Game Theory

Lecture 02:

- Basic Solution Concepts
  - Strategic Dominance
  - Rationalizability

Dominant Strategies

- Example: Prisoner's Dilemma.
  - Two people arrested for a crime, placed in separate rooms, and the authorities are trying to extract a confession.

prisoner 1 / prisoner 2	Confess	Don't confess
Confess	$(-4, -4)$	$(-1, -5)$
Don't confess	$(-5, -1)$	$(-2, -2)$

- What will the outcome of this game be?
- Regardless of what the other player does, playing "Confess" is better for each player.
- The action "Confess" **strictly dominates** the action "Don't confess"
- Prisoner's dilemma paradigmatic example of a self-interested, rational behavior not leading to jointly *(socially) optimal result*.

Dominant Strategy Equilibrium

- Compelling notion of equilibrium in games would be dominant strategy equilibrium, where each player plays a dominant strategy.

Definition

(Dominant Strategy) *A strategy $s_i \in S_i$ is dominant for player i if*

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) \qquad \text{for all } s_i' \in S_i \text{ and for all } s_{-i} \in S_{-i}.$$

Definition

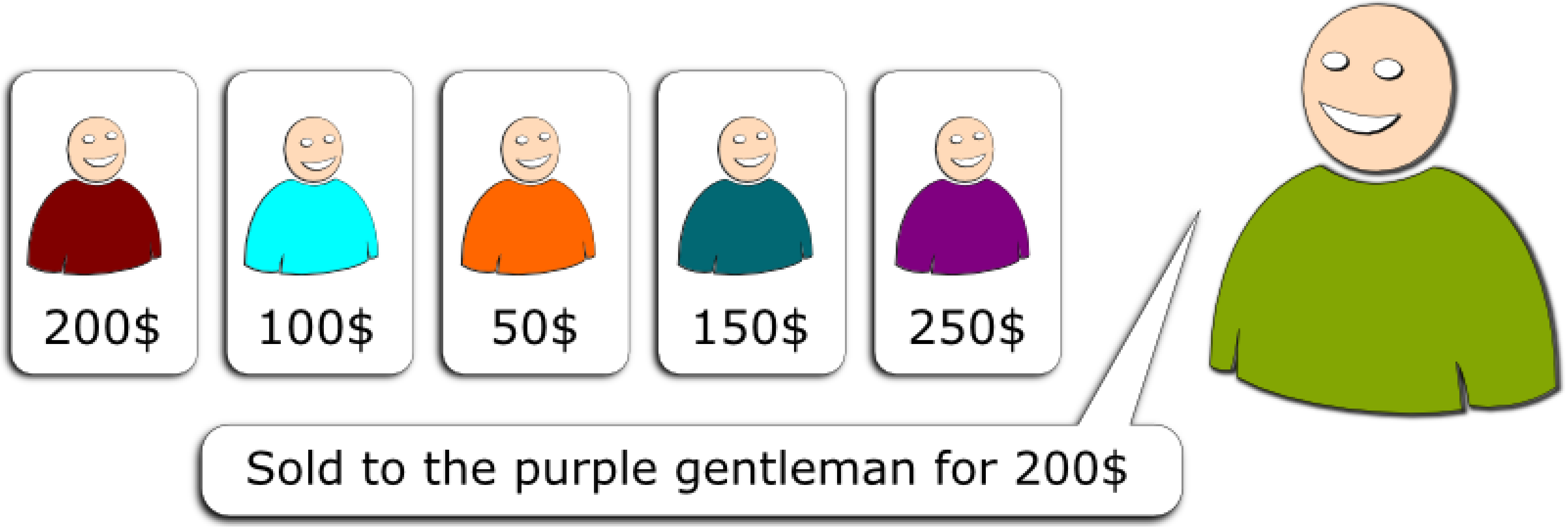
(Dominant Strategy Equilibrium) *A strategy profile s^* is the dominant strategy equilibrium if for each player i, s_i^* is a dominant strategy.*

- These notions could be defined for strictly dominant strategies as well.

