f(x) \right| &\leq \frac{1}{h} \int_x^{x+h} |f(t) - f(x)| dt. \end{aligned}$$

This proves the theorem. ■

*Remark.* The converse of Theorem 5.2 may not be true in general.

**5.3 Theorem.** Every point of continuity of an integrable function  $f$  is a Lebesgue point of  $f$ .

*Proof.* Let  $f$  be continuous at  $x_0$ . Then, for every  $\epsilon > 0$ , there is a  $\delta > 0$  such that

$$|f(t) - f(x_0)| < \epsilon, \quad \text{for } |t - x_0| < \delta.$$

For  $|h| < \delta$ , we have

$$\frac{1}{h} \int_x^{x+h} \{f(t) - f(x)\} < \epsilon,$$

and the theorem is proved. ■

**5.4 Definition.** The set of all Lebesgue points in  $[a, b]$  of  $f$  is called the **Lebesgue set** of the function  $f$ .

**6 ABSOLUTELY CONTINUOUS FUNCTIONS**

We now introduce a special class of functions of bounded variation, namely, the class of absolutely continuous functions. This class is important for a number of applications and helps to obtain results already proved under less stringent conditions. In particular, as we shall see in the next section, it characterises the class of indefinite integrals.

**6.1 Definition.** A real-valued function  $f$  defined on an interval  $[a, b]$  is **absolutely continuous** on  $[a, b]$ , if for each  $\epsilon > 0$  there exists a  $\delta > 0$  such that

$$\sum_{i=1}^n |f(x_i) - f(x_{i-1})| < \epsilon,$$

for every finite pairwise disjoint collection  $\{]x_i, x'_i[: i = 1, 2, \dots, n\}$  of open intervals in  $[a, b]$  with  $\sum_{i=1}^n |x'_i - x_i| < \delta$ .

*Remarks*

1. Every absolutely continuous function is uniformly continuous, as may be seen by choosing a single interval  $]x, x'[\subset [a, b]$ . However, a uniformly continuous function need not be absolutely continuous; for example, the Cantor function is uniformly continuous in  $[0, 1]$  but not absolutely continuous. The function  $f$  in Problem 6 for  $\alpha = \beta = 1$ ) is also an example of a continuous function which is absolutely continuous.

2. In Definition 6.1, we may replace 'finite' with 'countably infinite'. In fact, suppose that given any  $\epsilon > 0$ , there is a  $\delta > 0$  such that

$$\sum_{i=1}^n |f(x_i) - f(x_{i-1})| < \epsilon' < \epsilon,$$

for any finite collection  $\{]x_i, x'_i[\}$  of pairwise disjoint intervals of total length less than  $\delta$ . Consider any countably infinite collection of pairwise disjoint intervals  $]y_i, y'_i[\subset [a, b]$  of total length less than  $\delta$ . Then, obviously

$$\sum_{i=1}^n |f(y_i) - f(y_{i-1})| < \epsilon',$$

for every  $n$ . Hence, letting  $n \rightarrow \infty$ , we get

$$\sum_{i=1}^{\infty} |f(y_i) - f(y_{i-1})| \leq \epsilon' < \epsilon.$$

**6.2 Theorem.** *If  $f$  is absolutely continuous on  $[a, b]$ , then  $f$  is of bounded variation on  $[a, b]$ .*

*Proof.* For  $\epsilon = 1$ , there is a  $\delta > 0$  such that for every finite collection  $\{]x_i, x'_i[, i = 1, 2, \dots, n\}$  of pairwise disjoint intervals in  $[a, b]$  with  $\sum_{i=1}^n |x'_i - x_i| < \delta$ , we have

$$\sum_{i=1}^n |f(x'_i) - f(x_i)| < 1.$$

Select a natural number

$$N > \frac{b-a}{\delta}.$$

Divide  $[a, b]$  by means of points

$$c = c_0 < c_1 < c_2 \dots < c_N = b$$

such that  $c_j - c_{j-1} < \delta$ , for  $j = 1, 2, \dots, N$ . Therefore, for every finite collection  $\{]x_i, x'_i]\}$  of pairwise disjoint subintervals in  $[c_{j-1}, c_j]$ , we have

$$\sum_i |f(x'_i) - f(x_i)| < 1$$

$$\Rightarrow T_{c_{j-1}}^{c_j}(f) \leq 1, j = 1, 2, \dots, N.$$

Hence

$$T_a^b(f) = \sum_{j=1}^N T_{c_{j-1}}^{c_j}(f) \leq N < \infty,$$

and this proves that  $f$  is a function of bounded variation. ■

*Note* The result in Theorem 6.2 does not hold good when  $a$  or  $b$  is allowed to be infinite.

**6.3 Corollary.** *If  $f$  is an absolutely continuous function, it has a derivative almost everywhere.*

*Proof.* It follows from Theorem 6.2 and Corollary 3.3. ■

*Remark.* The converse of Theorem 6.2 need not be true. There are functions of bounded variation which are not absolutely continuous; one such function is the Cantor function, cf. II-10.6.

**6.4 Theorem.** *Let  $f$  be a function of bounded variation on  $[a, b]$ . Then  $f$  is absolutely continuous on  $[a, b]$  if and only if the variation function  $v_f$ , given by  $v_f(x) = T_a^x(f)$ , is absolutely continuous on  $[a, b]$ .*

*Proof.* The absolute continuity of  $v_f$  clearly implies that of  $f$  since  $|f(x_i) - f(x_{i-1})| \leq v_f(x_i) - v_f(x_{i-1})$  for  $a \leq x_{i-1} < x_i < b$ .

On the other hand, assume  $f$  to be absolutely continuous. Then given an  $\epsilon > 0$ , there exists a  $\delta > 0$  such that, for any finite collection  $C = \{[x_i, x'_i], i = 1, 2, \dots, n\}$  of pairwise disjoint intervals in  $[a, b]$  with  $\sum_{i=1}^n |x'_i - x_i| < \delta$ , we have

$$\sum_{i=1}^n |f(x'_i) - f(x_i)| < \epsilon.$$

For each  $i$ , let

$$P_i = \{x_i = a_0^i < a_1^i < a_2^i < \dots < a_{m_i}^i = x'_i\}$$

be a partition of  $[x_i, x'_i]$ . Since

$$\sum_{i=1}^n \sum_{j=1}^{m_i} |a_j^i - a_{j-1}^i| = \sum_{i=1}^n |x'_i - x_i| < \delta,$$

we have

$$\sum_{i=1}^n \sum_{j=1}^{m_i} |f(a_j^i) - f(a_{j-1}^i)| < \epsilon.$$

Fixing the collection  $C$  but varying the partitions  $P_i$  of each  $[x_i, x'_i]$ , we have, upon taking the supremum over all such partitions  $P_i$ ,

$$\sum_{i=1}^n T_{x_i}^{x'_i}(f) < \epsilon$$

$$\Rightarrow \sum_{i=1}^n v_f(x'_i) - v_f(x_i) < \epsilon.$$

This proves  $v_f$  is absolutely continuous. ■

**6.5 Theorem.** *If  $f$  and  $g$  are absolutely continuous functions on  $[a, b]$  and  $\lambda$  is any constant, then so are the functions: (i)  $f+g$ ; (ii)  $\lambda f$ ; (iii)  $fg$ ; and (iv)  $f/g$  ( $g \neq 0$ ).*

*Proof.* The proof is an immediate consequence of Definition 6.1 and the obvious properties of absolute value. ■

*Note.* From Theorems 6.2 and 6.5 together with the remark that follows Theorem 6.2, we note that the set of all absolutely continuous functions on  $[a, b]$  is a proper subspace of  $BV[a, b]$ , the linear space of all functions of bounded variation on  $[a, b]$ .

**6.6 Theorem.** *If  $f$  is absolutely continuous on  $[a, b]$  and  $f' = 0$  a.e., then  $f$  is a constant function.*

*Proof.* Let  $c \in [a, b]$  be arbitrary. Then it is enough to prove that  $f(c) = f(a)$ . Let  $E = \{x \in [a, c] : f'(x) = 0\}$ . Then  $E \subset [a, c]$ .

Let  $\epsilon$  and  $\eta$  be arbitrary positive numbers. Then, each  $x \in E$  is the left endpoint of an arbitrary small interval  $[x, x+h] \subset [a, c]$  such that

$$|f(x+h) - f(x)| < \eta h \quad (h > 0).$$

Thus, the set of all such intervals forms a Vitali cover of  $E$ . From among these intervals, by Lemma 2.3, we can choose a finite collection  $\{[x_i, y_i] : i = 1, 2, \dots, N\}$  of pairwise disjoint intervals which covers all of  $E$  except for a set of measure less than  $\delta$ , where  $\delta > 0$  is the number corresponding to  $\epsilon > 0$  in the definition of the absolute continuity of  $f$ . If we label  $x_i$  so that  $x_i \leq x_{i+1}$ , we have

$$y_0 = a \leq x_1 < y_1 \leq x_2 < y_2 \dots < y_N \leq c = x_{N+1}.$$

Also, the complement of the union of these intervals is again the union of a finite number of disjoint intervals  $\{[y_i, x_{i+1}] : i = 0, 1, \dots, N\}$ . Their combined length is not greater than  $\delta$ , i.e.

$$\sum_{i=0}^N |x_{i+1} - y_i| < \delta.$$

Now, we note that

$$\begin{aligned} \sum_{i=1}^N |f(y_i) - f(x_i)| &\leq \eta \sum_{i=1}^N (y_i - x_i) \\ &< \eta(c - a), \end{aligned}$$

by the way the intervals  $\{[x_i, y_i]\}$  were constructed; and

$$\sum_{i=0}^N |f(x_{i+1}) - f(y_i)| < \epsilon,$$

by the absolute continuity of  $f$ . Hence

$$\begin{aligned} |f(c) - f(a)| &= \left| \sum_{i=0}^N [f(x_{i+1}) - f(y_i)] + \sum_{i=1}^N [f(y_i) - f(x_i)] \right| \\ &\leq \epsilon + \eta(b - a). \end{aligned}$$

Since  $\epsilon$  and  $\eta$  are arbitrary positive numbers, the result follows by letting  $\epsilon \rightarrow 0$  and  $\eta \rightarrow 0$ . ■

**6.7 Corollary.** *If the derivatives of two absolutely continuous functions are equivalent, then the functions differ by a constant.*

**Problems**

14. Let  $f$  be defined by  $f(0) = 0$  and  $f(x) = x^2 \sin(1/x)$  for  $0 < x \leq 1$ . Then prove that  $f$  is an absolutely continuous function.
15. Work out an example of a function which is continuous and monotone but not absolutely continuous.
16. Show that if  $f$  is nonnegative and absolutely continuous function on  $[a, b]$ , then so is  $f^p$  for  $p \geq 1$ .
17. Let  $f$  and  $g$  be absolutely continuous functions on  $[a, b]$ . Is  $f \circ g$  so? Justify.
18. Let  $f$  be defined on  $[0, 1]$ . Show that  $f$  need not be absolutely continuous on  $[0, 1]$  even if  $f$  is so on  $[\epsilon, 1]$  for every  $\epsilon > 0$  and continuous at 0.
19. Show that an absolutely continuous function  $f$  defined on  $[a, b]$  transforms:
  - (a) sets of measure zero into sets of measure zero; and
  - (b) measurable sets into measurable sets.
20. A function  $f$  is said to be **Lipschitzian** (or to satisfy a **Lipschitz condition**) on  $[a, b]$  if there is a constant  $K$  such that

$$|f(x) - f(t)| \leq K |x - t|, \text{ for all } x, t \in [a, b].$$

Show that if  $f$  is Lipschitzian on  $[a, b]$ , it is absolutely continuous on  $[a, b]$ .

[Hint: With  $\epsilon > 0$  given, take  $\delta = \epsilon/K$  in Definition 6.1.]

21. A monotone function  $f$  on  $[a, b]$  is called **singular** if  $f' = 0$  a.e. Show that any monotone increasing function is the sum of an absolutely continuous function and a singular function.
22. Let  $f$  be absolutely continuous on  $[a, b]$ . Show that

$$T_a^b(f) = \int_a^b |f'| \quad \text{and} \quad P_a^b(f) = \int_a^b [f']^+.$$

**7 INTEGRAL OF THE DERIVATIVE**

We first give a result which is more precise than that in Theorem 4.1 in regard to an indefinite integral of an integrable function.

**7.1 Theorem.** *Let  $f$  be an integrable function on  $[a, b]$ . Then the indefinite integral  $F$  of  $f$  is absolutely continuous on  $[a, b]$ .*

*Proof.* Let  $\epsilon > 0$  be given. Then there is a  $\delta > 0$  such that for every measurable set  $A \subset [a, b]$  with  $m(A) < \delta$ , we have (cf. V-5.10)

$$\int_A |f| < \epsilon,$$

since the integrability of  $f$  implies that of  $|f|$ . Thus, for any finite collection  $\{[x_i, x'_i] : i = 1, 2, \dots, N\}$  of pairwise disjoint open intervals in  $[a, b]$  with  $\sum_{i=1}^N (x'_i - x_i) < \delta$ , we have

$$\begin{aligned} \sum_{i=1}^N \left| \int_{x_i}^{x'_i} f(t) dt \right| &\leq \sum_{i=1}^N \int_{x_i}^{x'_i} |f(t)| dt < \epsilon \\ \Rightarrow \sum_{i=1}^N |F(x'_i) - F(x_i)| &< \epsilon. \end{aligned}$$

Hence, it follows that  $F$  is an absolutely continuous function. ■

We now prove the converse of Theorem 7.1.

**7.2 Theorem.** *If  $F$  is an absolutely continuous function on  $[a, b]$ , then  $F$  is an indefinite integral of its derivative; more precisely:*

$$F(x) = \int_a^x f(t) dt + C,$$

where  $f = F'$  and  $C$  is a constant.

Alternatively, we may state the theorem as: *If  $F$  is absolutely continuous function on  $[a, b]$ , then  $F'$  is integrable over  $[a, b]$ , and*

$$\int_a^x F'(t) dt = F(x) - F(a).$$

*Proof.* The function  $F$ , being absolutely continuous, is of bounded variation, and so we may write

$$F = F_1 - F_2,$$

where  $F_1$  and  $F_2$  are monotone increasing functions. Also,  $F'$  exists a.e. on  $[a, b]$  and

$$F' = F'_1 - F'_2$$

$$\Rightarrow |F'| \leq |F'_1| + |F'_2|.$$

This, in view of Theorem 2.5, gives

$$\begin{aligned} \int_a^b |F'| &\leq \int_a^b |F'_1| + \int_a^b |F'_2| \\ &\leq F_1(b) - F_1(a) + F_2(b) - F_2(a) \\ &< \infty \end{aligned}$$

$$\Rightarrow F' \text{ is integrable over } [a, b].$$

Write

$$G(x) = \int_a^x F'(t) dt.$$

Then, by Theorem 7.1,  $G$  is an absolutely continuous function on  $[a, b]$  and so is the function  $H = F - G$ . But, it may be noted, in view of Theorem 4.4, that

$$H' = F' - G' = 0 \text{ a.e.}$$

It follows from Corollary 6.7 that  $H$  is a constant function (say)  $A$ . Hence

$$F(x) = \int_a^x F'(t) dt + A.$$

Taking  $x = a$ , we get  $A = F(a)$ . This establishes the result. ■

It follows from Theorems 7.1 and 7.2 that: *A function  $f$  is absolutely continuous on  $[a, b]$  if and only if it is an indefinite integral of an integrable function on  $[a, b]$ .*

*Remark.* If the restriction that  $F$  is absolutely continuous is removed, then  $F'$  need not be integrable (Lebesgue) even.

**7.3 Example.** Let  $F: [0, 1] \rightarrow \mathbb{R}$  be a function defined by

$$F(x) = \begin{cases} x^2 \sin\left(\frac{\pi}{x^2}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

The derivative  $F'$  exists on  $[0, 1]$  while  $F'$  is not integrable on  $[0, 1]$ . In fact,

$$\int_0^1 \frac{1}{x} \left| \cos\left(\frac{\pi}{x^2}\right) \right| dx = \infty.$$

*Note.* The class of absolutely continuous functions over  $[a, b]$  is identical with the class of functions obtained by integrating Lebesgue integrable functions over  $[a, b]$  except that the corresponding functions in two classes differ at the most by a constant.

As an application of Theorems 7.1 and 7.2, we prove a result on integration by parts which is similar to that for the Riemann integral.

