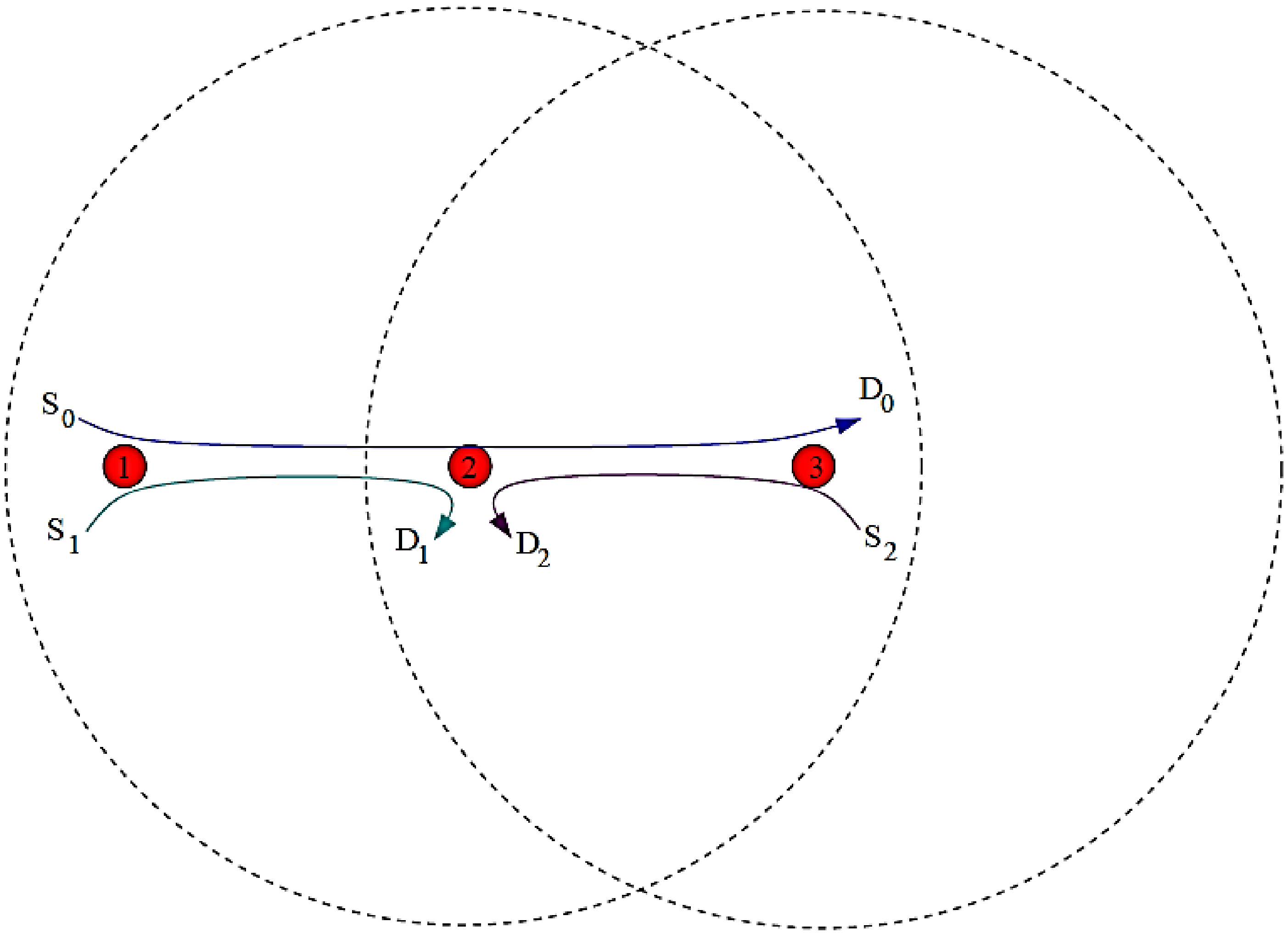
Lecture 02

Introduction to Convex Programming

Example Resource Allocation in Wireless Networks



We assume that hops are of unit capacity, i.e., if scheduled, a hop could transfer packets at the rate of 1 packet per time unit.