Example

Second-Price Auction

- bidders write down bids on pieces of paper
- auctioneer awards the good to the bidder with the highest bid
- that bidder pays the amount bid by the second-highest bidder



Utility = true value - payment

If bidders report truthfully, then the auction maximizes the social welfare

$$\sum_{i=1}^{n} v_i x_i,\ \textit{where}\ x_i\ \textit{is 1 if}\ i\ \textit{wins and 0 if}\ i\ \textit{loses, subject to} \sum_{i=1}^{n} x_i \leq 1$$

Example (Cont'd)

Theorem

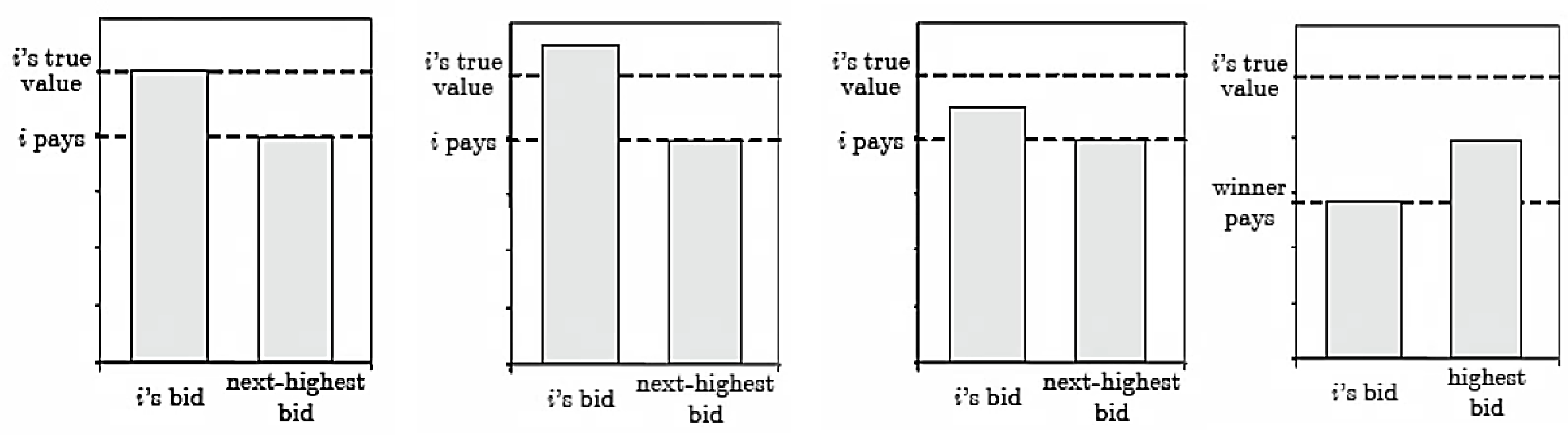
Truth-telling is a dominant strategy in a second-price auction.

Proof.

Assume that the other bidders bid in some arbitrary way. We must show that i's best response is always to bid truthfully. We'll break the proof into two cases:

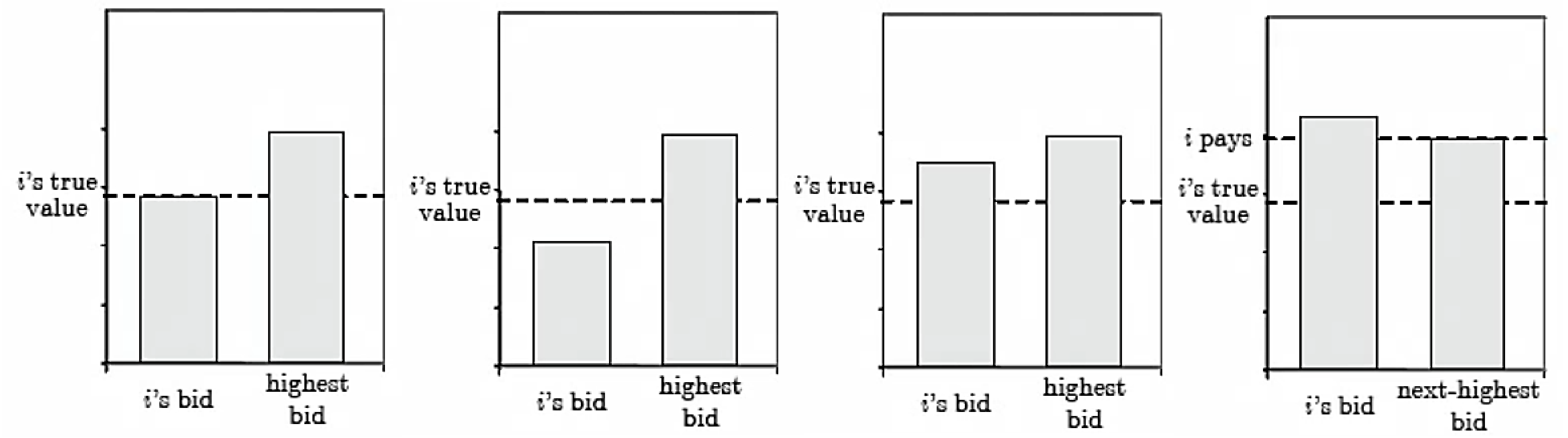
1. Bidding honestly, i would win the auction
2. Bidding honestly, i would lose the auction