**7.4 Theorem.** *Let  $f$  and  $g$  be integrable functions over  $[a, b]$ . Suppose*

$$F(x) = \int_a^x f(t) dt + F(a)$$

and

$$G(x) = \int_a^x g(t) dt + G(a),$$

for all  $x \in [a, b]$ . Then

$$\int_a^b F(t)g(t) dt + \int_a^b f(t)G(t) dt = F(b)G(b) - F(a)G(a).$$

*Proof.* By Theorem 7.1,  $F$  and  $G$  are both absolutely continuous functions on  $[a, b]$  and hence so is  $FG$ , by Theorem 6.5. Using Theorem 7.2, we have

$$\begin{aligned} \int_a^b (FG)' &= (FG)(b) - (FG)(a) \\ &= F(b)G(b) - F(a)G(a). \end{aligned}$$

Also, by Theorem 4.4,  $F' = f$  a.e. and  $G' = g$  a.e. in  $[a, b]$ , and therefore

$$(FG)' = FG' + F'G = Fg + fG \text{ a.e. in } [a, b].$$

Hence

$$\int_a^b F(t)g(t) dt + \int_a^b f(t)G(t) dt = F(b)G(b) - F(a)G(a). \blacksquare$$

**7.5 Corollary.** *If  $f$  and  $g$  are absolutely continuous functions on  $[a, b]$ , then*

$$\int_a^b f(t)g'(t) dt + \int_a^b f'(t)g(t) dt = f(b)g(b) - f(a)g(a).$$

*Proof.* Since  $f$  and  $g$  are absolutely continuous,  $f'$  and  $g'$  are integrable over  $[a, b]$ . Also

$$f(x) = \int_a^x f'(t) dt + f(a)$$

and

$$g(x) = \int_a^x g'(t) dt + g(a).$$

The result now follows from Theorem 7.4.  $\blacksquare$

## VII

# The Lebesgue $L^p$ Spaces

Many of the classical spaces in analysis consist of measurable functions and most of the important norms on such spaces have been defined by integrals. The Lebesgue  $L^p$ -spaces are among such important classes. A complete understanding of these spaces needs a thorough understanding of the Lebesgue theory of measure and integration which we have developed in the preceding chapters. We are now fully prepared to introduce the  $L^p$  spaces. These spaces have remarkable properties and are of enormous importance in analysis as well as its applications.

### 1 NOTION OF BANACH SPACES

Let  $X$  be a linear space\* over a field of real or complex numbers. A norm on  $X$  is a real-valued function  $\|\cdot\|$  on  $X$  which has the properties:

1.  $\|x\| \geq 0$   
and  $\|x\| = 0 \Leftrightarrow x = 0$ ,  $\forall x \in X$
2.  $\|x + y\| \leq \|x\| + \|y\|$ ,  $\forall x, y \in X$
3.  $\|\alpha x\| = |\alpha| \|x\|$ ,  $\forall x \in X$  and  $\alpha \in \mathbb{R}$  (or  $\mathbb{C}$ ).

**1.1 Definition.** A linear space  $X$  equipped with a norm  $\|\cdot\|$  on it is called a **normed linear space** or simply a **normed space**.

The normed space just defined is denoted by  $(X, \|\cdot\|)$ . If no confusion is likely to occur, we may denote it simply by  $X$ .

A norm  $\|\cdot\|$  on  $X$  defines a metric  $d$  on  $X$  given by

$$d(x, y) = \|x - y\|.$$

\*For the definition of linear spaces and their basic properties, one may refer to any standard book on Linear Algebra.

It is simple to verify, from properties (1)-(3) of the norm, that  $d$  does define a metric on  $X$ . The metric  $d$  thus defined is called the **metric induced by the norm**.

The notion of convergence for sequences and the related concepts in normed spaces follow readily from the corresponding definitions in metric spaces and the fact that now we have

$$d(x, y) = \|x - y\|.$$

**1.2 Definition.** A sequence  $\{x_n\}$  in a normed space  $X$  is said to **converge** to an element  $x \in X$  if given an  $\epsilon > 0$ , there is an  $N$  such that

$$\|x_n - x\| < \epsilon, \quad \forall n > N.$$

We write  $x_n \rightarrow x$  or  $\lim_{n \rightarrow \infty} x_n = x$ .

**1.3 Definition.** A sequence  $\{x_n\}$  in a normed space  $X$  is a **Cauchy sequence** if given an  $\epsilon > 0$ , there is an  $N$  such that

$$\|x_n - x_m\| < \epsilon, \quad \forall n, m > N.$$

One may verify that each convergent sequence in a normed space is a Cauchy sequence, but the converse may not be true.

**1.4 Definition.** A normed space is said to be **complete** if every Cauchy sequence in it is convergent, i.e. if for every Cauchy sequence  $\{x_n\}$  in  $X$ , there is an element  $x$  in  $X$  such that  $x_n \rightarrow x$ .

**1.5 Definition.** A complete normed linear space is called a **Banach space**.

### 1.6 Examples

1. The space  $\mathbb{Q}$  of all rational numbers with the norm  $\|\cdot\|$  defined by  $\|x\| = |x|$ ,  $x \in \mathbb{Q}$  is a normed space, but not a Banach space.
2. The spaces  $\mathbb{R}$  and  $\mathbb{C}$  (the real numbers and the complex numbers) are Banach spaces with the norm  $\|\cdot\|$  given by  $\|x\| = |x|$ ,  $x \in \mathbb{R}$  (or  $\mathbb{C}$ ).
3. Let  $p$ ,  $1 \leq p < \infty$ , be a real number. Denote by  $l^p$  the space of all sequences  $x = (x_1, x_2, x_3, \dots)$  of scalars such that  $\sum_{n=1}^{\infty} |x_n|^p < \infty$ . The space  $l^p$  is a Banach space with the norm defined by

$$\|x\|_p = \left( \sum_{n=1}^{\infty} |x_n|^p \right)^{1/p}.$$

4. The space  $l^\infty$  of all bounded sequences  $x = (x_1, x_2, x_3, \dots)$  of scalars is a Banach space under the norm given by

$$\|x\|_\infty = \sup_n |x_n|.$$

5. The space  $P[a, b]$  of all polynomials defined on  $[a, b]$  is a normed space under the norm given by

$$\|x\| = \max_{t \in [a, b]} |x(t)|.$$

But  $P[a, b]$  is not a Banach space.

6. The space  $C[a, b]$  of all continuous functions defined on  $[a, b]$  is a Banach space under the norm given by

$$\|x\| = \max_{t \in [a, b]} |x(t)|.$$

Sequences were available to us even in the metric spaces (general). In a normed space, we may proceed an important step further and use the series as follows.

Let  $\{x_k\}$  be a sequence in a normed space  $X$ . Then we can associate with  $\{x_k\}$ , the sequence  $\{s_n\}$  of partial sums given by  $s_n = \sum_{k=1}^n x_k$ . And we can discuss the behaviour of the series  $\sum_{k=1}^{\infty} x_k$  in regard to its convergence or non-convergence accordingly as the sequence  $\{s_n\}$  of its partial sums is so.

**1.7 Definition.** A sequence  $\{x_k\}$  in a normed space  $X$  is said to be **summable** to the sum  $s$  if the sequence  $\{s_n\}$  of the partial sums of the series  $\sum_k x_k$  converges to  $s$  in  $X$ , i.e.

$$\|s_n - s\| \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

or

$$\left\| \sum_{k=1}^n x_k - s \right\| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

If this is the case, we write

$$s = \sum_{k=1}^{\infty} x_k.$$

The sequence  $\{x_k\}$  is said to be **absolutely summable** if

$$\sum_{k=1}^{\infty} \|x_k\| < \infty.$$

*Note.* We know that for a sequence of real numbers absolute summability implies summability. But this is not true in general for sequences in normed spaces. However, we prove the following.

**1.8 Theorem.** *A normed space  $X$  is complete if and only if every absolutely summable sequence is summable.*

*Proof.* Assume that  $X$  is a complete normed space. Let  $\{x_n\}$  be an absolutely summable sequence in  $X$ . Then

$$\sum_{n=1}^{\infty} \|x_n\| = M < \infty.$$

Thus, for each  $\epsilon > 0$ , there is an  $N$  such that

$$\sum_{n=N}^{\infty} \|x_n\| < \epsilon.$$

Let  $s_n = \sum_{k=1}^n x_k$  be the partial sums of the series  $\sum_{n=1}^{\infty} x_n$ . Then, we have

$$\begin{aligned} \|s_n - s_m\| &= \left\| \sum_{k=m+1}^n x_k \right\| \\ &\leq \sum_{k=m+1}^n \|x_k\| \\ &\leq \sum_{k=N}^{\infty} \|x_k\| < \epsilon, \end{aligned}$$

for  $n \geq m > N$ . Thus  $\{s_n\}$  is a Cauchy sequence in  $X$  and must converge to some element (say)  $s$  in  $X$ , since  $X$  is complete. Hence  $\{x_n\}$  is summable in  $X$ .

Conversely, suppose each absolutely summable sequence in  $X$  is summable in  $X$ . Let  $\{x_n\}$  be a Cauchy sequence in  $X$ . Then, for each  $k$ , there is an integer  $n_k$  such that

$$\|x_n - x_m\| < \frac{1}{2^k}, \quad \forall n, m > n_k.$$

We may choose  $n_k$  such that  $n_{k+1} > n_k$ . Then  $\{x_{n_k}\}$  is a subsequence of  $\{x_n\}$ . Set

$$\begin{aligned} y_1 &= x_{n_1} \\ y_2 &= x_{n_2} - x_{n_1} \\ &\cdot \\ &\cdot \\ y_k &= x_{n_k} - x_{n_{k-1}} \end{aligned}$$

We note that

1.  $\sum_{i=1}^k y_i = x_{n_k}$
2.  $\|y_k\| < \frac{1}{2^k}, k > 1$

and as such

$$\sum \|y_k\| \leq \|y_1\| + \sum_{k=2}^{\infty} 2^{1-k} = \|y_1\| + 1 < \infty.$$

Thus, the sequence  $\{y_k\}$  is absolutely summable and hence summable to some element (say)  $x$  in  $X$ . It remains to prove that  $\lim_{n \rightarrow \infty} x_n = x$ .

Since  $\{x_n\}$  is a Cauchy sequence, given an  $\epsilon > 0$ , there is an  $N$  such that

$$\|x_n - x_m\| < \frac{\epsilon}{2}, \quad \forall n, m > N.$$

Further, since  $x_{n_k} \rightarrow x$ , there is a  $K$  such that

$$\|x_{n_k} - x\| < \frac{\epsilon}{2}, \quad \forall k \geq K.$$

Choose  $k$  so large that  $k > K$  and  $n_k > N$ . Hence

$$\begin{aligned} \|x_n - x\| &\leq \|x_n - x_{n_k}\| + \|x_{n_k} - x\| \\ &< \epsilon, \quad \forall n > N \end{aligned}$$

$$\Rightarrow x_n \rightarrow x. \blacksquare$$

**1.9 Definition.** Let  $X$  be a normed space over a field  $\mathbb{R}$  (or  $\mathbb{C}$ ). A mapping  $f: X \rightarrow \mathbb{R}$  (or  $\mathbb{C}$ ) is called a **linear functional** on  $X$  if  $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$ , for all  $x, y \in X$  and  $\alpha, \beta \in \mathbb{R}$  (or  $\mathbb{C}$ ).

**1.10 Definition.** A linear functional  $f$  on a normed space  $X$  is said to be **bounded** if there is a constant  $K > 0$  such that

$$|f(x)| \leq K \|x\|, \quad \forall x \in X. \quad (1)$$

The smallest constant  $K$  for which (1) holds is called the **norm** of  $f$ , written  $\|f\|$ . In fact, we have

$$\|f\| = \sup \left\{ \frac{|f(x)|}{\|x\|} : x \neq 0 \text{ and } x \in X \right\},$$

or equivalently

$$\|f\| = \sup \{ |f(x)| : x \in X \text{ and } \|x\| = 1 \}.$$

Also, we note that

$$|f(x)| \leq \|f\| \|x\|, \quad \forall x \in X.$$

## 2 THE CLASSES $L^p$

If  $f$  is a measurable function on  $E$ , then  $|f|^p$  is so for each  $p$ ,  $0 < p < \infty$ . Designate by  $L^p(E)$ , the class of all  $p$ -integrable functions over  $E$ , i.e.,

$$L^p(E) = \left\{ f : \int_E |f|^p < \infty \right\}.$$

### 2.1 Examples

1. Let  $E = [0, 16]$  and  $f : E \rightarrow \mathbb{R}$  be a function defined by  $f(x) = (x)^{-1/4}$ . Then  $f \in L^1(E)$  but  $f \notin L^4(E)$ .
2. Let  $E = \left[0, \frac{1}{2}\right]$  and the function  $f : E \rightarrow \mathbb{R}$  be defined by  $f(x) = \left[ x \log^2 \left( \frac{1}{x} \right) \right]^{-1}$ . Then  $f \in L^1(E)$ .
3. Let  $E = ]0, \infty[$  and the function  $f : E \rightarrow \mathbb{R}$  be defined by  $f(x) = (1+x)^{-1/2}$ . Then  $f \in L^p(E)$  for each  $p$ ,  $2 < p < \infty$ .

It is easy to verify that  $L^p(E)$  is a linear space over  $\mathbb{R}$ . Indeed, we observe that:

1.  $f, g \in L^p(E) \Rightarrow f+g \in L^p(E)$ , since
 
$$\begin{aligned} |f+g|^p &\leq 2^p \max \{|f|^p, |g|^p\} \\ &\leq 2^p(|f|^p + |g|^p). \end{aligned}$$
2.  $f \in L^p(E)$  and  $\alpha \in \mathbb{R} \Rightarrow \alpha f \in L^p(E)$ .

Furthermore, if  $f \in L^p(E)$ , then the inequalities

$$\begin{cases} 0 \leq f^+ \leq |f| \\ 0 \leq f^- \leq |f| \end{cases}$$

imply that  $f^+$ ,  $f^-$  and  $|f|$  are also in  $L^p(E)$ .

In order to define  $L^\infty(E)$ , let  $f$  be a real-valued and measurable function on a set  $E$  with  $m(E) > 0$ . A real number  $M$  is said to be an **essential bound** for the function  $f$  if

$$|f(x)| \leq M \text{ a.e. on } E.$$

A function  $f$  is said to be **essentially bounded** if it has an essential bound. In other words, a function  $f$  defined on  $E$  is essentially bounded if it is bounded except possibly on a set of measure zero. The **essential supremum** of  $f$  on  $E$  is defined by

$$\text{ess sup } |f(x)| = \inf \{M : |f(x)| \leq M \text{ a.e. on } E\},$$

or equivalently

$$\text{ess sup } |f(x)| = \inf \{M : m(\{x \in E : |f(x)| > M\}) = 0\}.$$

If  $f$  does not have any essential bound, then its essential supremum is defined to be  $\infty$ .

Let us designate by  $L^\infty(E)$  the class of all those measurable functions defined on  $E$  which are essentially bounded on  $E$ , i.e.

$$L^\infty(E) = \{f : \text{ess sup } |f| < \infty\}.$$

It is not difficult again to verify that  $L^\infty(E)$  is a linear space over  $\mathbf{R}$ .

## 2.2 Examples

1. Every bounded function on  $E$  is in  $L^\infty(E)$ .

2. The function  $f : [a, b] \rightarrow \mathbf{R}$  given by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is irrational} \\ \infty & \text{if } x \text{ is rational} \end{cases}$$

is in  $L^\infty[a, b]$ .

Let us now define a function  $\|\cdot\|_p : L^p(E) \rightarrow \mathbf{R}$ ,  $0 < p \leq \infty$ , as follows:

$$\|f\|_p = \left( \int_E |f|^p \right)^{1/p}, \quad 0 < p < \infty$$

$$\|f\|_\infty = \text{ess sup } |f|.$$

*Note.* We shall establish in §4 that if  $1 \leq p \leq \infty$ , then  $\|\cdot\|_p$  defines a norm on  $L^p(E)$ ,  $1 \leq p \leq \infty$ . In case  $1 \leq p < \infty$ ,  $\|\cdot\|_p$  is called a  $p$ -norm on  $L^p(E)$ .

A relationship between  $\|\cdot\|_p$  ( $1 \leq p < \infty$ ) and  $\|\cdot\|_\infty$  will soon be established in Theorem 2.4 which, in fact, would also justify the notation  $L^\infty(E)$  for the class of essentially bounded functions on  $E$ . Before, we do so let us prove some results in the form of a lemma.

**2.3 Lemma.** *Let  $f \in L^\infty(E)$ . Then:*

(a)  $|f(x)| \leq \|f\|_\infty$  a.e. on  $E$ .

(b)  $\|f\|_\infty = \sup \{M : m(\{x \in E : |f(x)| \geq M\}) \neq 0\}$ .

*Proof.* (a) Let  $\eta = \|f\|_\infty$ . Note that

$$\{x \in E : |f(x)| \geq \eta\} = \bigcup_{n=1}^{\infty} \left\{ x \in E : |f(x)| > \eta + \frac{1}{n} \right\}.$$

Since the union of a countable collection of sets of measure zero has measure zero, we have

$$m(\{x \in E : |f(x)| \geq \|f\|_\infty\}) = 0$$

$$\Rightarrow |f(x)| \leq \|f\|_\infty \text{ a.e. on } E.$$

(b) It is obvious from the definition of  $\|f\|_\infty$ . ■

**2.4 Theorem.** *Let  $E$  be a measurable set with  $m(E) < \infty$ . Then  $L^\infty(E) \subset L^p(E)$  for each  $p$  with  $1 \leq p < \infty$ . Furthermore, if  $f \in L^\infty(E)$ , then*

$$\|f\|_\infty = \lim_{p \rightarrow \infty} \|f\|_p.$$

*Proof.* Let  $f \in L^\infty(E)$  and  $\eta = \|f\|_\infty$ . Then

$$|f(x)|^p \leq \eta^p \text{ a.e. on } E$$

$$\Rightarrow \int_E |f|^p \leq \eta^p \cdot m(E).$$

Therefore,  $f \in L^p(E)$  and consequently  $L^\infty(E) \subset L^p(E)$ . Furthermore, we note that

$$\|f\|_p \leq \eta [m(E)]^{1/p}.$$

Since  $[m(E)]^{1/p} \rightarrow 1$  as  $p \rightarrow \infty$ , we obtain

$$\limsup_{p \rightarrow \infty} \|f\|_p \leq \eta.$$

On the other hand, let  $|f(x)| \geq \alpha$  on a set  $F$  with  $m(F) > 0$ . Then

$$\|f\|_p \geq \alpha [m(F)]^{1/p}$$

$$\Rightarrow \liminf_{p \rightarrow \infty} \|f\|_p \geq \alpha.$$

This verifies that (cf. Lemma 2.3)

$$\sup \{ \alpha : m(\{x \in E : |f(x)| \geq \alpha\}) \neq 0 \} \leq \liminf_{p \rightarrow \infty} \|f\|_p.$$

Hence, the result follows from

$$\eta \leq \liminf_{p \rightarrow \infty} \|f\|_p \leq \limsup_{p \rightarrow \infty} \|f\|_p \leq \eta. \blacksquare$$

**2.5 Corollary.**  $L^\infty(E) \subset \bigcap_{p \geq 1} L^p(E)$ , and the norm on  $L^\infty(E)$  is equal to the limit of  $\|f\|_p$  as  $p \rightarrow \infty$  provided  $m(E) < \infty$ .