Wireless Interference Model. The dashed circles indicate the interference regions of the two destination nodes. Due to interference, only one node can transmit at a time. We have to schedule the fraction of time allocated to the two possible hops as well as the rate allocated to each flow.

Example Resource Allocation in Wireless Networks

- In the previous slide, there are three flows in progress and two of the nodes act as destinations.
- Around each destination, we draw a circle that is called either the reception range or interference region, associated with that destination.
- A transmitter has to lie within the reception range of a receiver for the transmission to be successful.
- In addition, no other node in the interference region can transmit if the receiver is to receive the transmitted data successfully.
- Thus, if node 2 is to successfully receive data, either node 3 or node 1 may transmit, but not both at the same time.

Example Resource Allocation in Wireless Networks

- We have to consider both the scheduling of the links to avoid interference as well as rate allocation to each flow.

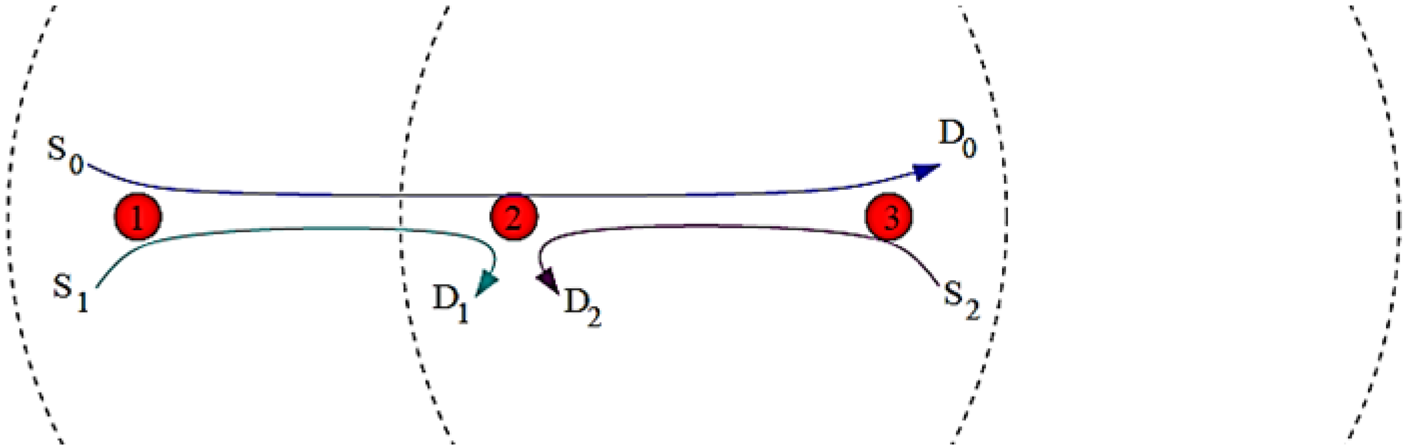
We incorporate the interference constraint by introducing new variables which indicate the fraction of time that each link is scheduled.

Let R_{ij} be the fraction of time that node i transmits to node j.

Also, define the vector $R = [R_{12}, R_{23}, R_{32}]^T$.

Since nodes 1 and 2 cannot transmit simultaneously, we have $R_{12} + R_{23} + R_{32} \leq 1$.