Example (Cont'd)



- Bidding honestly, i is the winner
- If i bids higher, he will still win and still pay the same amount
- If i bids lower, he will either still win and still pay the same amount…or lose and get utility of zero.

Example (Cont'd)



- Bidding honestly, i is not the winner
- If i bids lower, he will still lose and still pay nothing
- If i bids higher, he will either still lose and still pay nothing…or win and pay more than his valuation.

Dominant and Dominated Strategies

- Though compelling, dominant strategy equilibria do not always exist, for example, as illustrated by the partnership or the matching pennies games we have seen before
- Nevertheless, in the prisoner's dilemma game, "confess, confess" is a dominant strategy equilibrium.
- We can also introduce the converse of the notion of dominant strategy, which will be useful next.

Definition

(Strictly Dominated Strategy) *A strategy $s_i \in S_i$ is strictly dominated for player i if there exists some $s_i' \in S_i$ such that*

$$u_i(s_i', s_{-i}) > u_i(s_i, s_{-i}) \qquad \text{for all } s_{-i} \in S_{-i}.$$ ★

Restricting attention to opponents' pure strategies

- In general we want to allow for players choosing mixed strategies.
- It seems we would actually want the definition of dominance to be that s_i' strictly dominates s_i if the inequality holds for all possible mixed strategies by her opponents, i.e. if

$$\forall \boldsymbol{\sigma}_{-i} \in \Sigma_{-i}, \quad u_i(\langle s_i', \boldsymbol{\sigma}_{-i} \rangle_i) > u_i(\langle s_i, \boldsymbol{\sigma}_{-i} \rangle_i). \qquad ★★$$

- *Prima facie,* the definition above looks more difficult to satisfy than ★ because the inequality must hold in a larger set of cases.

But, the two definitions are equivalent! Let's see why.

- Clearly, satisfaction of the inequality in ★★ implies satisfaction in ★ because the set of deleted pure-strategy profiles $\boldsymbol{S_{-i}}$ is included in the set of deleted mixed-strategy profiles $\boldsymbol{\Sigma_{-i}}$.

Restricting attention to opponents' pure strategies

Arguing the other direction; i.e.

$$\forall \mathbf{s}_{-i} \in S_{-i}, \quad u_i(\langle s_i', \mathbf{s}_{-i}\rangle_i) > u_i(\langle s_i, \mathbf{s}_{-i}\rangle_i). \Rightarrow \forall \boldsymbol{\sigma}_{-i} \in \Sigma_{-i}, \quad u_i(\langle s_i', \boldsymbol{\sigma}_{-i}\rangle_i) > u_i(\langle s_i, \boldsymbol{\sigma}_{-i}\rangle_i).$$

Note that $u_i(\langle s_i', \boldsymbol{\sigma}_{-i}\rangle_i)$ is a convex combination of $u_i(\langle s_i', \mathbf{s}_{-i}\rangle_i)$ terms, one for each $\mathbf{s}_{-i} \in S_{-i}$.

$$u_i(\langle s_i', \boldsymbol{\sigma}_{-i}\rangle_i) = \sum_{\mathbf{s}_{-i} \in S_{-i}} \left(\prod_{j \in I \setminus \{i\}} \sigma_j(s_j) \right) u_i(\langle s_i', \mathbf{s}_{-i}\rangle_i)$$

Now assume that s_i' strictly dominates s_i

Then we replace each $u_i(\langle s_i', \mathbf{s}_{-i}\rangle_i)$ term by something smaller, viz. $u_i(\langle s_i, \mathbf{s}_{-i}\rangle_i)$.