*Remark.* The result in Theorem 2.4 need not be true if  $m(E) = \infty$ .

**2.6 Example.** Consider the function  $f: \mathbb{R} \rightarrow \mathbb{R}$  given by  $f(x) = C$  ( $C \neq 0$ ) for all  $x$ . Clearly  $f \in L^\infty(\mathbb{R})$  while  $f \notin L^p(\mathbb{R})$  for any  $p$  with  $0 < p < \infty$ .

Most of the results for the spaces  $L^p(E)$  we prove in the remaining sections hold good for any measurable set  $E$  with  $m(E) < \infty$ . How

ever, for the sake of simplicity, we take  $E = [a, b]$ . We shall denote by  $L^p$  the space  $L^p[a, b]$  unless specified otherwise.

### 3 THE HÖLDER AND MINKOWSKI INEQUALITIES

In the study of  $L^p$  spaces, an essential role is played by another space  $L^q$  whose exponent  $q$  is associated with  $p$  by the relation  $\frac{1}{p} + \frac{1}{q} = 1$ , where  $p$  and  $q$  are nonnegative extended real numbers. Two such numbers  $p$  and  $q$  are termed as (mutually) conjugate numbers. We adhere to the convention  $1/\infty = 0$  so that 1 and  $\infty$  are conjugate exponents (numbers). It is clear that  $q = p(p-1)^{-1} > 1$ . If  $p = 2$ , then  $q = 2$ . But if  $p \neq 2$ , then  $q \neq 2$ .

Before establishing that  $\|\cdot\|_p$  defines a norm on  $L^p$ , we shall prove some important inequalities which are immensely useful in doing so. In fact, Riesz's main tools in the study of  $L^p$  spaces were the inequalities we shall obtain in Theorems 3.2 and 3.3. To obtain these inequalities, we will require the following inequality which is a generalization of the inequality between arithmetic and geometric means.

**3.1 Lemma.** *Let  $0 < \lambda < 1$ . Then*

$$\alpha^\lambda \beta^{1-\lambda} \leq \lambda \alpha + (1-\lambda)\beta$$

*holds good for every pair of nonnegative real numbers  $\alpha$  and  $\beta$  with equality only if  $\alpha = \beta$ .*

*Proof.* The inequality is trivial if either  $\alpha = 0$  or  $\beta = 0$ . Hence, assume that  $\alpha > 0$  and  $\beta > 0$ . Consider the function  $\varphi$  defined for a nonnegative real number  $t$  by

$$\varphi(t) = (1-\lambda) + \lambda t - t^\lambda.$$

Then,  $\varphi'(t) = \lambda(1 - t^{\lambda-1})$ , and so  $t = 1$  is the only possible point for the extrema of  $\varphi$ . It is verified that  $\varphi$  attains its maximum at  $t = 1$ . Thus

$$\varphi(t) \geq \varphi(1) = 0$$

$$\Rightarrow 1 - \lambda + \lambda t \geq t^\lambda.$$

Setting  $t = \alpha/\beta$ , the inequality follows. The equality holds good only for  $t = 1$  which is obtained only if  $\alpha = \beta$ . ■

**3.2 Theorem (Riesz-Hölder Inequality).** *Let  $p$  and  $q$  be nonnegative extended real numbers such that  $\frac{1}{p} + \frac{1}{q} = 1$ . If  $f \in L^p$  and  $g \in L^q$ , then  $f \cdot g \in L^1$  and*

$$\int |fg| \leq \|f\|_p \|g\|_q.$$

Equality holds if and only if, for some nonzero constants  $A$  and  $B$ , we have  $A |f|^p = B |g|^q$  a.e.

*Proof.* When  $p=1$ , then  $q=\infty$ , and the inequality is available trivially in this case. Indeed, if  $\|g\|_\infty = M$ , then  $|g| \leq M$  a.e., and so

$$|fg| \leq M |f| \text{ a.e.}$$

Thus  $fg \in L^1$ , and by integrating, we get

$$\int |fg| \leq M \int |f| \leq \|f\|_1 \|g\|_\infty.$$

Now, assume that  $1 < p < \infty$  and consequently  $1 < q < \infty$ . The inequality is trivial if either  $f=0$  a.e. or  $g=0$  a.e. So assume  $f \neq 0$  a.e. and  $g \neq 0$  a.e. This gives that  $\|f\|_p > 0$  and  $\|g\|_q > 0$ . Now, applying Lemma 3.1 with

$$\left\{ \begin{array}{l} \lambda = \frac{1}{p} \\ \alpha = \left( \frac{|f(t)|}{\|f\|_p} \right)^p \\ \beta = \left( \frac{|g(t)|}{\|g\|_q} \right)^q, \end{array} \right.$$

we obtain

$$\frac{|f(t)g(t)|}{\|f\|_p \|g\|_q} \leq \frac{1}{p} \frac{|f(t)|^p}{(\|f\|_p)^p} + \frac{1}{q} \frac{|g(t)|^q}{(\|g\|_q)^q} \tag{2}$$

This gives that  $fg \in L^1$ , and by integrating, we find that

$$\frac{\int |fg|}{\|f\|_p \|g\|_q} \leq \frac{1}{p} + \frac{1}{q} = 1.$$

Hence

$$\int |fg| \leq \|f\|_p \|g\|_q. \tag{3}$$

Equality in (2) would occur if  $\alpha = \beta$ , and, consequently, in (3) if  $\alpha = \beta$  holds a.e. In other words, if

$$\|g\|_q^q |f(t)|^p = \|f\|_p^p |g(t)|^q. \blacksquare$$

*Note.* The inequality (3) is homogeneous, i.e., it holds for  $af$  and  $bg$  with  $a, b \in \mathbb{R}$  whenever it is so for  $f$  and  $g$ .

The Riesz-Hölder inequality implies that if  $f \in L^p$  and  $g \in L^q$ ,  $(1/p) + (1/q) = 1$  then  $fg \in L^1$ . It is not true in general that the product of two integrable functions is also an integrable function. In fact, we have the following.

*Remark.* The condition that  $p$  and  $q$  are mutually conjugate exponents cannot be dropped in Theorem 3.2, see Problem 3. For the converse of Theorem 3.2, see Problem 14.

**3.3 Theorem (Riesz-Minkowski Inequality).** *Let  $1 \leq p \leq \infty$ . Then for every pair  $f, g \in L^p$ , the following inequality holds:*

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p.$$

*Proof.* The case for  $p = 1$  is straightforward. If  $p = \infty$ , we note that

$$\begin{cases} |f| \leq \|f\|_\infty \text{ a.e.} \\ |g| \leq \|g\|_\infty \text{ a.e.} \end{cases}$$

$\Rightarrow$

$$|f+g| \leq \|f\|_\infty + \|g\|_\infty \text{ a.e.},$$

and hence the result follows in this case also. Thus, we now assume that  $1 < p < \infty$ .

Since  $L^p$  is a linear space,  $f+g \in L^p$ . Also, we have

$$\int |f+g|^p \leq \int |f+g|^{p-1} |f| + \int |f+g|^{p-1} |g|.$$

Let  $1 < q < \infty$  be such that  $\frac{1}{p} + \frac{1}{q} = 1$ . Then, since  $(p-1)q = p$ , observe that

$$\int (|f+g|^{p-1})^q = \int |f+g|^p,$$

and therefore  $|f+g|^{p-1} \in L^q$ . As such, by Theorem 3.2, both

$$|f+g|^{p-1} |f| \quad \text{and} \quad |f+g|^{p-1} |g|$$

belong to  $L^1$ , and the Riesz-Hölder Inequality leads to

$$\int |f+g|^{p-1} |f| \leq \|f\|_p \|(|f+g|^{p-1})\|_q,$$

and

$$\int |f+g|^{p-1} |g| \leq \|g\|_p \|(|f+g|^{p-1})\|_q.$$

But

$$\|(|f+g|^{p-1})\|_q = \left( \int |f+g|^{(p-1)q} \right)^{1/q} = (\|f+g\|_p)^{p/q},$$

since  $(p-1)q=p$ . Hence

$$\int |f+g|^p \leq (\|f\|_p + \|g\|_p)(\|f+g\|_p)^{p/q}.$$

If  $\|f+g\|_p$  is nonzero finite, the result follows by dividing both sides by  $\|f+g\|_p^{p/q}$ . In case  $\|f+g\|_p=0$ , there is nothing to prove while in case  $\|f+g\|_p=\infty$ , we either have  $\|f\|_p=\infty$  or  $\|g\|_p=\infty$  in view of the relation  $|f+g| \leq |f|+|g|$ , and the result is obviously true again. ■

*Remark.* Equality holds in Theorem 3.3 if and only if one of the functions  $f$  and  $g$  is a multiple of the other.

*Note.* For the special case  $p=q=2$ , the inequality in Theorem 3.2 is known as the **Cauchy Schwarz Inequality**. Cauchy (1821) first proved the inequality (Cauchy's inequality) for square summable sequences. Indeed, if  $\{a_n\}$  and  $\{b_n\}$  are sequences of numbers (real or complex) such that  $\sum_{n=1}^{\infty} |a_n|^2 < \infty$  and  $\sum_{n=1}^{\infty} |b_n|^2 < \infty$ , then

$$\sum_{n=1}^{\infty} |a_n b_n| \leq \left( \sum_{n=1}^{\infty} |a_n|^2 \right)^{1/2} \left( \sum_{n=1}^{\infty} |b_n|^2 \right)^{1/2}.$$

This inequality was generalized to integrals by A. Schwarz (1885). However, the same generalization had already been obtained by a Russian mathematician Victor Bunyakovsky (1859) which remained unnoticed by Western mathematicians. O. Hölder (1889) extended Cauchy's inequality for the general values of  $p$  and  $q$  by establishing, for sequences  $\{a_n\}$  and  $\{b_n\}$  with  $\sum_{n=1}^{\infty} |a_n|^p < \infty$  and  $\sum_{n=1}^{\infty} |b_n|^q < \infty$ , that

$$\sum_{n=1}^{\infty} |a_n b_n| \leq \left( \sum_{n=1}^{\infty} |a_n|^p \right)^{1/p} \left( \sum_{n=1}^{\infty} |b_n|^q \right)^{1/q}$$

where  $(1/p)+(1/q)=1$ . The latter inequality is then generalized to the case of integrals by F. Riesz (1910).

Likewise, the inequality in Theorem 3.3 was first proved for finite sums of numbers by a German mathematician Hermann Minkowski (1896) and then generalized to the case of integrals of functions by F. Riesz (1910).

In view of the above observations, it is more appropriate to name the inequalities in Theorems 3.2 and 3.3, the **Riesz-Hölder Inequality** and the **Riesz-Minkowski Inequality** respectively. However, these are

popularly known as the Hölder Inequality and the Minkowski Inequality.

*Remark.* The inequalities in Theorems 3.2 and 3.3, for  $0 < p < 1$ , need not be true. In fact, for  $0 < p < 1$ , we prove the following.

**3.4 Theorem (Riesz-Hölder Inequality for  $0 < p < 1$ ).** *Let  $0 < p < 1$  and  $q$  be the conjugate exponent of  $p$ . If  $f \in L^p$  and  $g \in L^q$ , then*

$$\int |fg| \geq \left( \int |f|^p \right)^{1/p} \left( \int |g|^q \right)^{1/q},$$

*provided*

$$\int |g|^q \neq 0.$$

*Proof.* First note that  $q < 0$  since  $p < 1$  and  $(1/p) + (1/q) = 1$ . Set

$$p = \frac{1}{P} \quad \text{and} \quad -\frac{p}{q} = \frac{1}{Q}.$$

Then  $P > 1$ ,  $Q > 1$  and  $(1/P) + (1/Q) = 1$ .

Further, setting

$$|fg| = F^P \quad \text{and} \quad |g|^q = G^Q$$

so that  $|f|^p = FG$ , we find that  $F \in L^P$  and  $G \in L^Q$ . Thus, the Riesz-Hölder Inequality in Theorem 3.2, is applicable to the functions  $F$  and  $G$  giving

$$\begin{aligned} \int FG &\leq \|F\|_P \|G\|_Q \\ \Rightarrow \int |f|^p &\leq \left( \int |fg| \right)^P \left( \int |g|^q \right)^{-p/q} \\ \Rightarrow \int |fg| &\geq \left( \int |f|^p \right)^{1/p} \left( \int |g|^q \right)^{1/q}. \blacksquare \end{aligned}$$

*Note.* The condition  $\int |g|^q \neq 0$  in Theorem 3.4 is necessary since  $q < 0$ .

**3.5 Theorem (Riesz-Minkowski Inequality for  $0 < p < 1$ ).** *Let  $0 < p < 1$  and  $f, g$  be in  $L^p$  such that  $f \geq 0$  and  $g \geq 0$ . Then*

$$\|f+g\|_p \geq \|f\|_p + \|g\|_p.$$

*Proof.* Observe that

$$(f+g)^p = \int f(f+g)^{p-1} + \int g(f+g)^{p-1}.$$

Now following exactly on the lines of the proof of Theorem 3.3, by using the Riesz-Hölder Inequality for  $0 < p < 1$ , the result is proved. ■

*Remark.* Strict inequality in Theorem 3.5 holds good.

**3.6 Example.** Take  $f = \chi_{[0, 1/2]}$  and  $g = \chi_{[1/2, 1]}$ . Clearly  $f, g \in L^p[0, 1]$  for any  $p$ ,  $0 < p < 1$ . Observe that  $\|f + g\|_p = 1$ , while

$$\|f\|_p + \|g\|_p = 2^{-1/p} + 2^{-1/p} = 2^{1-(1/p)} < 1.$$

#### 4 $L^p$ BANACH SPACES

We are now prepared to show that  $\|\cdot\|_p$  defines a norm on  $L^p$ . In fact, for  $1 \leq p \leq \infty$ , the function  $\|\cdot\|_p : L^p \rightarrow \mathbb{R}$  satisfies the following conditions:

1.  $\|f\|_p \geq 0$
2.  $\|f\|_p = 0$  if and only if  $f = 0$  a.e.
3.  $\|\alpha f\|_p = |\alpha| \|f\|_p$ ,  $\alpha$  is a real number
4.  $\|f + g\|_p \leq \|f\|_p + \|g\|_p$ .

Conditions (1) and (3) are immediate from the definition of  $\|\cdot\|_p$ ; the condition (2) follows from Problem 4 in Chapter V, while condition (4) is available from the Riesz-Minkowski Inequality.

Unfortunately the definition of  $\|\cdot\|$  on  $L^p$  fails to satisfy the norm requirement that  $\|f\|_p = 0 \Rightarrow f = 0$ . As such,  $\|\cdot\|_p$  is not a norm on  $L^p$ . However, to avoid this difficulty, we do not distinguish between **equivalent functions**, i.e. the functions that are equal almost everywhere. In that situation, we regard the space  $L^p$  consisting of equivalence classes of functions; for example, 0 will represent the class of functions each of which is equivalent to zero. Thus  $\|\cdot\|_p$  now defines a norm on  $L^p$ , regarded as the space of equivalence classes, and therefore,  $L^p$  becomes a normed space. If one refers to  $L^p$  as a normed space, it is in reality the space of equivalence classes  $\tilde{f}$  of functions. But this should not pose any problem since, for any  $\tilde{f} \in L^p$ , the norm is given by

$$\|\tilde{f}\|_p = \|g\|_p,$$

where  $g$  is any function in the equivalence class  $\tilde{f}$ , and this definition does not depend on the choice of the function  $g$  in the class  $\tilde{f}$ . In actual practice, the equivalence classes are relegated to the background, and the elements of  $L^p$  are thought of as functions, where any two functions are regarded identical if they are equivalent.