Thus, the network utility maximization problem becomes



$$
\begin{aligned}
\max_x \sum_{i=0}^{2} U_i(x_i) & \\
x_0 + x_1 &\leq R_{12} \\
x_0 &\leq R_{23} \\
x_2 &\leq R_{32} \\
R_{12} + R_{23} + R_{32} &\leq 1 \\
x, R &\geq 0
\end{aligned}
$$

Central concept: convexity

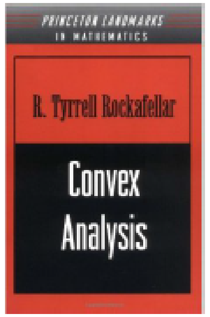
Historically, linear programs were the focus in optimization

Initially, it was thought that the important distinction was between linear and nonlinear optimization problems. But some nonlinear problems turned out to be much harder than others ...

Now it is widely recognized that the right distinction is between convex and nonconvex problems

Your supplementary textbooks for the course:

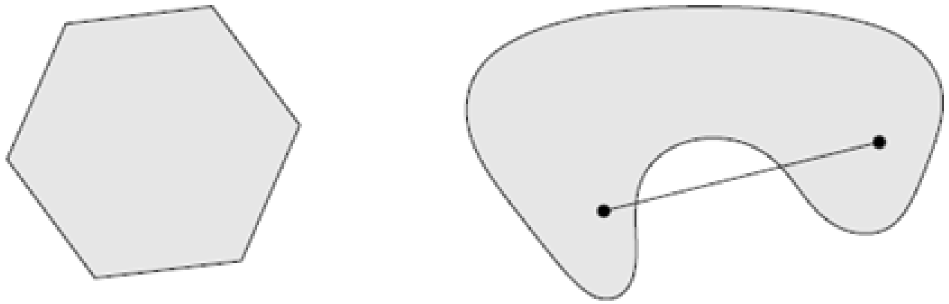
Boyd and Vandenberghe (2004) 

, Rockafellar (1970) 