The result is equal to $u_i(\langle s_i, \boldsymbol{\sigma}_{-i}\rangle_i)$,

In symbolic terms,

$$u_i(\langle s_i', \boldsymbol{\sigma}_{-i}\rangle_i) = \sum_{\mathbf{s}_{-i} \in S_{-i}} \left(\prod_{j \in I \setminus \{i\}} \sigma_j(s_j) \right) u_i(\langle s_i', \mathbf{s}_{-i}\rangle_i) > \sum_{\mathbf{s}_{-i} \in S_{-i}} \left(\prod_{j \in I \setminus \{i\}} \sigma_j(s_j) \right) u_i(\langle s_i, \mathbf{s}_{-i}\rangle_i) = u_i(\langle s_i, \boldsymbol{\sigma}_{-i}\rangle_i).$$

Mixed-strategy dominance

- Are there cases in which a pure strategy is dominated by some mixed strategy $\sigma_i' \in \Sigma_i$ of player *i*'s but is not dominated by any pure strategy? The answer is yes.

Example: **A mixed strategy can dominate where no pure strategy can.**

	l: $[q]$	r: $[1-q]$
U: $[p]$	6, 0	0, 6
M: $[1-p]$	0, 6	6, 0
D	2, 0	2, 0

- Consider the mixed strategy for Row in which she plays $\sigma_R' = p \circ U \oplus (1-p) \circ M$,

$$\left.\begin{aligned} u_R(\sigma_R';\ l) &= 6p + 0\cdot(1-p) > u_R(D;\ l) = 2, \\ u_R(\sigma_R';\ r) &= 0\cdot p + 6(1-p) > u_R(D;\ r) = 2. \end{aligned}\right\} \quad p \in \left(\tfrac{1}{3}, \tfrac{2}{3}\right).$$

Mixed-strategy dominance

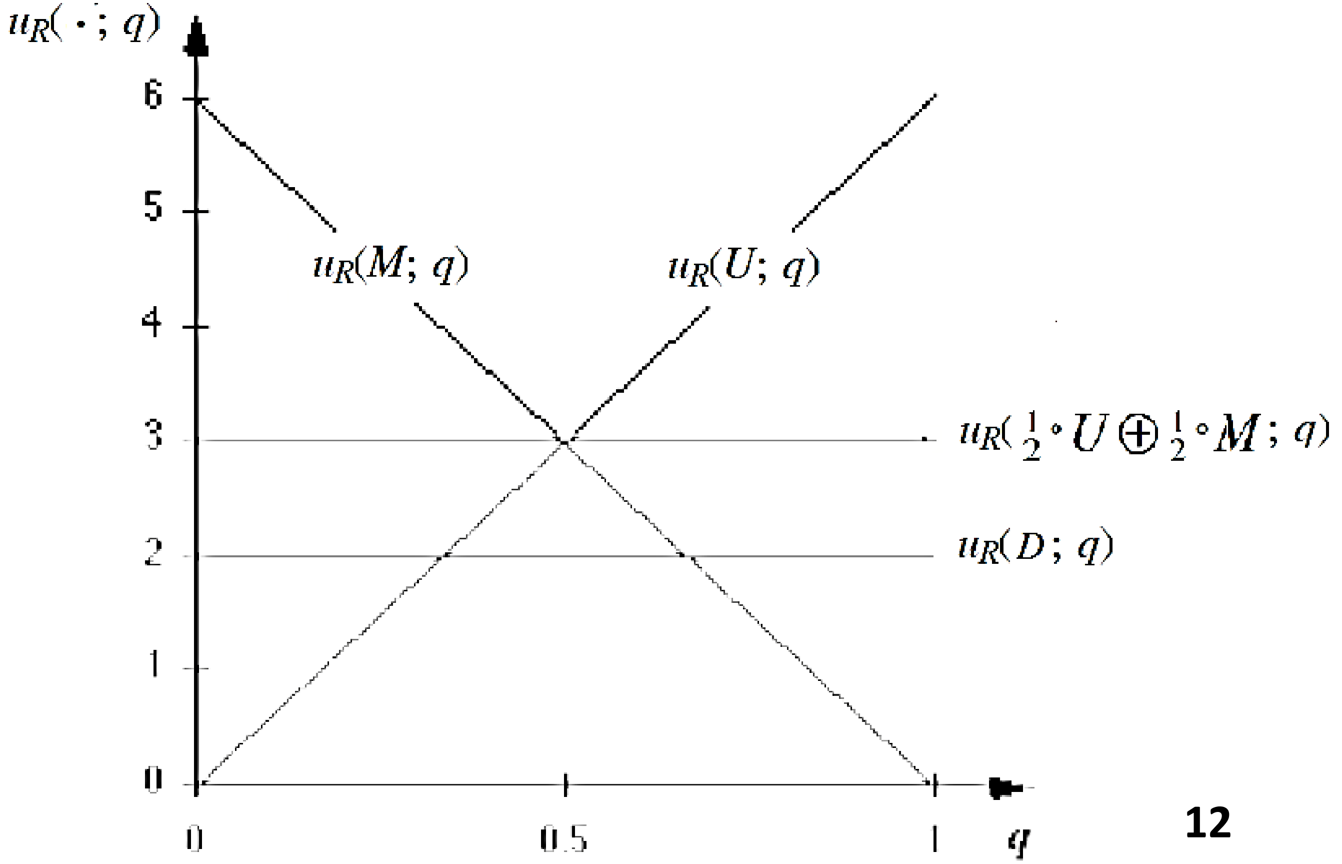
	l: [q]	r: [$1-q$]
U: [p]	6,0	0,6
M: [$1-p$]	0,6	6,0
D	2,0	2,0