The norm  $\|\cdot\|_p$  on  $L^p$  induces in a natural way a metric  $d$  on it given by

$$d(f, g) = \|f - g\|_p.$$

**4.1 Theorem (Riesz-Fischer Theorem).** *The normed spaces  $L^p$ ,  $1 \leq p \leq \infty$ , are complete.*

*Proof.* To prove the result for the case  $p = \infty$ , let  $\{f_n\}$  be a Cauchy sequence in  $L^\infty$ . Then

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty,$$

except on a set  $A_{n,m} \subset [a, b]$  with  $m(A_{n,m}) = 0$ . If  $A = \bigcup_{n,m} A_{n,m}$ , then  $m(A) = 0$  and

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty,$$

for all  $n$  and  $m$ , and for all  $x \in [a, b] - A$ . Therefore, it follows that  $\{f_n\}$  converges uniformly to a bounded limit  $f$  outside  $A$  and the result is proved by observing the fact that the convergence in  $L^\infty$  is equivalent to uniform convergence outside a set of measure zero. Now, assume  $1 \leq p < \infty$ . It is enough to show that each absolutely summable sequence in  $L^p$  is summable in  $L^p$  to some element in  $L^p$  (cf. Theorem 1.8).

Let  $\{f_n\}$  be a sequence in  $L^p$  with

$$\sum_{n=1}^{\infty} \|f_n\|_p = M < \infty.$$

Define a sequence  $\{g_n\}$  of functions, where

$$g_n(x) = \sum_{k=1}^n |f_k(x)|.$$

Observe, for each  $x$  that  $\{g_n(x)\}$  is an increasing sequence of (extended) real numbers and as such must converge to an extended real number  $g(x)$  (say), i.e.,  $g_n(x) \rightarrow g(x)$ , for each  $x \in [a, b]$ . Since the functions  $g_n$  are measurable, the function  $g$  is so. Also, in view of Riesz-Minkowski Inequality (cf. Theorem 3.3), we note that

$$\begin{aligned} \|g_n\|_p &= \left\| \sum_{k=1}^n |f_k| \right\|_p \\ &\leq \sum_{k=1}^n \|f_k\|_p < M \end{aligned}$$

$$\Rightarrow \int |g_n|^p \leq M^p(b-a).$$

Therefore, since  $g_n \geq 0$ , by Fatou's Lemma, we have

$$\int g^p \leq M^p(b-a).$$

This verifies that  $g^p$  is integrable, and hence  $g(x)$  is finite a.e. on  $[a, b]$ . Thus, we find that, for each  $x$  for which  $g(x)$  is finite, the sequence  $\{f_n(x)\}$  is an absolutely summable sequence of real numbers, and therefore must be summable to a real number (say)  $s(x)$ . Let us set

$$s(x) = 0,$$

for those  $x$  where  $g(x) = \infty$ . Then the function  $s$  so defined is the limit a.e. of the partial sums

$$s_n(x) = \sum_{k=1}^n f_k(x),$$

i.e.,  $s_n(x) \rightarrow s(x)$  a.e. Hence  $s$  is a measurable function. Further

$$\begin{aligned} |s_n(x)| &\leq \sum_{k=1}^n |f_k(x)| \\ &= g_n(x) \\ &\leq g(x), \end{aligned}$$

which implies  $|s(x)| \leq g(x)$ . Therefore,  $s \in L^p$  since  $g \in L^p$ , and

$$|s_n(x) - s(x)|^p \leq 2^p(g(x))^p.$$

But  $2^p g^p$  is an integrable function and  $|s_n(x) - s(x)|^p \rightarrow 0$  a.e. So, by the Lebesgue Dominated Convergence Theorem (cf. V-6.6), we have

$$\int |s_n - s|^p \rightarrow 0$$

$$\Rightarrow \|s_n - s\|_p \rightarrow 0.$$

Hence the sequence  $\{f_n\}$  is summable in  $L^p$  and has the sum  $s$  in  $L^p$ . This proves the theorem completely. ■

*Remark.* It is worthwhile to point out that the space  $C[a, b]$  is a normed space under the norm  $\|\cdot\|_p$  but not a Banach space. However, it can be noted that the completion of  $C[a, b]$  under  $\|\cdot\|_p$  is  $L^p[a, b]$ , for each  $p$  with  $1 \leq p < \infty$  (cf. Theorem 6.4).

Though  $\|\cdot\|_p$  does not define a norm on  $L^p$  for  $0 < p < 1$  since the inequality in Theorem 3.3 is reversed in this case (cf. Theorem 3.5), we still have the following.

**4.2 Theorem.** *If  $0 < p < 1$ , then  $L^p$  is a complete metric space with metric  $\rho$  defined by*

$$\rho(f, g) = \|f - g\|_p^p, \quad \forall f, g \in L^p.$$

**5 CONVERGENCE IN THE MEAN**

We have already dealt with several notions of the convergence of a sequence of real-valued functions; namely, pointwise convergence, convergence almost everywhere, uniform convergence, almost uniform convergence and convergence in measure. However, the concept of the norm permits us to formulate another type of convergence in the spaces  $L^p$ ,  $1 \leq p \leq \infty$ , with the aid of almost the same expression as used in the case of real numbers. More precisely, we have the following.

**5.1 Definition.** A sequence  $\{f_n\}$  of functions in  $L^p$ ,  $1 \leq p \leq \infty$ , is said to converge to  $f \in L^p$  in the norm of  $L^p$  if for each  $\epsilon > 0$ , there exists a positive integer  $N$  such that  $\|f_n - f\|_p \rightarrow 0$ . This type of convergence is usually referred to as **convergence in the mean of order  $p$**  when  $1 \leq p < \infty$  and **nearly uniform convergence** when  $p = \infty$ .

As usual, one may define a  $p$ -mean Cauchy sequences of functions in  $L^p$ .

**5.2 Theorem.** Let  $\{f_n\}$  be a sequence in  $L^p$  which converges in the mean of order  $p$  to  $f$  in  $L^p$ . Then:

- (a) If the sequence  $\{f_n\}$  converges in the mean of order  $p$  to  $g$ , then  $f = g$  a.e.
- (b) The sequence  $\{f_n\}$  is a  $p$ -mean Cauchy sequence.
- (c)  $\lim_{p \rightarrow \infty} \|f_n\|_p = \|f\|_p$ , in particular the sequence  $\{f_n\}$  is bounded with respect to the norm  $\|\cdot\|_p$ .

*Remark.* The converse of (b) is true, since  $L^p$  is a complete normed space, while that of (c) need not be true (cf. Example 2 in 5.3).

In general, convergence in the mean implies nor is implied by the pointwise convergence (or convergence almost everywhere).

**5.3 Examples**

1. For each  $n \in \mathbb{N}$ , consider the function  $f_n: ]0, 1[ \rightarrow \mathbb{R}$  given by

$$f_n(x) = \begin{cases} n & \text{if } 0 < x \leq \frac{1}{n} \\ 0 & \text{if } \frac{1}{n} < x < 1. \end{cases}$$

It can easily be verified that  $\lim_{n \rightarrow \infty} f_n(x) = 0$  for each  $x \in ]0, 1[$  while  $\|f_n\|_p \rightarrow \infty$  as  $n \rightarrow \infty$  for  $p > 1$ .

2. For each  $n \in \mathbf{N}$ , consider the function  $f_n: \mathbf{R} \rightarrow \mathbf{R}$  given by  $f_n = \chi_{[n, n+1]}$ . Note that  $f_n(x) \rightarrow 0$  as  $n \rightarrow \infty$  for each  $x \in \mathbf{R}$ . On the other hand

$$\begin{aligned} f_n &\in L^p(\mathbf{R}), & 1 \leq p &\leq \infty \\ \Rightarrow & \|f_n\|_p = 1, & \forall n \in \mathbf{N} \\ \Rightarrow & \|f_n\|_p \not\rightarrow 0 & \text{as } n \rightarrow \infty \text{ for any } p, 1 \leq p \leq \infty. \end{aligned}$$

3. Consider the sequence  $\{f_n\}$  of functions as defined in IV-12.6. Observe that

$$\|f_n\|_p \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Hence  $\{f_n\}$  converges in the mean of order  $p$  to zero. But  $\{f_n\}$  does not converge for any  $x \in [0, 1]$ .

Towards the relationship between pointwise convergence and convergence in the mean of order  $p$ , we prove that the following results.

**5.4 Theorem.** *Let  $\{f_n\}$  be a sequence in  $L^p$ ,  $1 \leq p < \infty$ , such that  $f_n \rightarrow f$  a.e. and that  $f \in L^p$ . If  $\lim_{n \rightarrow \infty} \|f_n\|_p = \|f\|_p$ , then  $\lim_{n \rightarrow \infty} \|f_n - f\|_p = 0$ .*

*Proof.* We assume, without any loss of generality, that each  $f_n \geq 0$  a.e. so that  $f$  is also so since the result in the general case follows by considering  $f = f^+ - f^-$ .

For any pair of nonnegative real numbers  $a$  and  $b$ , we have

$$|a - b|^p \leq 2^p (|a|^p + |b|^p), \quad 1 \leq p < \infty.$$

Taking  $a = f_n$  and  $b = f$ , we get

$$2^p (|f_n|^p + |f|^p) - |f_n - f|^p \geq 0 \text{ a.e.}$$

Thus, using Fatou's Lemma and the hypothesis, we get

$$\begin{aligned} 2^p \int |f|^p &= \int \lim_{n \rightarrow \infty} [2^{p-1} (|f_n|^p + |f|^p) - |f_n - f|^p] \\ &\leq \liminf_{n \rightarrow \infty} \int [2^{p-1} (|f_n|^p + |f|^p) - |f_n - f|^p] \\ &= 2^{p-1} \lim_{n \rightarrow \infty} \int |f_n|^p + 2^{p-1} \int |f|^p + \liminf_{n \rightarrow \infty} \left( - \int |f_n - f|^p \right) \\ &= 2^p \int |f|^p - \limsup_{n \rightarrow \infty} \int |f_n - f|^p. \end{aligned}$$

Since  $\int |f|^p < \infty$ , it follows that

$$\limsup_{n \rightarrow \infty} \int |f_n - f|^p \leq 0.$$

Therefore

$$\limsup_{n \rightarrow \infty} \int |f_n - f|^p = \liminf_{n \rightarrow \infty} \int |f_n - f|^p = 0,$$

so that

$$\lim_{n \rightarrow \infty} \int |f_n - f|^p = 0.$$

Hence

$$\lim_{n \rightarrow \infty} \|f_n - f\|_p = 0. \blacksquare$$

**5.5 Theorem.** *Let  $\{f_n\}$  be a sequence in  $L^p$ ,  $1 \leq p < \infty$ , with  $\|f_n - f\|_p \rightarrow 0$  as  $n \rightarrow \infty$ . Then, there is a subsequence  $\{f_{n_k}\}$  of  $\{f_n\}$  such that  $f_{n_k} \rightarrow f$  a.e.*

*Proof.* Since the convergence in the mean of order  $p$ ,  $1 \leq p < \infty$ , implies the convergence in measure (see Problem 7), the result follows in view of IV-12.7.

## 6 PROPERTIES OF $L^p$ SPACES

**6.1 Theorem.** *Let  $0 < q < p \leq \infty$ . Then  $L^p \subset L^q$ , and there exists a constant  $K > 0$  such that*

$$\|f\|_q \leq K \|f\|_p, \quad \forall f \in L^p.$$

*Proof.* Clearly, the result holds good for each  $(0 < q < \infty$  when  $p = \infty$ . Thus, assume that  $0 < q < p < \infty$ . Let  $f \in L^p$ . Then  $f^q \in L^{\frac{p}{q}}$ . Set  $\lambda = p/q$ . Clearly  $\lambda > 1$  and choose  $\mu$  such that  $(1/\lambda) + (1/\mu) = 1$ . Then

$$\begin{aligned} \int_a^b |f|^q &= \int_a^b |f|^{q \cdot 1} \\ &\leq \left( \int_a^b |f|^{q\lambda} \right)^{1/\lambda} \left( \int_a^b 1 \right)^{1/\mu} \\ &= \left( \int_a^b |f|^p \right)^{q/p} (b-a)^{1/\mu}. \end{aligned}$$

Thus  $f \in L^q$  which proves that  $L^p \subset L^q$ . Further, if we set  $K^q = b-a$ , then

$$\|f\|_q \leq K \|f\|_p. \blacksquare$$

*Remark.* The result in Theorem 6.1 holds good for any measurable set  $E$  with  $m(E) < \infty$ . However, in case  $m(E) = \infty$ , the result  $L^p(E) \subset L^q(E)$ , for  $0 < q < p$ , need not be true.

**6.2 Example.** Take  $E = ]1, \infty[$  and define a function  $f: E \rightarrow \mathbb{R}$  by

$$f(x) = x^{-1/q}, \quad 1 \leq q < \infty.$$

Clearly  $f \in L^p(E)$  if  $p > q$  while  $f \notin L^q(E)$ .

**6.3 Theorem.** Let  $0 < q < p < \infty$ . If  $f \in L^p \cap L^q$ , then  $f \in L^r$  for all  $q < r < p$ .

*Proof.* For each  $r$ ,  $q < r < p$ , we can find a  $t$ ,  $0 < t < 1$ , such that

$$r = tq + (1-t)p.$$

Now

$$f \in L^p \cap L^q \Rightarrow f \in L^p \text{ and } f \in L^q$$

$$\Rightarrow |f|^{p(1-t)} \in L^{1/(1-t)} \text{ and } |f|^{tq} \in L^{1/t}.$$

Moreover, we note that  $1/t > 1$  and the exponents  $1/t$  and  $1/(1-t)$  are conjugate to each other. Therefore, by Riesz-Hölder inequality (cf. Theorem 3.2), we have

$$|f|^r = |f|^{tq} |f|^{(1-t)p} \in L^1,$$

which verifies that  $f \in L^r$ . ■

The following theorem will be of considerable value as we continue our investigation on the properties of Lebesgue integrable functions.

**6.4 Theorem.** Let  $f \in L^p$ ,  $1 \leq p < \infty$  and  $\epsilon > 0$ . Then there exists:

- (a) A step function  $\psi$  such that  $\|f - \psi\|_p < \epsilon$ .
- (b) A continuous function  $g$  such that  $\|f - g\|_p < \epsilon$ .

**6.5 Corollary.** (a) The space  $C[a, b]$  of all continuous functions  $f: [a, b] \rightarrow \mathbb{R}$  is dense in  $L^p$  for each  $p$  ( $1 \leq p < \infty$ ) and

(b) The family of all step functions defined on  $[a, b]$  is dense in  $L^p$  for each  $p$  ( $1 \leq p < \infty$ ).

*Remark.* If we replace the interval  $[a, b]$  by an infinite interval or more generally by any measurable set  $E$  with  $m(E) = \infty$ , then a function  $f \in L^p$  may not be approximated by a continuous function. Note, for instance, that a constant function on  $\mathbb{R}$  is continuous but not integrable. As such, the space  $C(E)$  of continuous functions on  $E$  need not even be a subset of  $L^1(E)$  or  $L^p(E)$ , while for approximation of  $f$  by step functions we have Theorem 6.6.

**6.6 Theorem.** *Let  $f \in L^p(\mathbb{R})$ ,  $1 \leq p < \infty$  and  $\epsilon > 0$ . Then there exists a step function  $\varphi$  on  $\mathbb{R}$  such that  $\|f - \varphi\|_p < \epsilon$ .*

**6.7 Corollary.** *The set of all step functions on  $\mathbb{R}$  is dense in  $L^p(\mathbb{R})$  for each  $p$ ,  $1 \leq p < \infty$ .*

**6.8 Theorem.** *For  $1 \leq p < \infty$ , the space  $L^p$  is separable.*

*Proof.* Consider the collection  $\mathcal{R}$  of all step functions having discontinuities only at rational points in  $[a, b]$  and assuming only rational values. Clearly  $\mathcal{R}$  is a countable set. But each step function can be approximated by a step function in  $\mathcal{R}$  with respect to the norm  $\|\cdot\|_p$  and also the family of all step functions is dense in  $L^p$  (cf. Theorem 6.4). Thus, the countable family  $\mathcal{R}$  is dense in  $L^p$ . Hence  $L^p$  is a separable space. ■

*Remark.* The situation is different when  $p = \infty$ .

**6.9 Theorem.**  *$L^\infty$  is not a separable space.*

*Proof.* We note that

$$\|\chi_{[a,c]} - \chi_{[a,d]}\|_\infty = 1, \quad c \neq d$$

$$\Rightarrow S_{1/2}(\chi_{[a,c]}) \cap S_{1/2}(\chi_{[a,d]}) = \emptyset, \quad c \neq d$$

where

$$S_{1/2}(\chi_{[a,c]}) = \{f \in L^\infty : \|f - \chi_{[a,c]}\|_\infty < \frac{1}{2}\}.$$

Let  $\mathcal{F}$  be any arbitrary set which is dense in  $L^\infty$ . Then, for each  $c$  with  $a < c < b$ , there is a function  $f_c \in \mathcal{F}$  such that

$$\|\chi_{[a,c]} - f_c\|_\infty < \frac{1}{2},$$

since  $\chi_{[a,c]} \in L^\infty$ . As such  $f_c \neq f_d$  for  $c \neq d$ . Hence  $\mathcal{F}$  must be uncountable. This proves that  $L^\infty$  is not a separable space. ■

**6.10 Corollary.** *No family of continuous functions is dense in  $L^\infty$ .*

However, we have the following.