Convex sets and functions

Convex set: $C \subseteq \mathbb{R}^n$ such that

$$x, y \in C \implies tx + (1-t)y \in C \text{ for all } 0 \le t \le 1$$

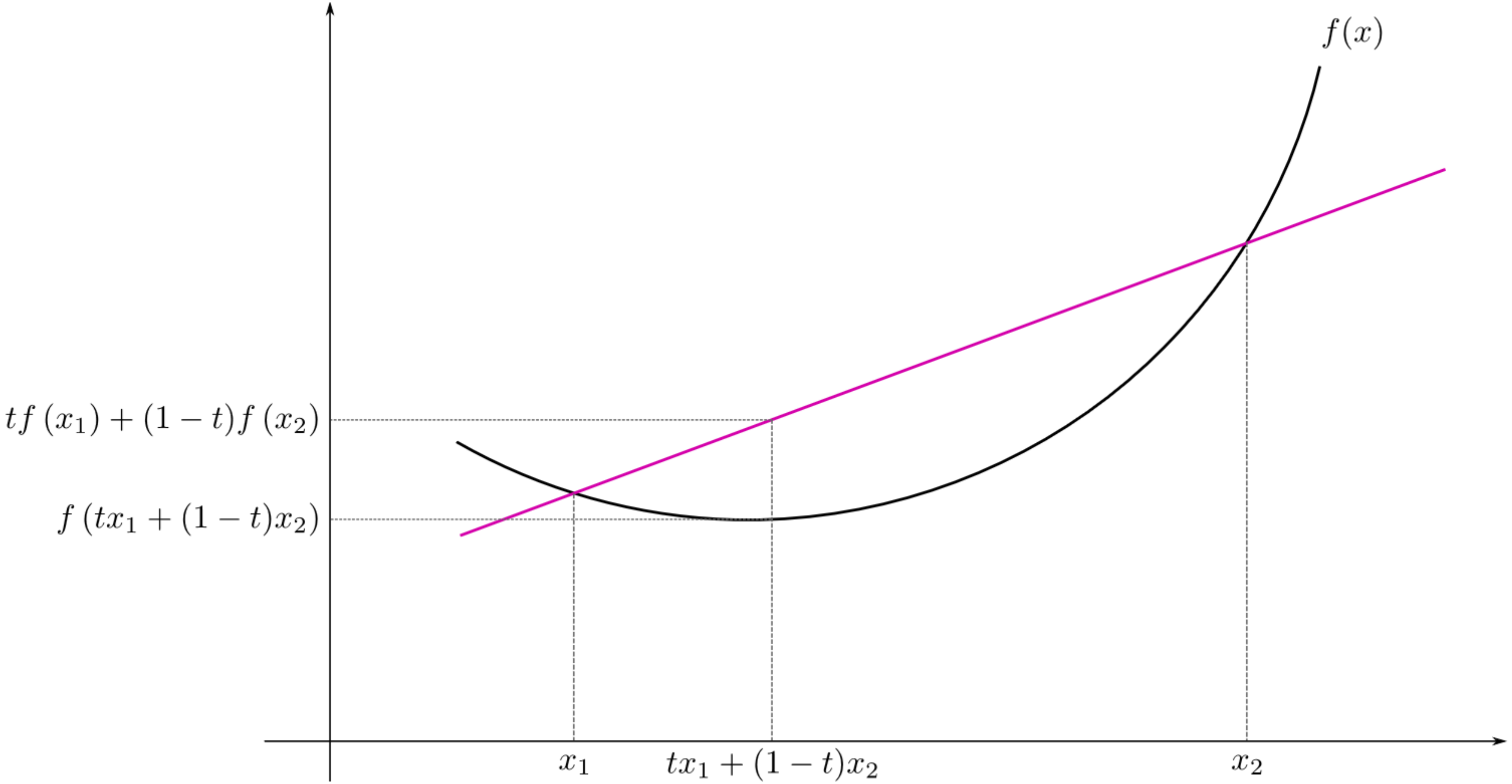


Convex function: $f : \mathbb{R}^n \to \mathbb{R}$ such that $\mathrm{dom}(f) \subseteq \mathbb{R}^n$ convex, and

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) \quad \text{for } 0 \le t \le 1$$

and all $x, y \in \mathrm{dom}(f)$

- In mathematics, a real-valued function defined on an n-dimensional interval is called convex if the **line segment between any two points on the graph of the function lies above or on the graph**.



Convex optimization problems

Optimization problem:

$$\begin{aligned} \min_{x \in D} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \leq 0,\ i = 1, \dots m \\ & h_j(x) = 0,\ j = 1, \dots r \end{aligned}$$

Here $D = \text{dom}(f) \cap \bigcap_{i=1}^{m} \text{dom}(g_i) \cap \bigcap_{j=1}^{r} \text{dom}(h_j)$, common domain of all the functions

This is a convex optimization problem provided the functions f and $g_i, i = 1, \dots m$ are convex, and $h_j, j = 1, \dots r$ are affine:

$$h_j(x) = a_j^T x + b_j, \quad j = 1, \dots r$$

Local minima are global minima

For convex optimization problems, local minima are global minima

Formally, if x is feasible—$x \in D$, and satisfies all constraints—and minimizes f in a local neighborhood,

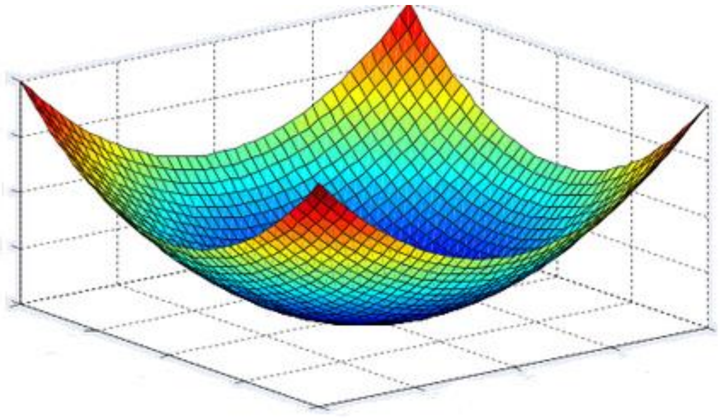
$$f(x) \le f(y) \ \text{ for all feasible } y, \ \|x - y\|_2 \le \rho,$$