- The intuition for the successful domination of Down by a mixture of Up and Middle can be more clearly explained when we consider Column's choice between left and right as a mixed strategy:

$$\sigma_C = q \circ l \oplus (1-q) \circ r.$$

$$u_R(U;\ q) = 6q + 0 \cdot (1-q) = 6q,$$

$$u_R(M;\ q) = 0 \cdot q + 6(1-q) = 6 - 6q.$$



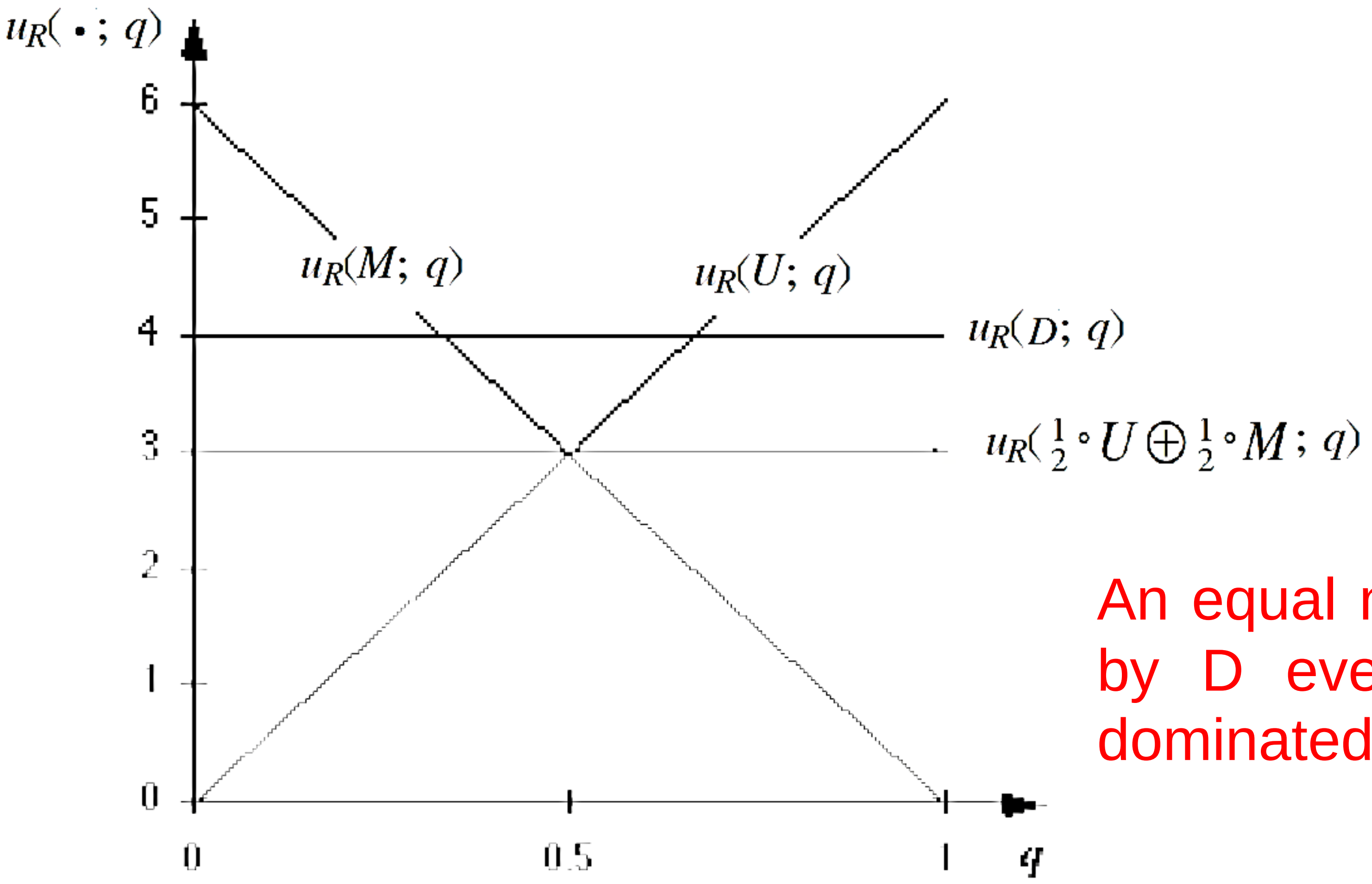
Dominated mixed strategies

- Any mixed strategy which puts positive probability on a dominated strategy is itself dominated.
  - It is easy to show that, if some mixed strategy σ_i has a dominated pure strategy in its support, you could construct another mixed strategy σ_i' which strictly dominates σ_i.
- However, this does not mean that any mixed strategy which puts positive probability only upon undominated pure strategies is necessarily undominated itself.
  - A non-degenerate mixed strategy σ_i can be dominated by another mixed strategy σ_i' (even by a pure strategy) even though σ_i puts no weight on dominated pure strategies.

Example: A mixed strategy over undominated pure strategies can be dominated.

- Consider the mixture $\sigma_R{}' = \frac{1}{2} \circ U \oplus \frac{1}{2} \circ M$