**6.11 Theorem.** *The family of step functions is dense in  $L^\infty$ .*

## 7 BOUNDED LINEAR FUNCTIONALS ON $L^p$ SPACES

Let  $p$  and  $q$  be two conjugate exponents. If  $g \in L^q$ ,  $1 \leq q \leq \infty$ , it follows from the Riesz-Hölder Inequality (cf. Theorem 3.2) that

$f \cdot g \in L^1$  for each  $f \in L^p$ . As such, for a fixed  $g \in L^q$ , one can define a function  $F_g: L^p \rightarrow \mathbb{R}$  by

$$F_g(f) = \int fg.$$

Clearly,  $F_g$  is a linear functional on the Banach space  $L^p$ . In fact, we now prove that it is bounded also.

**7.1 Theorem.** *Let  $p$  and  $q$  ( $1 \leq p, q \leq \infty$ ) be two conjugate exponents and  $g \in L^q$ . Then the linear functional defined by*

$$F_g(f) = \int fg$$

is a bounded linear functional on  $L^p$  such that  $\|F_g\| = \|g\|_q$ .

*Proof.* First consider the case when  $p = \infty$  and  $q = 1$ . Observe, by the Riesz-Hölder Inequality (cf. Theorem 3.2), that

$$|F_g(f)| \leq \|g\|_1 \|f\|_\infty, \quad \forall f \in L^p.$$

Thus, it follows that  $F_g$  is a bounded linear functional on  $L^p$  and that

$$\|F_g\| \leq \|g\|_1.$$

To prove the reverse inequality, let\*

$$f = \operatorname{sgn} g.$$

Clearly,  $f \in L^\infty$  and satisfies  $\|f\|_\infty = 1$ . Therefore

$$F_g(f) = \int fg = \int |g| = \|g\|_1$$

$$\Rightarrow \|F_g\| = \|g\|_1.$$

Let us now consider the case when  $1 < p < \infty$ . Again, by the Riesz-Hölder Inequality,

$$|F_g(f)| \leq \|g\|_q \|f\|_p; \quad \forall f \in L^p.$$

Therefore,  $F_g$  is a linear functional on  $L^p$  and satisfies  $\|F_g\| \leq \|g\|_q$ . Further, to obtain the reverse inequality, let

$$f = |g|^{q-1} \operatorname{sgn} g.$$

Clearly,  $f$  is a measurable function and  $|f|^p = |g|^{p(q-1)} = |g|^q$ . This verifies that  $f \in L^p$ . Also, since

$$f \cdot g = (|g|^{q-1} \operatorname{sgn} g) \cdot g = |g|^q,$$

\* $\operatorname{sgn} g(x) = \begin{cases} 1 & \text{if } g(x) \geq 0 \\ -1 & \text{if } g(x) < 0. \end{cases}$

we note that

$$\begin{aligned} F_g(f) &= \int fg = \int |g|^q \\ &= \left( \int |g|^q \right)^{1/p} \left( \int |g|^q \right)^{1/q} \\ &= \left( \int |f|^p \right)^{1/p} \left( \int |g|^q \right)^{1/q} \\ &= \|f\|_p \|g\|_q \end{aligned}$$

$$\Rightarrow \|F_g\| \geq \|g\|_q.$$

Hence the proof is complete. ■

We are interested to know if the converse of Theorem 7.1 is true in the sense that every bounded linear functional on  $L^p$  is representable as in Theorem 7.1. For this, we prove the following.

**7.2 Theorem (Riesz Representation Theorem.** *Let  $F$  be a bounded linear functional on  $L^p$ ,  $1 \leq p < \infty$ . Then there is a function  $g$  in  $L^q$  such that*

$$F(f) = \int fg,$$

and that  $\|F\| = \|g\|_q$ .

The proof of Theorem 7.2 needs the following lemma.

**7.3 Lemma.** *Let  $g$  be an integrable function on  $[a, b]$  and  $K$  be a constant such that*

$$\left| \int fg \right| \leq K \|f\|_p,$$

for all bounded measurable functions  $f$ . Then  $g \in L^q$  and  $\|g\|_q \leq K$ .

*Proof.* First we consider the case when  $p=1$  and  $q=\infty$ . Let  $\epsilon > 0$  be given, and let

$$E = \{x \in [a, b] : |g(x)| \geq K + \epsilon\}.$$

Set  $f = (\text{sgn } g)\chi_E$ . Then  $f$  is a bounded measurable function such that  $\|f\|_1 = m(E)$ . Therefore

$$\begin{aligned} Km(E) &= K \|f\|_1 \geq \left| \int fg \right| \\ &= \left| \int g(\text{sgn } g)\chi_E \right| \\ &= \int_E |g| \geq (K + \epsilon)m(E) \end{aligned}$$

$\Rightarrow m(E) = 0$ , since  $\epsilon > 0$  is arbitrary.

Hence  $\|g\|_\infty \leq K$ .

Let us now assume that  $1 < p < \infty$ . Define a sequence  $\{g_n\}$  of bounded measurable functions, where

$$g_n(x) = \begin{cases} g(x) & \text{if } |g(x)| \leq n \\ 0 & \text{if } |g(x)| > n. \end{cases}$$

If we get  $f_n = |g_n|^{q/p} \operatorname{sgn} g_n$ , then each  $f_n$  is a bounded measurable function such that

$$\|f_n\|_p = (\|g_n\|_q)^{q/p} \quad \text{and} \quad |g_n|^q = f_n \cdot g_n = f_n \cdot g.$$

Therefore

$$(\|g_n\|_q)^q = \int f_n g \leq K \|f_n\|_p = K (\|g_n\|_q)^{q/p}$$

$$\Rightarrow (\|g_n\|_q)^{q - q/p} \leq K$$

$$\Rightarrow (\|g_n\|_q \leq K, \text{ since } q - q/p = 1$$

$$\Rightarrow \int |g_n|^q \leq K^q.$$

But  $|g_n|^q \rightarrow |g|^q$  a.e. Thus, by Fatou's Lemma, we have

$$\int |g|^q \leq \liminf_{n \rightarrow \infty} \int |g_n|^q \leq K^q.$$

Hence,  $g \in L^q$  and  $\|g\|_q \leq K$ .

*Proof of Theorem 7.2.* We shall obtain the proof of this theorem in four stages.

*Stage 1.* Suppose  $f = \chi_t$ ,  $t \in [a, b]$ , where  $\chi_t$  denotes the characteristic function of the interval  $[a, t]$ .

Set  $\varphi(t) = F(\chi_t)$ . Clearly,  $\varphi$  defines a real-valued function on  $[a, b]$ . We first show that  $\varphi$  is an absolutely continuous function. Let  $\{x_i, x'_i\}$  be any finite collection of non-overlapping subintervals of  $[a, b]$  such that  $\sum_i |x'_i - x_i| < \delta$ . If we set.

$$f = \sum_i (\chi_{x'_i} - \chi_{x_i}) \operatorname{sgn} \{\varphi(x'_i) - \varphi(x_i)\},$$

then  $(\|f\|_p)^p < \delta,$

and so

$$\begin{aligned} \sum_i |\varphi(x'_i) - \varphi(x_i)| &= F(f) \\ &\leq \|F\| \cdot \|f\|_p \\ &< \|F\| \cdot \delta^{1/p}. \end{aligned}$$

Thus  $\sum_I |\varphi(x_i') - \varphi(x_i)| < \epsilon$ , for any finite collection of intervals of total length less than  $\delta \left( = \frac{\epsilon^p}{\|F\|^p} \right)$  and as such  $\varphi$  is absolutely continuous on  $[a, b]$ .

By Theorem VI-7.2, there is an integrable function  $g$  on  $[a, b]$  such that

$$\varphi(t) = \int_0^t g, \quad \forall t \in [a, b].$$

Therefore

$$F(\chi_t) = \int g \chi_t.$$

*Stage 2.* Suppose  $f$  is a step function. Since every step function on  $[a, b]$  can be expressed as a linear combination of the form  $\sum c_i \chi_{t_i}$  with the exception of a finite number of points and  $F$  is a linear functional, we have

$$F(f) = \int gf.$$

*Stage 3.* Suppose  $f$  is a bounded measurable function on  $[a, b]$ . By Theorem IV-11.1, there is a sequence  $\{\psi_n\}$  of step functions such that  $\psi_n \rightarrow f$  a.e. Since the sequence  $\{ |f - \psi_n|^p \}$  is uniformly bounded and  $f - \psi_n \rightarrow 0$  a.e., the Bounded Convergence Theorem gives  $\|f - \psi_n\|_p \rightarrow 0$  as  $n \rightarrow \infty$ , and therefore

$$\begin{aligned} |F(f) - F(\psi_n)| &= |F(f - \psi_n)| \\ &\leq \|F\| \|f - \psi_n\|_p \\ \Rightarrow F(f) &= \lim_{n \rightarrow \infty} F(\psi_n) \\ &= \lim_{n \rightarrow \infty} \int g \psi_n. \end{aligned}$$

But, since  $|g\psi_n| \leq \eta |g|$ , where  $\eta$  is the uniform bound of  $\{\psi_n\}$ , by Lebesgue Dominated Convergence Theorem, we have

$$\int fg = \lim_{n \rightarrow \infty} \int g \psi_n.$$

Hence  $\int fg = F(f)$ , for each bounded measurable function  $f$ . Furthermore, since  $|F(f)| \leq \|F\| \|f\|_p$ , we have  $g \in L^q$  and  $\|g\|_q \leq \|F\|$ , in view of Lemma 7.3.

**Stage 4.** Finally, suppose  $f \in L^p$  is any arbitrary function. Let  $\epsilon > 0$  be given. Then, by Theorem 6.4, there is a step function  $\psi$  such that  $\|f - \psi\|_p < \epsilon$ . Since  $\psi$  is bounded, we have

$$F(\psi) = \int \psi g.$$

Therefore

$$\begin{aligned} \left| F(f) - \int fg \right| &= \left| F(f) - F(\psi) + \int \psi g - \int fg \right| \\ &\leq \left| F(f - \psi) \right| + \left| \int (\psi - f)g \right| \\ &\leq \|F\| \cdot \|f - \psi\|_p + \|g\|_q \cdot \|f - \psi\|_p \\ &< (\|F\| + \|g\|_q)\epsilon. \end{aligned}$$

Since  $\epsilon > 0$  is arbitrary, letting  $\epsilon \rightarrow 0$ ,

$$F(f) = \int fg.$$

The equality  $\|F\| = \|g\|_q$  follows from Theorem 7.1. ■

### Problems

- For  $p$ ,  $1 \leq p < \infty$ , Prove that
  - $g \in L^p$  and  $|f| \leq |g| \Rightarrow f \in L^p$ .
  - $f, g \in L^p, \Rightarrow fg \in L^{p/2}$ .
- If  $f \in L^p$  and  $g \in L^q$  where  $p, q > 0$ , then show that  $fg \in L^r$ , for a suitable  $r$ . [Hint: Set  $p' = p\lambda$  and  $q' = q\lambda$  where  $1/p + 1/q = \lambda$ . Apply the Riesz-Hölder Inequality for functions  $|f|^{1/\lambda} \in L^{p'}$  and  $|g|^{1/\lambda} \in L^{q'}$ .]
- Let  $f \in L^1$  such that  $f$  is not equivalent to any bounded function. Show that there is a  $g \in L^1$  such that  $fg \notin L^1$ .
- If  $f$  is continuous on  $[a, b]$ , show that

$$\lim_{p \rightarrow \infty} \|f\|_p = \sup_{t \in [a, b]} |f(t)|.$$

- Prove Theorem 4.2.
- Work out the details of the proof of Theorem 5.2.
- Let  $\{f_n\}$  be in  $L^p, 1 \leq p < \infty$ . Show that if  $\lim_{n \rightarrow \infty} \|f_n - f\|_p = 0$  holds in  $L^p$ , then  $f_n \xrightarrow{m} f$ .
- Let  $p$  and  $q$  be conjugate exponents and let  $f_n \rightarrow f$  in  $L^p$  and  $g_n \rightarrow g$  in  $L^q$ . Prove that  $f_n g_n \rightarrow fg$  in  $L^1$ .

9. Let  $\{f_n\}$  be a sequence of functions in  $L^\infty$ . Prove that  $\{f_n\}$  converges to  $f$  in  $L^\infty$  if and only if there is a set  $E$  of measure zero such that  $\{f_n\}$  converges to  $f$  uniformly on  $[a, b] - E$ .
10. Prove Theorems 6.4 and 6.6.
11. Prove Corollary 6.10 [Hint: Apply the method of contradiction and use Theorem 6.9]
12. Let  $f \geq 0$  be such that  $f \in L^p$ ,  $p > 0$ , and let  $f_n = \min(f, n)$ . Show that  $f_n \in L^p$  and  $\lim_{n \rightarrow \infty} \|f - f_n\|_p = 0$ .
13. A sequence  $\{f_n\}$  in  $L^p$  is said to **converge weakly** to  $f$  in  $L^p$  if, for every function  $g \in L^q$  with  $1/p + 1/q = 1$ .

$$\lim_{n \rightarrow \infty} \int g f_n = \int g f.$$

Prove that if  $\{f_n\}$  converges in the mean of order  $p$  to  $f$  in  $L^p$ ,  $1 \leq p \leq \infty$ , it also converges weakly to  $f$  in  $L^p$ .

14. Let  $f$  be a real-valued measurable function on  $[a, b]$ ,  $1 \leq p \leq \infty$  and  $(1/p) + (1/q) = 1$ . Then show that

$$\|f\|_p = \sup \int_a^b f g,$$

where supremum is taken over all real-valued functions  $g$  with

$$\|g\|_q \leq 1 \quad \text{and} \quad \int_a^b f g < \infty.$$

15. Let  $\{f_n\}$  be a sequence of functions in  $L^p$ ,  $1 < p < \infty$ , which converge a.e. to a function  $f$  in  $L^p$  and suppose there is a constant  $K$  such that  $\|f_n\|_p \leq K$  for all  $n$ . Prove that for each function  $g$  in  $L^q$

$$\int f g = \lim_{n \rightarrow \infty} \int f_n g.$$

Discuss if the result is true for  $p = 1$ .

16. Let  $\|f_n - f\|_p \rightarrow 0$  as  $n \rightarrow \infty$ , where  $\{f_n\}$  is a sequence in  $L^p$ , and  $f \in L^p$ , ( $1 \leq p < \infty$ ). Suppose  $\{g_n\}$  is a sequence of measurable functions such that  $|g_n| \leq M$  for some constant  $M$ , for all  $n$  and  $g_n \rightarrow g$  a.e. Show that  $\|g_n f_n - g f\|_p \rightarrow 0$  as  $n \rightarrow \infty$ .
17. A continuous function  $f: \mathbb{R} \rightarrow \mathbb{R}$  is said to have **compact support** if there is a compact set  $K \subset \mathbb{R}$  such that  $f(x) = 0$  whenever  $x \notin K$ . The class of all such functions on  $\mathbb{R}$  is denoted by  $C_c(\mathbb{R})$ . Further, if  $E$  is a measurable set in  $\mathbb{R}$ , define the class  $C_c(E) = \{f \chi_E: f \in C_c(\mathbb{R})\}$ . Prove that

- (a)  $C_c(\mathbf{R}) \subset L^p(\mathbf{R})$  and  $C_c(E) \subset C(E) \cap L^p(E)$ , where  $C(E)$  is the class of functions defined and continuous on  $E$ .
- (b)  $C_c(E) = C(E)$ , if  $E$  is compact and, in particular,  $C_c[a, b] = C[a, b]$ .
- (c)  $C_c(E)$  is dense in  $L^p(E)$  for  $1 \leq p < \infty$ .
18. Let  $C_c^\infty(\mathbf{R})$  be the space of infinitely differentiable functions on  $\mathbf{R}$  with compact support and let  $C_c^\infty(E) = \{f\chi_E : f \in C_c^\infty(\mathbf{R})\}$ ,  $E$  being a measurable set in  $\mathbf{R}$ . Prove that  $C_c^\infty(E)$  is dense in  $L^p(E)$  if  $1 \leq p < \infty$ .
19. Prove Theorem 6.11.

# Appendix I

## Existence of Riemann Integral

We give here a criterion for a bounded real-valued function defined on the interval  $[a, b]$  to be Riemann integrable therein. Before doing so, we first introduce the rudiments of the Riemann theory of integration.

Let  $f$  be a real-valued function defined on a bounded interval  $I$ . We define the **oscillation** of  $f$  over  $I$ , denoted by  $w(I)$ , as

$$w(I) = \text{lub}_{x \in I} f(x) - \text{glb}_{x \in I} f(x).$$

Define the oscillation of  $f$  at  $c \in I$ , denoted by  $w(c)$ , as

$$w(c) = \text{glb}_J w(J),$$

where  $J$  ranges over all bounded open subintervals of  $I$  containing  $c$ . Clearly  $w(J) \geq 0$  for any  $J$  and as such  $w(c) \geq 0$  for all  $c \in I$ .

**Proposition 1.** A bounded function  $f: [a, b] \rightarrow \mathbf{R}$  is continuous at a point  $c$  if and only if  $w(c) = 0$ .

*Proof.* It is easy to verify that the oscillation of  $f$  over  $J$  is the supremum of the set of numbers  $|f(x_1) - f(x_2)|$ , where  $x_1$  and  $x_2$  are any two points of  $J$ . Let  $f$  be continuous at  $c$ . Then given an  $\epsilon > 0$ , there exists a bounded open interval  $J$  containing  $c$  such that

$$|f(x) - f(c)| < \epsilon/2, \quad \forall x \in J.$$

In particular, if  $x_1$  and  $x_2$  are any two points of  $J$ ,

$$|f(x_1) - f(x_2)| \leq |f(x_1) - f(a)| + |f(x_2) - f(a)|$$

$$< \epsilon$$

$$\Rightarrow w(J) < \epsilon.$$

Hence  $w(c) = 0$ .