then

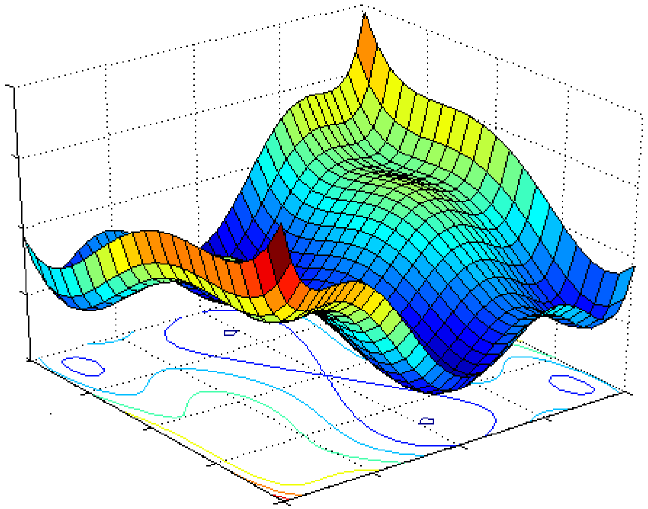
$$f(x) \le f(y) \ \text{ for all feasible } y$$

This is a very useful fact and will save us a lot of trouble!

Convex



Nonconvex



Theorem (Local minima are global minima)

For a convex optimization problem, if x is feasible and minimizes f in a local neighborhood,

$$f(x) \le f(y) \text{ for all feasible } y, \|x - y\|_2 \le \rho,$$

then $f(x) \le f(y)$ for all feasible y.

Proof: Suppose $\exists z \in D$ and is feasible such that $f(z) < f(x)$.

According to the definition of local minima, we have $\|z - x\|_2 > \rho$.

We let $y = tx + (1-t)z$, where $0 \le t \le 1$.

Because D is a convex set, according to its definition we also have $y \in D$.

Then for each $i = 1, ..., m$,

$$g_i(tx + (1-t)z) \leq tg_i(x) + (1-t)g_i(z) \leq 0. \tag{1.1}$$

For each $j = 1, ..., r$,

$$h_i(tx + (1-t)z) = 0 \tag{1.2}$$

From (1.1) and (1.2), we conclude y is also feasible.

If we let t large enough (close to 1 but less than 1) such that $\|x - y\|_2 \leq \rho$, we obtain

$$f(y) = f(tx + (1-t)z) \leq tf(x) + (1-t)f(z) < f(x),$$

which is contradictory to the local minima definition.

So by proof of contradiction, we conclude the local minima are also global minima.

■

Appendix I:

Vector Norms

Norms

A ***norm*** of a vector $\|x\|$ is informally a measure of the "length" of the vector. For example, we have the commonly-used Euclidean or ℓ_2 norm,

$$\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}.$$

Note that $\|x\|_2^2 = x^T x$.

More formally, a norm is any function $f : \mathbb{R}^n \to \mathbb{R}$ that satisfies 4 properties:

1. For all $x \in \mathbb{R}^n$, $f(x) \geq 0$ (non-negativity).
2. $f(x) = 0$ if and only if $x = 0$ (definiteness).
3. For all $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, $f(tx) = |t| f(x)$ (homogeneity).
4. For all $x, y \in \mathbb{R}^n$, $f(x + y) \leq f(x) + f(y)$ (triangle inequality).

Norms

Other examples of norms are the ℓ_1 norm,

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$

and the ℓ_∞ norm,

$$\|x\|_\infty = \max_i |x_i|.$$

In fact, all three norms presented so far are examples of the family of ℓ_p norms, which are parameterized by a real number $p \geq 1$, and defined as

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}.$$