	l: $[q]$	r: $[1-q]$
U: $[p]$	6,0	0,6
M: $[1-p]$	0,6	6,0
D	4,0	4,0



An equal mixture of U and M is dominated by D even though neither U nor M is dominated!

Domination and never-a-best-response

Consider a strategy $\sigma_i \in \Sigma_i$ for player $i \in I$ and beliefs $\boldsymbol{\sigma}_{-i} \in \Sigma_{-i}$ which player i holds about the actions of the other players.

we say that σ_i is *never a best response* for i if

$$\boxed{\forall \boldsymbol{\sigma}_{-i} \in \Sigma_{-i},\ \exists \sigma_i' \in \Sigma_i,\ u_i(\langle \sigma_i', \boldsymbol{\sigma}_{-i} \rangle_i) > u_i(\langle \sigma_i, \boldsymbol{\sigma}_{-i} \rangle_i).} \quad **$$

If σ_i is a dominated strategy for player i, then there exists a strategy $\sigma_i' \in \Sigma_i$ which is better-for-i than σ_i regardless of the actions $\boldsymbol{\sigma}_{-i}$ of the other players; i.e.

$$\boxed{\exists \sigma_i' \in \Sigma_i,\ \forall \boldsymbol{\sigma}_{-i} \in \Sigma_{-i},\ u_i(\langle \sigma_i', \boldsymbol{\sigma}_{-i} \rangle_i) > u_i(\langle \sigma_i, \boldsymbol{\sigma}_{-i} \rangle_i).} \quad *$$

From (*) you can easily deduce (**); i.e.

a dominated strategy is never a best response.