On the other hand, let  $w(c)=0$ . Given an  $\epsilon > 0$ , there exists a bounded open interval  $J$  containing  $c$  such that  $w(J) < \epsilon$ . This, further, implies

$$|f(x_1) - f(x_2)| < \epsilon, \quad \forall x_1, x_2 \in J.$$

In particular,

$$|f(x) - f(c)| < \epsilon, \quad \forall x \in J$$

Hence  $f$  is continuous at  $c$ . ■

**Proposition 2.** Let  $f$  be a bounded function defined on  $[a, b]$ . Then  $f$  is Riemann integrable if and only if, for each  $\epsilon > 0$  there exists a partition  $P$  of  $[a, b]$  such that

$$S(P) - s(P) < \epsilon.$$

*Proof.* The proof is left to the reader. ■

**Proposition 3.** If  $w(x) < \lambda$  for each  $x \in [a, b]$ , then there exists a partition  $P$  of  $[a, b]$  such that

$$S(P) - s(P) < \lambda(b - a).$$

*Proof.* For each  $x \in [a, b]$ , there is a bounded open interval  $J_x$  containing  $x$  such that  $w(J_x) < \lambda$ . Since  $[a, b]$  is closed and bounded, by the Heine-Borel Theorem, a finite number of intervals  $J_x$  will cover  $[a, b]$ . Let  $P$  be the set of the end points of these  $J_x$ . Let  $I_1, I_2, \dots, I_n$  be the component intervals of  $P$ . Then  $w(I_i) < \lambda$ , for all  $i = 1, 2, \dots, n$ . Hence

$$\begin{aligned} S(P) - s(P) &= \sum_{i=1}^n (M_i - m_i)(x_i - x_{i-1}) \\ &= \sum_{i=1}^n w(I_i)(x_i - x_{i-1}) \\ &< \lambda(b - a), \quad I_i = ]x_{i-1}, x_i[. \quad \blacksquare \end{aligned}$$

Now the main result is as follows.

**Theorem.** A bounded function defined on  $[a, b]$  is Riemann integrable if and only if it is continuous almost everywhere.

*Proof.* Let  $f$  be Riemann integrable over  $[a, b]$  and

$$E = \{x \in [a, b]: w(x) = 0\}.$$

We shall show that  $m(E) = 0$ . Observe that

$$E = \bigcup_{m=1}^{\infty} E_m,$$

where

$$E_m = \left\{ x \in [a, b] : w(x) \geq \frac{1}{m} \right\}.$$

It is enough to show that  $m(E_m) = 0$  for each  $m$ .

Fix  $m$ . Since  $f$  is Riemann integrable over  $[a, b]$ , there exists in view of Proposition 2 a partition  $P$  of  $[a, b]$  such that

$$S(P) - s(P) < \frac{\epsilon}{2m}. \quad (1)$$

Thus

$$\begin{aligned} \sum_{i=1}^n w(I_i) (x_{i+1} - x_i) &= \sum_{i=1}^n (M_i - m_i) (x_{i+1} - x_i) \\ &= S(P) - s(P) \\ &< \frac{\epsilon}{2m}. \end{aligned} \quad (2)$$

Now,  $E_m = E'_m \cup E''_m$ , where  $E'_m$  is the set of points of  $E_m$  that are points of the partition  $P$  and  $E''_m = E - E'_m$ . Obviously,  $E'_m \subset \bigcup_{k=1}^p J_k$ , where  $J_k$ 's are open intervals such that the sum of their lengths does not exceed  $\epsilon/2$ . Further, if  $x \in E''_m$ , then  $x$  belongs to some  $I_i$ . Hence

$$w(I_i) \geq w(x) \geq \frac{1}{m}.$$

If we denote by  $I_1, I_2, \dots, I_r$  those of the intervals  $I_i$  of  $P$  each of which contains a point of  $E''_m$ , then, in view of (2), the sum of lengths of  $I_j$  ( $j = 1, 2, \dots, r$ ) does not exceed  $\epsilon/2$ . Since  $E'_m \subset \bigcup_{k=1}^p J_k$  and

$E''_m \subset \bigcup_{k=1}^r I_k$ , it follows that  $E_m$  is of measure zero.

To prove the converse, suppose that  $f$  is continuous a.e. in  $[a, b]$ . Since  $m(E_m) = 0$  for each  $m$ ,  $E_m \subset \bigcup_{i=1}^{\infty} I_i$ , where each  $I_i$  is an open interval of  $[a, b]$  and\*

$$\sum_{i=1}^{\infty} l(I_i) < \frac{\epsilon}{2w([a, b])}.$$

It is clear that  $E_m$  is a closed set in  $\mathbb{R}$  and hence in  $[a, b]$ . Therefore, by the Heine-Borel Theorem, there is a finite number of intervals  $I_1, I_2, \dots, I_r$ , which covers  $E_m$ . Observe that

$$[a, b] - \bigcup_{k=1}^r I_k$$

\*We may assume that  $w([a, b]) > 0$ .

can be expressed as a union of closed intervals  $J_1, J_2, \dots, J_p$ . Thus

$$[a, b] = \left( \bigcup_{k=1}^r I_{i_k} \right) \cup \left( \sum_{i=1}^p J_i \right).$$

Since no interval  $J_i (i = 1, 2, \dots, p)$  contains a point of  $E_m$ , there exists, by Proposition 3, a partition  $P_i$  of  $J_i$  such that

$$S(P_i) - s(P_i) < \frac{l(J_i)}{m}.$$

Set  $P = P_1 \cup P_2 \cup \dots \cup P_p$ . Clearly  $P$  is a partition of  $[a, b]$ . Hence

$$\begin{aligned} S(P) - s(P) &= \sum_{i=1}^p \{S(P_i) - s(P_i)\} + \sum_{k=1}^r (M_{i_k} - m_{i_k})(x_{i_k} - x_{i_{k-1}}) \\ &< \frac{1}{m} \sum_{i=1}^p l(J_i) + \sum_{k=1}^r w(I_{i_k})(x_{i_k} - x_{i_{k-1}}) \\ &< \frac{b-a}{m} + w([a, b]) \sum_{k=1}^r (x_{i_k} - x_{i_{k-1}}) \\ &< \frac{b-a}{m} + w([a, b]) \cdot \frac{\epsilon}{2w([a, b])} \\ &< \epsilon, \end{aligned}$$

by a suitable choice of  $m$  for each  $\epsilon > 0$ . Hence, by Proposition 2,  $f$  is Riemann integrable over  $[a, b]$ . ■

## Appendix II

### Nowhere Differentiable Continuous Functions

We give here functions which are continuous on  $\mathbf{R}$  but nowhere differentiable.

**Example 1 (Van der Waerden).** Let  $F: \mathbf{R} \rightarrow \mathbf{R}$  be a function defined by

$$F(x) = \sum_{k=0}^{\infty} \frac{f_k(x)}{10^k}, \quad (1)$$

where

$f_0(x)$  = distance from  $x$  to the nearest integer

$f_k(x) = f_0(10^k x)$ ,  $k = 0, 1, 2, \dots$

We observe that:

- (a) Each  $f_k$  is a continuous function on  $\mathbf{R}$ .
- (b)  $F$  is continuous on  $\mathbf{R}$ .

Since

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{f_k(x)}{10^k} &\leq \sum_{k=0}^{\infty} \frac{1}{2 \cdot 10^k} \quad (x \in \mathbf{R}) \\ &= \frac{5}{9} < \infty, \end{aligned}$$

by the Weierstrass  $M$ -test, the series (1) converges uniformly on  $\mathbf{R}$ . Therefore, it follows that  $F$  is continuous on  $\mathbf{R}$  because it is the uniform limit of continuous functions.

(c)  $F$  is nowhere differentiable.

Let  $a \in \mathbf{R}$  be any point. It is enough to show that  $F'(a)$  does not exist.

Suppose  $a = a_0 \cdot a_1 a_2 a_3 \dots a_n \dots$ . For  $n \in \mathbb{N}$ , let

$$x_n = a_0 \cdot a_1 a_2 a_3 \dots a_{n-1} b_n a_{n+1} \dots,$$

where  $b_n = a_{n+1}$  if  $a_n \neq 4$  or  $9$  and  $b_n = a_{n-1}$  if  $a_n = 4$  or  $9$ .  
Therefore

$$x_n - a = \pm 10^{-n}$$

$$\Rightarrow \lim_{n \rightarrow \infty} x_n = a.$$

Thus, for any  $n \in \mathbb{N}$ , we note that

$$\begin{aligned} f_k(x_n) - f_k(a) &= \pm 10^{k-n} & (k=0, 1, 2, \dots, (n-1)) \\ f_k(x_n) - f_k(a) &= 0 & (k \geq n) \end{aligned}$$

Hence

$$\begin{aligned} \frac{F(x_n) - F(a)}{x_n - a} &= \sum_{k=0}^{\infty} \frac{f_k(x_n) - f_k(a)}{10^k(x_n - a)} \\ &= \sum_{k=0}^{n-1} \frac{\pm 10^{k-n}}{10^k(\pm 10^{-n})} \\ &= \sum_{k=0}^{n-1} \pm 1. \end{aligned}$$

This verifies that

$$\lim_{n \rightarrow \infty} \left[ \frac{F(x_n) - F(a)}{x_n - a} \right]$$

does not exist.

**Example 2 (Weierstrass).** Let  $F : \mathbb{R} \rightarrow \mathbb{R}$  be a function defined by

$$F(x) = \sum_{n=0}^{\infty} \frac{\cos 3^n x}{2^n}, \quad (x \in \mathbb{R}). \quad (2)$$

Since the series (2) is uniformly convergent on  $\mathbb{R}$  and each term in the series is a continuous function on  $\mathbb{R}$ , it follows that the function  $F$  is continuous on  $\mathbb{R}$ . On the other hand, observe that the series

$$-\sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n \sin 3^n x,$$

obtained by differentiating the series (2) term by term, is divergent when  $x$  is not a multiple of  $\pi$ . This indicates that  $F$  is nowhere differentiable.

*Note.* One can construct several examples on the lines of Examples 1 and 2 above. For instance, in Example 1, one may replace the function  $F$  by the function defined by

$$F(x) = \sum_{k=0}^{\infty} \frac{f_k(x)}{u^k}, \quad (x \in \mathbf{R})$$

where

$$f_0(x) = \text{distance from } x \text{ to the nearest integer}$$

$$f_k(x) = f_0(u^k x), \quad k = 0, 1, 2, \dots;$$

while in Example 2, replace the function  $F$  by the function defined by

$$F(x) = \sum_{n=0}^{\infty} b^n \cos(a^n \pi x),$$

where  $a$  is an odd natural number and  $b$  a real number such that

$$0 < b < 1 \quad \text{and} \quad ab > 1 + \frac{3\pi}{2}.$$

At this stage, we may point out that most of the continuous functions are nowhere differentiable. This is justified by the Baire category theorem: "The set  $\mathbf{R}$  is not of the first category."\* (The proof of this theorem is beyond the scope of this book.)

\*A set which is the union of a denumerable number of nowhere dense sets is called a set of the first category.

## Appendix III

# The Development of the Notion of the Integral\*

Gentlemen:

Foregoing technical developments, we are going to examine as a whole the successive modifications and enrichments of the notion of the integral and the appearance of other concepts used in recent research concerning functions of real variables.

Before Cauchy, there was no definition of the integral in the actual sense of the word "definition". One was limited to saying which areas it was necessary to add or subtract to obtain the integral.

For Cauchy a definition was necessary, because with him appeared the concern for rigour which is characteristic of modern mathematics. Cauchy defined continuous functions and the integrals of these functions in nearly the same way as we do now. To arrive at the integral of  $f(x)$ , it sufficed for him to form the sums (see Figure III.1).

$$S = \sum f(\xi_i) (x_{i+1} - x_i) \quad (1)$$

which surveyors and mathematicians have used for centuries for approximating areas, and to deduce from this the integral

$$\int_a^b f(x) dx$$

by passage to the limit.

Although this passage was obviously legitimate to those starting with a notion of area, Cauchy had to prove that the sum  $S$  actually

\*This is the English translation of Henri Lebesgue's lecture delivered in a conference La Societe Mathematique at Copenhagen on May 8, 1926 and which is included as an appendix in the book Soo Bong Chae, *Lebesgue Integration*, Marcel Dekker, New York, No. 58 (1980). We wish to acknowledge our thanks to Marcel Dekker, for permitting us to include the same in this book.

tended toward a limit under the conditions which he considered. An analogous necessity is imposed each time one replaces an experimental notion with a purely logical definition. It should be added that the interest of the defined object is no longer evident; it can only result from the study of the properties of this object.

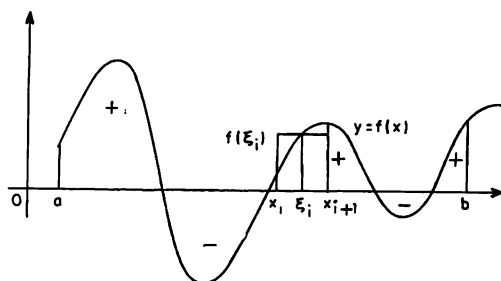


Fig. III.1.

This is the price of logical progress. What Cauchy did is considerable enough to have a philosophical meaning. It is often said that Descartes reduced geometry to algebra. I would say more readily that, by employing coordinates, he reduced all geometries to that of the straight line and that this geometry, in giving us the notions of continuity and irrational number, has permitted algebra to attain its actual scope.

In order that the reduction of all geometries to the geometry of the straight line be achieved, it was necessary to eliminate a certain number of notions related to geometries of higher dimensions such as length of a curve, area of a surface, and volume of a body. Precisely here lies the progress which Cauchy realized. After him, it sufficed that arithmeticians construct the linear continuum with the aid of natural numbers to accomplish the arithmetization of the science.

And now, should we limit ourselves to doing analysis? No. Indeed, all that we will do can be translated into arithmetical language, but if one were to refuse to have direct, geometric, intuitive insights, if one were reduced to pure logic which does not permit a choice among every thing that is exact, one would hardly think of many questions, and certain notions, the majority of those notions which we are going to examine today, for example, would escape us completely.

For a long time, certain discontinuous functions were integrated; Cauchy's definition still applied to these integrals, but it was natural to investigate, as Riemann did, the exact scope of this definition.

If  $f_i$  and  $\bar{f}_i$  designate the lower and upper bounds of  $f(x)$  on  $(x_i, x_{i+1})$ , then  $S$  lies between

$$\underline{S} = \sum f_i(x_{i+1} - x_i) \quad \text{and} \quad \bar{S} = \sum \bar{f}_i(x_{i+1} - x_i)$$

Riemann showed that it suffices that

$$\bar{S} - \underline{S} = \sum (\bar{f}_i - f_i) (x_{i+1} - x_i)$$

tends toward zero for a particular sequence of partitions of  $(a, b)$  into smaller and smaller intervals  $(x_i, x_{i+1})$  in order for Cauchy's definition to apply. Darboux added that the usual passages to the limit by  $\underline{S}$  and  $\bar{S}$  always give two definite numbers

$$\int_a^b f(x) dx \quad \bar{\int}_a^b f(x) dx$$

These numbers are in general different and are equal only when the Cauchy-Riemann integral exists.

From the logical point of view, these are very natural definitions, are they not? Nevertheless, one could say that they are use less in the practical sense. Riemann's definition, in particular, has the disadvantage that it applies only rarely and, in a sense, by chance.

Indeed, it is evident that partitioning of  $(a, b)$  into smaller and smaller intervals  $(x_i, x_{i+1})$  makes the differences  $\bar{f}_i - f_i$  smaller and smaller if  $f(x)$  is continuous, and by virtue of this continuous process it is clear that this partitioning causes  $\bar{S} - S$  to tend toward zero if there are only a few points of discontinuity. However, there is no reason to hope that the case will be the same for a function discontinuous everywhere. So, in effect, taking smaller and smaller intervals  $(x_i, x_{i+1})$ , that is to say, values of  $f(x)$  related to values of  $x$  which come closer and closer together, in no way guarantees that one takes values of  $f(x)$  whose differences become less and less.

Let us proceed according to the goal to be attained: to gather or group values of  $f(x)$  which differ by little. It is clear, then, that we must partition not  $(a, b)$ , but rather the interval  $(f, \bar{f})$  bounded by the lower and upper bounds of  $f(x)$  on  $(a, b)$ . We do this with the aid of numbers  $y_i$  differing among themselves by less than  $\epsilon$ ; we are led, for example, to consider values of  $f(x)$  defined by

$$y_i \leq f(x) \leq y_{i+1}$$

The corresponding values of  $x$  form a set  $E_i$ ; in the case of Fig. III.2, this set  $E_i$  is made up of four intervals. With a certain continuous function  $f(x)$ , it might be formed by an infinite number of intervals.

With an arbitrary function, it might be very complicated. But, no matter—it is this set  $E_i$  which plays the role analogous to that of the interval  $(x_i, x_{i+1})$  in the definition of the integral of continuous functions, since it makes known to us the values of  $x$  which give to  $f(x)$  values differing very little.

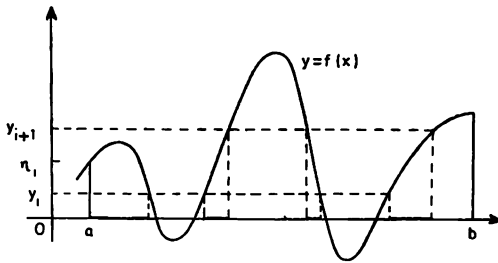


Fig. III.2.

If  $\eta_i$  is any number chosen between  $y_i$  and  $y_{i+1}$ ,

$$y_i \leq \eta_i \leq y_{i+1}$$

the values of  $f(x)$  for the points of  $E_i$  differ from  $\eta_i$  by less than  $\epsilon$ . The number  $\eta_i$  will play the role which was assumed by  $f(\xi_i)$  in (1); as for the role of the length or measure  $x_{i+1} - x_i$  of the interval  $(x_i, x_{i+1})$ , this will be played by a measure  $m(E_i)$ , which we will assign to the set  $E_i$  in a moment. We form in this manner the sum

$$S = \sum \eta_i m(E_i) \quad (2)$$

But first let us look at what we have just done and, in order to understand it better, repeat it in other terms.

The geometers of the seventeenth century considered the integral of  $f(x)$ —the word “integral” had not yet been invented, but that is hardly important—as the sum of an infinity of indivisibles,<sup>1</sup> each of which is an ordinate, positive or negative, of  $f(x)$ . Very well! We have simply grouped the indivisibles of comparable size; we have, as one says in algebra, made the collection or reduction of similar terms. It may again be said that, with Riemann’s procedure, one

<sup>1</sup>In the context of areas, *indivisibles* are “infinitely narrow” rectangles of “infinitesimal” area. Leibniz used the symbol  $dx$  to denote the “width” of an indivisible, so that the “area” of an indivisible of length  $y$  was given by the product  $y dx$ . He then introduced the symbol  $\int y dx$  for the “sum” or “integral” of the areas of the indivisibles which gives the area of a given region.

attempted to sum the indivisibles by taking them in the order in which they were furnished by the variation of  $x$ . One operated as did a merchant without a method who counted coins and bills randomly in the order in which they fell into his hand, while we operate like the methodical merchant who says

I have  $m(E_1)$  pennies worth  $1 \cdot m(E_1)$ .

I have  $m(E_2)$  nickels worth  $5 \cdot m(E_2)$ .

I have  $m(E_3)$  dimes worth  $10 \cdot m(E_3)$ .

etc., and thus I have altogether

$$S = 1 \cdot m(E_1) + 5 \cdot m(E_2) + 10 \cdot m(E_3) + \dots$$

The two procedures will certainly lead the merchant to the same result because, as rich as he might be, he has only a finite number of bills to count; but for us who have to sum an infinity of indivisibles, the difference between the two methods of adding is capital.

Let us now occupy ourselves with the definition of the number  $m(E_i)$  attached to  $E_i$ . The analogy between this measure and a length, or the same with a number of bills, leads us naturally to say that, in the example in Fig. III.2,  $m(E_i)$  will be the sum of the lengths of the four intervals constituting  $E_i$ , and that, in an example in which  $E_i$  is formed from an infinity of intervals,  $m(E_i)$  will be the sum of the lengths of all these intervals. In the general case, it leads us to proceed as follows: We enclose  $E_i$  in a finite or countably infinite number of intervals, and let  $l_1, l_2, \dots$  be the lengths of these intervals. We certainly want

$$m(E_i) \leq l_1 + l_2 + \dots$$

If we look for the greatest lower bound of the second member for all possible systems of intervals which can serve to cover  $E_i$ , this will be an upper bound for  $m(E_i)$ . For this reason we denote it by  $\overline{m(E_i)}$  and we have

$$m(E_i) \leq \overline{m(E_i)} \quad (3)$$

If  $C$  is the set of points of  $(a, b)$  not included in  $E_i$ , we have similarly

$$m(C) \leq m(C)$$

Now, we obviously wish to have

$$m(E_i) + m(C) = m((a, b)) = b - a$$

Therefore we must have

$$m(E_i) \geq b - a - \overline{m(C)} \quad (4)$$

The inequalities (3) and (4) give then the upper and lower bounds of  $m(E_i)$ . One can easily see that these two inequalities are never contradictory. When the lower and upper bounds of  $E_i$  are equal,  $m(E_i)$  is defined, and we say then that  $E_i$  is measurable.<sup>2</sup>

A function  $f(x)$  for which the sets  $E_i$  are measurable for all  $y_i$  is called measurable. For such a function, formula (2) defines a sum  $S$ . One can easily prove that, when one varies the choice of  $y_i$  in such a way that  $\epsilon$  tends toward zero, the sum  $S$  approaches a definite limit which is by definition

$$\int_a^b f(x) dx.^3$$

This first extension of the notion of integral led to many others. Let us suppose that it is a question of integrating a function  $f(x, y)$  in two variables. We proceed exactly as before. We assign to it sets  $E_i$  which are now points in the plane and no longer the points on a line. To these sets we must now attribute a plane measure; this measure is deduced from the area of the rectangles

$$\alpha \leq x \leq \beta, \quad \gamma \leq y \leq \delta$$

entirely in the same manner as linear measure is deduced from the length of intervals. With the measure defined, formula (2) will give the sum  $S$ , from which the integral is deduced by passage to the limit.

The definition which we have considered thus extends itself immediately to functions of several variables. Here is another extension which applies equally well whatever the number of variables, but which I state only in the case where it is a question of integrating  $f(x)$  on  $(a, b)$ .

I have said that it is a matter of forming the sum of indivisibles represented by the various ordinates of the points  $x, y=f(x)$ . A moment ago, we grouped these indivisibles according to their size. Let

<sup>2</sup>The method of defining the measure of sets used here is that of C. Jordan (*Cours d'Analyse de l'Ecole Polytechnique*, Vol. 1) but with this modification essential to our aim; that we enclose the set  $E_i$  to be measured in intervals which may be infinite in number, whereas Jordan always used only a finite number of intervals. This use of a countable infinity in place of integers is suggested by the endeavors of Borel who, moreover, himself used this idea in particular for a definition of measure (*Lecons sur la théorie des fonctions*).

<sup>3</sup>*Comptes Rendus Acad. Sci. Paris*, 129, 1909. Definitions equivalent to that of the text were proposed by various authors. The most interesting are due to W.H. Young (*Philos. Trans. Roy. Soc. London*, 204, 1905, *Proc. London Math. Soc.*, 1910). See also, for example, the notes by Borel and by F. Riesz (*Comptes Rendus Acad. Sci. Paris*, 154, 1912),

us now restrict ourselves to grouping them according to their sign; we will have to consider the plane set  $E_p$  of those points, the ordinates of which are positive, and the set  $E_n$  of points with negative ordinates. For the simple case in which  $f(x)$  is continuous, before Cauchy, as I recalled in the beginning, one wrote

$$\int_a^b f(x) dx = \text{area}(E_p) - \text{area}(E_n)$$

This leads us to write

$$\int_a^b f(x) dx = m_s(E_p) - m_s(E_n)$$

$m_s$  designating a plane measure. This new definition is equivalent to the preceding one. It brings us back to the intuitive method before Cauchy, but the definition of measure has given it a solid logical foundation.

We thus know two ways of defining the integral of a function of one or more variables, and that we know without having to consider the more or less complicated form of the domain of integration, because the domain  $D$  intervenes only as follows: The sets  $E_i$  of our first definition and the sets  $E_p, E_n$  of the second are formed by taking values of the function  $f$  only on the points of  $D$ .

Since the choice of the domain of integration  $D$  enters only in the formation of the sets  $E_i$ , or  $E_p$  and  $E_n$ , it is clear that we could just as well agree to form these sets  $E_i, E_n, E_p$  by taking into consideration only the values assumed by  $f$  on the points of a given set  $E$ , and we will have hence defined the integral of  $f$  extended to the set  $E$ .

In order to make precise the scope of this new extension of the notion of integral, let us recall that our definitions require that  $f$  be measurable, that is to say, that the sets  $E_i$  be measurable for the first definition, and that  $E_p$  and  $E_n$  be measurable for the second, and, in view of this,  $E$  must also be measurable. We thus know how to define the integral extended to a measurable set of a measurable and bounded function on this set. I have, in effect, implicitly supposed thus far that we are dealing with bounded functions.

What would have to be changed in the first manner of definition if the function  $f$  to be integrated were not bounded? The interval  $(f, \bar{f})$  would no longer be finite; an infinity of numbers  $y_i$  would be needed to divide it into intervals of length at most equal to  $\epsilon$ , so there would be an infinity of sets  $E_i$  and the sum  $S$  of formula (2) would now be a series. In order not to be stopped at the outset, we must assume that the series  $S$  is convergent for the first choice of the numbers  $y_i$ ,

that we would make; but, if  $S$  exists for one choice of  $y_i$ , it exists for all choices of  $y_i$ , and the definition of the integral applies without modification.

The name of summable functions has been given to all functions which can be integrated by the indicated procedures, that is to say, to all measurable functions for which the sums  $S$  have a meaning. Every bounded measurable function is summable; and as no one has up to now succeeded in naming a nonmeasurable function, one could say that, up to now, practically every bounded function has an integral. On the contrary, there exists very simple unbounded functions which are not summable. Thus, one must not be astonished that our notion of integral still reveals itself insufficient in certain questions.

We have just extended the notion of integral to unbounded functions by starting with the first of our definitions; the second leads to the same result. But for this it is necessary to enlarge the notion of measure in such a way that it applies not only to bounded sets, which we thus far considered solely, but also to sets of points extending to infinity. I mention this second method of proceeding only because it is also related to another extension of the definite integral in which the interval, the domain, the set on which the integral is extended, is no longer presumed finite, as we have done up to now, but may go to infinity.

I limit myself to just an indication, because I will not be considering in what follows this extension of the integral concept. It is for the same reason that I am content with mentioning briefly the research, still very original, undertaken by a young man killed in the war, R. Gateaux, who intended to define the operation of integration for functions of infinitely many variations. This research, which was continued by Paul Lévy and by Norbert Wiener, is not without relation to the axiomatic studies undertaken by M. Fréchet and by P.J. Daniell with the aim of extending the notion of integral to abstract sets.<sup>4</sup> Fréchet and Daniell proposed furthermore to apply to abstract sets not only the definitions of which I have spoken thus far, but also a further extension of the definite integral, to which we shall be led soon by the notion of indefinite integral, which we are now going to examine.

<sup>4</sup>R. Gateaux, *Bull. Soc. Math. France*, 1919; P. Lévy, *Leçons d'analyse fonctionnelle*, 1922; N. Wiener, *Proc. London Math. Soc.*, 1922; M. Fréchet, *Bull. Soc. Math. France*, 1915; P.J. Daniell, *Ann. of Math.*, 1918 and 1919.

One ordinarily calls the indefinite integral of a function  $f(x)$  the function  $F(x)$  defined by

$$F(x) = C + \int_a^x f(x) dx \quad (5)$$

We do not adhere to this name but give rather to the words "indefinite integral" their original meaning. Originally, the two names "definite integral" and "indefinite integral" applied to the same expression  $\int_a^b f(x) dx$ . But the integral was called "definite" when it was a question of a given, determined, or defined interval  $(a, b)$ ; and the integral was "indefinite" when  $(a, b)$  was variable, undetermined, undefined, or, if one wishes, indefinite.

It is, in short, by a veritable abuse of language that one calls  $F(x)$  the indefinite integral of  $f(x)$ . If we remark in addition that when one studies  $F(x)$  it is always to obtain properties of  $\int_a^b f(x) dx$ , that it is actually  $\int_a^b f(x) dx$  which one studies through  $F(x)$ , one will be led to say: I call the indefinite integral of  $f(x)$  the function

$$\varphi(a, b) = \int_a^b f(x) dx = F(b) - F(a) \quad (5')$$

There are between an indefinite integral and the corresponding definite integral the same relations and same differences as between a function and a particular value taken on by this function. Furthermore, if we represent by  $D$  the interval  $(a, b)$  of integration, we may say that the indefinite integral is a function, the argument of which is the domain  $D$ ,

$$\psi(D) = \varphi(a, b)$$

From these reflections it clearly results that, relative to a function of two variables  $f(x, y)$ , one must not take for the indefinite integral, as is sometimes done, the function

$$F(X, Y) = c_1(x) + c_2(y) + \int_a^x \int_\beta^y f(x, y) dx dy \quad (6)$$

If one limits oneself to considering rectangular domains

$$a \leq x \leq b, \quad c \leq y \leq d$$

one must take for the indefinite integral the function of four variables

$$\varphi(a, b; c, d) = F(b, d) + F(a, c) - F(a, d) - F(b, c) \quad (7)$$

But if one wishes to consider all the domains of integration, since the most general domain cannot be determined by a finite number of parameters, however large the number, it becomes necessary to give up ordinary functions to represent the correspondence between a domain  $D$  and the integral extended to this domain and to study directly the function

$$\psi(D) = \iint_D f(x, y) dx dy$$

for which the argument  $D$  is a domain. It is this function which we will call the indefinite integral of  $f(x, y)$ . Or rather, since we have also defined the integral of  $f$  extended to a measurable set  $E$ , we will consider the indefinite integral as a set function which will have been defined for all measurable sets.<sup>5</sup>

In all that has been said up to now, there are, to be sure, only questions of language or of naming; but those questions would not have been asked if we had not acquired a new concept. It is for this reason that one should not be surprised that the new language has allowed one to give all possible meaning to facts perceived first of all in the case of the function  $F(x)$  of formula (5). One has succeeded, in particular, in characterizing set functions which are indefinite integrals by two properties: complete additivity and absolute continuity.<sup>6</sup>

When a set function possesses these two properties, it is the indefinite integral of a function  $f$  which depends on 1, 2, 3, . . . variables according to whether the sets  $E$  are formed with the aid of the points on a line, in a plane, in ordinary space, etc. In order to have a uniform language and notation, let us say that  $f$  is a point function,  $f(P)$ ; We write

$$\psi(E) = \int_E f(P) dm(E) \quad (8)$$

The function  $f(P)$  is entirely determined by  $\psi(E)$  to the extent that one can arbitrarily modify  $f$  on the points of an arbitrary set of measure zero without its ceasing to have  $\psi(E)$  for an indefinite integral. And one can obtain  $f(P)$  starting with  $\psi(E)$ , except on points of a set of measure zero, by the following procedure.

<sup>5</sup>*Ann. Sci. del' Ecole Normale Supérieure*, 1910.

<sup>6</sup>These terms are due respectively to de la Vallée-Poussin (*Intégrales de Lebesgue, fonctions d'ensemble, classe de Baire*, Paris, 1916) and G. Vitali (*R. Acc. Sci. Torino*, 1908). A function of a measurable set is absolutely continuous if, when  $E$  varies in such a way that  $m(E)$  tends toward zero,  $\psi(E)$  also tends toward zero. "Complete additivity" is a synonym for "countable additivity".

Let  $P$  be the point at which we wish to calculate  $f$ ; we take for the domain of integration  $\Delta$  an interval with center  $P$ , or a circle with center  $P$ , or a sphere with centre  $P$ —according to whether we are dealing with the case of the line, plane, or space—and we form the ratio  $\psi(\Delta)/m(\Delta)$ . Then, let  $\Delta$  tend to zero and we have

$$\lim_{\Delta \rightarrow 0} \frac{\psi(\Delta)}{m(\Delta)} = f(P) \quad (9)$$

This result evidently generalizes the classical theorem according to which,  $f(x)$  is continuous, the function  $F(x)$  of formula (5) admits  $f$  as its derivative; our procedure of calculating  $f(P)$  is indeed, in effect, a sort of differentiation of the set function  $\psi(E)$ .

This manner of differentiation was considered quite a long time ago. Cauchy<sup>7</sup> calls “coexistent quantities” those quantities determined at the same time, that is, by the same conditions. If, for example, one has a nonhomogeneous body, nonhomogeneous in composition and density, and if one considers a domain  $D$  of this body, the volume of  $D$ , the mass of  $D$ , the quantity of heat necessary to elevate by one degree the temperature of  $D$  supposed isolated, all are coexistent quantities. These are functions  $V(D)$ ,  $M(D)$ ,  $Q(D)$  of the domain.

It is not by happy chance that we arrive here at functions of domains. If one reflects on it, one quickly sees that every magnitude of physics is related not to a point, but to an extended body, that it is a function of a domain, at least insofar that it is a matter of directly measurable magnitudes. The body to be considered will not, however, always be a body of our customary space; it could be a body in a purely mathematically conceived space if, in the determination of the envisaged magnitude, there intervene nonspatial variables such as time, temperature, etc. But this is of little importance; directly measurable magnitude—mass, quantity of heat, quantity of electricity, for example—are functions of a domain and not functions of a point.

Physics meanwhile also considers magnitudes associated with points, such as speed, tension, density, specific heat; but these are derived magnitudes which one defines accurately most often by the ratio or the limit of the ratio of two coexistent quantities:

$$\text{Density} = \frac{\text{Mass}}{\text{Volume}}; \quad \text{Specific heat} = \frac{\text{Quantity of heat}}{\text{Mass}}$$

that is to say, by taking the derivative of a magnitude with respect to a coexistent quantity.

<sup>7</sup>*Exercices d'analyse et de physique mathématique*, Vol. 2, Paris, 1840-1847, pp. 188-229.

Thus physics, and consequently geometry, leads to the consideration of functions of a domain and their differentiation just as does analysis of functions of real variables. Similarly the functions of a domain have, in physics, a somewhat more primordial role than point functions. Why then do physicists not speak of these functions? Because mathematicians have not yet studied them and because algebra has notation neither for the domains, nor for the functions of domains. Thus one sees the physicist limit himself to considering special domains depending only on certain parameters, in such a way that the domain function to be considered is reduced to a function of parameters. This is, moreover, exactly what a mathematician does when, instead of considering the definite integral of  $f(x, y)$  in all of its generality, he limits himself to considering the functions  $F(X, Y)$ ,  $\varphi(a, b; c, d)$  of formulas (6) and (7).

We remark furthermore that formula (8) establishes a connection between the set functions  $\psi(E)$ , which are indefinite integrals, and point functions  $f(P)$ , which are dependent upon algebra. This formula (8) thus furnishes a sort of notation for certain set functions. But when one examines the two conditions required for a function to be an indefinite integral, one cannot doubt that physical quantities are among the class of functions susceptible to the notation.

These reflections on the nature of physical quantities may have allowed you to understand more precisely the interest and the importance of the notions which we have encountered. They show, in particular, that the operation of differentiation which appears in formula (9) is not the only one to be considered, that one can always consider the differentiation of a function  $\psi(E)$  with respect to a coexistent function  $p(E)$ , whether or not it is the measure  $m(E)$ .

One question now quickly comes to mind: Can one also replace the function  $m(E)$  with a given function  $p(E)$  in the definition of the integral? In this there is no difficulty. We will first replace formula (2) by

$$S = \sum \mu_i p(E_i)$$

if first the sets  $E_i$  belong to the family of those sets for which the function  $p(E)$  is defined, that is, the function to be integrated must be measurable with respect to  $p(E)$  in order for the series  $s$  to be convergent, that is,  $f$  must be summable with respect to  $p(E)$ . This being presumed, the definition of the integral of  $f(p)$  with respect to  $p(E)$ ,

$$\int f(P) dp(E)$$

is obtained as before if the function  $p(E)$  and possesses a certain property which one expresses in saying that  $p(E)$  must be of bounded variation.<sup>8</sup>

We have just arrived at a new and very considerable extension of the notion of integral in taking the formal point of view of the mathematician; the point of view of the physicist leads even more naturally to the same result, at least for continuous functions  $f(P)$ . One could similarly say that the physicists have always considered only integrations with respect to domain functions.

Suppose, for example, that one wishes to calculate the quantity of heat  $\varphi(D)$  necessary to elevate by one degree the temperature of a body  $D$  of which we spoke above. One must divide  $D$  into small partial bodies  $D_1, D_2, \dots$  of masses  $M(D_1), M(D_2), \dots$ , choose from each a point  $P_1, P_2, \dots$  and choose for an approximate value of  $\varphi(D)$  the sum

$$f(P_1)M(D_1) + f(P_2)M(D_2) + \dots$$

$f(P)$  designating the specific heat at  $P$ . This is to say that we are calculating  $\varphi(D)$  by the formula

$$\varphi(D) = \int_D f(P) dM(E).$$

In its general form the new integral was defined only in 1913 by Radon; it was, meanwhile, known since 1894 for the particular case of a continuous function of a single variable. But its first inventor, Stieltjès, was led to it by research in analysis and arithmetic, and he presented it in a purely analytical form which masked its physical significance so much that it required much effort to understand and

<sup>8</sup> $p(E)$  is said to be of bounded variation if, in whichever manner one partitions  $E$  into a countably infinite number of pairwise disjoint sets  $E_1, E_2, \dots$ , the series  $\sum |p(E_i)|$  is convergent.

The notion of functions of bounded variation was first introduced by C. Jordan for functions of one variable.

The only set functions  $p(E)$  to be considered in these theories are additive functions, that is, those for which one has

$$p(E_1 + E_2 + \dots) = p(E_1) + p(E_2) + \dots$$

$E_1, E_2, \dots$  being pairwise disjoint. If the additivity is complete, that is, if the sequence  $E_1, E_2, \dots$  can be chosen arbitrarily,  $p(E)$  is necessarily of bounded variation. In effect, the order of the sets being unimportant, the series  $p(E_1) + p(E_2) + \dots$  must remain convergent whatever the order; that is, the series  $\sum |p(E_i)|$  is convergent.

No attempts have been made up to now to get rid of the condition that  $p(E)$  be of bounded variation. One ought to remark besides that if  $p(E)$  were not of bounded variation, one could find a continuous function  $f(p)$  for which, nevertheless, our definition of integral would not apply.

recognize what is now evident. The history of these efforts cites the names of *F. Riesz*, *H. Lebesgue*, *W.H. Young*, *M. Fréchet*, *C. de la Vallée-Poussin*; it shows that we competed in ingenuity and in perspicacity, but also in blindness.<sup>9</sup>

And yet, mathematicians always considered Stieltjes-Radon integrals. The curvilinear integral  $\int_C f(x, y) dx$  is one of these integrals, relative to a function defined in terms of the length of the projection onto the  $x$  axis of a arcs of  $C$ ; the integral  $\iint_S f(x, y, z) dx dy$  involves in the same way a set function defined in terms of areas of  $S$  projected onto the  $xy$  plane.

In truth, these integrals most often present themselves in groups

$$\int_C f(x, y)dx + g(x, y)dy$$

$$\int_S f(x, y, z)dx dy + g(x, y, z)dy dz + h(x, y, z) dz dx$$

If one thinks also of integrals considered for the definition of lengths of curves or areas of surfaces,

$$\int_C (dx^2 + dy^2 + dz^2)^{1/2} \quad \iint_S [(dx dy)^2 + (dy dz)^2 + (dz dx)^2]^{1/2}$$

one will be led to say that it is also convenient to study modes of integration in which there appear several set functions  $p_1(E)$ ,  $p_2(E)$ , . . . . This study remains entirely for the future, although Hellinger and Toeplitz have utilized certain summations with respect to several set functions.<sup>10</sup>

We have thus for considered integration, definite or indefinite, as an operation furnishing a number, defined or variable, by a sort of generalized addition. We are placed with the point of view of quadratures. But one may also consider the integration of a continuous function as furnishing a function, just like the most simple of integrations of differential equations. It is this point of view of primitive functions which we will now consider.

Finding the primitive function  $F(x)$  of a given function  $f(x)$  is finding the function, determined to an additive constant, when it exists,

<sup>9</sup>J. Radon, *Sitz, Kais. Ak. Wiss. Vienna*, Vol. 122, Section IIa, 1913; T.J. Stieltjes, *Ann. Fac. Sci. Toulouse*, 1894; F. Riesz, *Comptes Rendus Acad. Sci. Paris*, 1909; H. Lebesgue, *ibid.*, 1910; W.H. Young, *Proc. London Math. Soc.*, 1913; M. Fréchet, *Nouv. Ann. des Math.*, 1909; de la Vallée-Poussin, *op. cit.*

<sup>10</sup>See, for example, *J. reine angew. Math.*, 144, 1914, pp. 212-238.

which admits  $f(x)$  as its derivative. It is this problem that we are going to study.

But first we remark that the preceding reflections lead to formulating the problem in a much more general fashion: Given a function  $f(P)$  which is the derivative with respect to a known function  $p(E)$  of an unknown function  $\psi(E)$ , find the primitive function  $\psi(E)$  of  $f(P)$ .

If, for example, we are dealing with a continuous function  $f(x)$  and if  $m(E)$  is the measure, the primitive function would no longer be the function  $F(x)$  of formula (5), but the indefinite integral  $\int_E f(x) dx$ .

I can only mention this general problem which has not been studied; I am content with remarking that the Stieltjes integral would be very insufficient for resolving it. This integral has, in effect, only been defined for the hypothesis that  $p(E)$  is of bounded variation, and one may certainly speak of differentiation with respect to a function  $p(E)$  with is not of bounded variation.

The theory of summable functions furnishes the following result related to the case in which  $p(E)$  is the measure  $m(E)$ : When the derivative  $f(P)$  is summable, the antiderivative of  $f$  is one of its primitive functions. I say one of its primitive functions because one still does not know very well now this general problem of primitive functions must be posed in order for it to be determined.<sup>11</sup>

Let us leave aside these questions, which I speak of only in order to show how much there remains to be done, and let us show how much has been done in the search for the primitive function  $F(x)$  of  $f(x)$ , thanks above all to Arnaud Denjoy.

I have just said that, when  $f(x)$  is summable, integration furnishes  $F(x)$  by formula (5). Suppose that, on  $(a, b)$ ,  $f(x)$  fails to be summable only at a single point  $c$ . Then integration gives us  $F(x)$  on  $(a, c - \epsilon)$  for arbitrarily small  $\epsilon$  and hence on the whole interval  $(a, c)$ ; it also gives  $F(x)$  on  $(c + \epsilon, b)$  and hence completely on  $(c, b)$ . And taking into account the continuity of  $F(x)$  at the point  $c$ , we have  $F(x)$  on the whole interval  $(a, b)$ . By such considerations of continuity,<sup>12</sup> one sees that, if one knows  $F(x)$  on every interval which contains no point of a set  $E$  in its interior or at its extremities, one can deduce  $F(x)$  by an operation which I shall designate by  $A$  on every interval adjacent to  $E$ , that is, on every interval having its end points in  $E$  but having no points of  $E$  in its interior.

<sup>11</sup>See on this subject the notes of Fubini and Vitali, appearing 1915-1916, in *Atti Rend. R. Acc. Lincei*.

<sup>12</sup>It is the introduction of these conditions of continuity which very considerably differentiates the problem of primitive functions from that of quadratures.

Suppose now that one knows  $F(x)$  on intervals  $(\alpha, \beta)$  adjacent to a set  $E$ , that the sum  $\sum[F(\beta) - F(\alpha)]$  is convergent, and that  $f(x)$  is summable on  $E$ .<sup>13</sup> Then it suffices to say that the primitive function must result from the contribution of  $E$  and the intervals adjacent to  $E$  in order to be led to the formula

$$F(x) - F(a) = \left\{ \int_E f(dx + \sum[F(\beta) - F(\alpha)] \right\}_a^x$$

the braces of the second member indicating that one must utilize there only points between  $a$  and  $x$ . From this formula there results the determination of  $F(x)$ , thanks to an operation with  $I$  will designate by  $B$ .

The preceding results mark the extreme points which I reached in my thesis, and I must say that I indicated them only somewhat by chance, because I did not at all suspect the importance given to them by Denjoy. Relying on Baire's results, Denjoy shows that, if  $f(x)$  is a derivative function on  $(a, b)$ , then

(1) The points for which  $f(x)$  is not summable form a set  $E_1$  which is not dense in  $(a, b)$ ; an operation  $O_1$  of type  $A$  determine  $F(x)$  on intervals adjacent to  $E_1$ .

(2) Next, there exists a set  $E_2$  formed from point of  $E_1$  and not dense in  $E_1$ , on the adjacent intervals of which one can calculate  $F(x)$  by an operation  $O_2$  of type  $B$ .

(3) Next, there exists a set  $E_3$  formed from points of  $E_2$  and not dense in  $E_2$ , on the adjacent intervals of which one can calculate  $F(x)$  by an operation  $O_3$  of type  $B$ , . . . .

If it turns out that after an infinite sequence of operation  $O_1, O_2, \dots$ , one has not yet found  $F(x)$  on the entire interval  $(a, b)$ , the points of  $(a, b)$  which are not interior points of intervals on which one has defined  $F(x)$  form a set  $E_\omega$ , and an operation of type  $A$ , the operation  $O_\omega$ , furnishes  $F$  on intervals adjacent to  $E_\omega$ . One considers next, if it is necessary, operations  $O_{\omega+1}, O_{\omega+2}, \dots$  of type  $B$ , followed by an operation  $O_{2\omega}$  of type  $A$ , followed by operations of type  $B$ , etc.

And Denjoy, using now classical arguments of Cantor and Bendixson, proves that this procedure will finally give us  $F(x)$  on the entire interval  $(a, b)$  after a finite or countably infinite number of operations.

<sup>13</sup>It is convenient to remark that these hypotheses are not contradictory, the same as if  $E$  is assumed to be the set of points on which  $f(x)$  is not summable in an interval  $(a, b)$  considered. For the determination of points of nonsummability on  $(a, b)$  it is necessary, in effect, to take into account all points of  $(a, b)$ , whether they belong to  $E$  or not; whereas summability on  $E$  is a condition occurring only on the points of  $E$ .

This operative procedure, certainly complicated, but just as natural, in principal, as those previously envisaged, was called by Denjoy "totalization".

Totalization solves entirely the problem of finding the primitive function  $F(x)$  of a given function  $f(x)$ ; it permits at the same time the determination of  $F(x)$  knowing only a derived number<sup>14</sup>  $f$  of  $F(x)$  and no longer its derivative. I shall not dwell on these beautiful results; the most important fact for us is that totalization, by a long detour, furnishes us with a new extension of the concept of definite integral. Every time, in effect, that totalization applies to a function  $f(x)$  and gives a corresponding function  $F(x)$ , we can attach to  $f(x)$  an integral, thanks to formulas (5) and (5').<sup>15</sup>

Gentlemen, I end now and thank you for your courteous attention; but a word of conclusion is necessary. This is, if you will, that a generalization made not for the vain pleasure of generalizing, but rather for the solution of problems previously posed, is always a fruitful generalization. The diverse applications which have already taken the concepts which we have just examined prove this super-abundantly.

<sup>14</sup>Dini's derivative.

<sup>15</sup>The detailed memoirs of Denjoy appeared from 1915 to 1917 in the *Journal de Math.*, in the *Bull. Soc. Math. France*, and in the *Ann. Sci. de l'École Normale Supérieure*.



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# List of Symbols and Notations

$\in$	belongs to
$\notin$	does not belong to
$\subset$	is a subset of, is contained in
$\supset$	contains
$\cup$	union
$\cap$	intersection
$\phi$	empty set
$\exists$	there exists
$\blacksquare$	Q.E.D.
$\forall$	for all
$\mathbf{N}$	set of natural numbers
$I$	set of integers
$\mathbf{Q}$	set of rational numbers
$\mathbf{R}$	set of real numbers, real number system
$I^+$	set of nonnegative integers
$\mathbf{N}^-$	set of negative integers
$\mathbf{Q}^+$	set of positive rational numbers
$\mathbf{Q}^-$	set of negative rational numbers
$\mathbf{R}^*$	set of extended real numbers, extended real number system
$\mathbf{R}^n$	$n$ -dimensional Euclidean space
$A - B$	the set of all points in $A$ which are not in $B$
$A \times B$	cartesian product of $A$ and $B$
$A \Delta B$	symmetric difference of $A$ and $B$
$A \sim B$	$A$ is equivalent to $B$
$A \not\sim B$	$A$ is not equivalent to $B$
$A^c$	the complement of $A$
$A^B$	the set of functions on $B$ with range in $A$
$A'$	the derived set of $A$
$\text{int}(A)$	interior of $A$
$\mathcal{P}(A)$	power set of $A$ , set of subsets of $A$
$\chi_A$	characteristic function of $A$

$c(A)$ or $\bar{A}$	closure of $A$
$\bar{A}$ or $\text{card}(A)$	cardinal number of $A$
$\aleph_0$	aleph nought
$c$	cardinal number of the continuum
$C$	cantor set
$C(\alpha)$	generalized cantor set
$\mathcal{F}_\sigma$ } $\mathcal{G}_\delta$ }	special Borel sets
$\mathcal{M}$	family of measurable sets
$\mathcal{B}$	class of Borel sets
$f: A \rightarrow B$	function (or mapping) with domain $A$ and range in $B$
$f^{-1}: B \rightarrow A$	inverse function (or mapping)
$f(A)$	image of $A$ under the function $f$
$f _A$	the restriction of $f$ to $A$
$g \circ f: A \rightarrow B$	composition of mappings
$f^{-1}(b)$	inverse image of $b$
$f^{-1}(B)$	inverse image of $B$
$f^+$	positive part of the function $f$
$f^-$	negative part of the function $f$
$f \sim g$	$f$ is equivalent to $g$
$f = g$	$f$ is equal to $g$
$[x]$	greatest integer not greater than $x$
$x \dot{+} y$	sum modulo 1 of $x$ and $y$
$xRy$	$x$ is related to $y$
$A \dot{+} y$	translate modulo 1 of $A$ by $y$
$l(I)$	length of the interval $I$
$m(E)$	measure of $E$
$m^*(E)$	outer measure of $E$
$m_*(E)$	inner measure of $E$
$D^+f(x)$	upper right derivative
$D_+f(x)$	lower right derivative
$D^-f(x)$	upper left derivative
$D_-f(x)$	lower left derivative
$BV[a, b]$	space of the functions of bounded variation on $[a, b]$
$v_f$	variation function
$T_a^b(f)$	total variation of $f$ on $[a, b]$
$C[a, b]$	space of continuous functions on $[a, b]$
$\ \cdot\ $	norm
$(X, \ \cdot\ )$	normed space $X$
$L^p(E)$	class of $p$ -integrable functions over $E$
$L^\infty(E)$	class of measurable essentially bounded functions on $E$

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