However, (**) does not simply imply (*);

Domination and never-a-best-response (Cont'd)

- In two-player games: never-a-best-response⇔dominated
  - See "Jim Ratlif's Notes" for a Proof.
- Three or more players: never-a-best-response ⇏ dominated
  - We show this by exhibiting a three-player game in which player 3 will have a strategy which is never a best response to any pair of mixed strategies by the two opponents yet this strategy will not be dominated by any other strategy of player 3's.

		$[q]$ l	$[1-q]$ r
$[p]$	U	9	0
$[1-p]$	D	0	0

A

		$[q]$ l	$[1-q]$ r
$[p]$	U	0	9
$[1-p]$	D	9	0

B

		$[q]$ l	$[1-q]$ r
$[p]$	U	0	0
$[1-p]$	D	0	9

C

		$[q]$ l	$[1-q]$ r
$[p]$	U	6	0
$[1-p]$	D	0	6

D

- To Show “D” is undominated, we need to prove:
  - It cannot be dominated by other pure strategies: A,B, and C.
  - We cannot find a mixture of A,B, and C that dominate “D”.
  - In general, if for each <u>alternative</u> strategy, we show there is at least one opponent profile against which “D” is undominated, we can safely rule out that <u>alternative</u> strategy.

A

		$[q]$ l	$[1-q]$ r
$[p]$	U	9	0
$[1-p]$	D	0	0

B

		$[q]$ l	$[1-q]$ r
$[p]$	U	0	9
$[1-p]$	D	9	0

C

		$[q]$ l	$[1-q]$ r
$[p]$	U	0	0
$[1-p]$	D	0	9

D

		$[q]$ l	$[1-q]$ r
$[p]$	U	6	0
$[1-p]$	D	0	6

- **D** is not dominated by **A** against **(D,r)**!
- **D** is not dominated by **B** against **(D,r)**!
- **D** is not dominated by **C** against **(U,*l*)**!

A

		$[q]$ l	$[1-q]$ r
$[p]$	U	9	0
$[1-p]$	D	0	0

B

		$[q]$ l	$[1-q]$ r
$[p]$	U	0	9
$[1-p]$	D	9	0

C

		$[q]$ l	$[1-q]$ r
$[p]$	U	0	0
$[1-p]$	D	0	9

D

		$[q]$ l	$[1-q]$ r
$[p]$	U	6	0
$[1-p]$	D	0	6

- Now, we argue that there is no mixture of A, B, and C that can dominate "D" for <u>every</u> profile of the opponents:
- Take the following general mixed strategy: $\sigma_3 = r \circ A \oplus (1-r-s) \circ B \oplus s \circ C$

  $r, s \geq 0$ and $r + s \leq 1$
- Consider the profile (U,l) of opponents:
  - By playing "D", agent 3 can achieve payoff **6**.
  - By playing "σ_3", agent 3 can reach **9r**.
  - Therefore, in order for to dominate "D", we should have: **r>2/3**.

		$[q]$ l	$[1-q]$ r
$[p]$	U	9	0
$[1-p]$	D	0	0

A

		$[q]$ l	$[1-q]$ r
$[p]$	U	0	9
$[1-p]$	D	9	0

B

		$[q]$ l	$[1-q]$ r
$[p]$	U	0	0
$[1-p]$	D	0	9

C

		$[q]$ l	$[1-q]$ r
$[p]$	U	6	0
$[1-p]$	D	0	6

D

$$\sigma_3 = r \circ A \oplus (1 - r - s) \circ B \oplus s \circ C$$

$$r, s \geq 0 \text{ and } r + s \leq 1$$

- Now, consider the profile (D,r) of opponents:
  - By playing “D”, agent 3 can achieve payoff **6**.
  - By playing "σ_3”, agent 3 can reach **9s**.
  - Therefore, in order for to dominate “D”, we should have: **s>2/3**.

Contradiction! We have r>2/3 and s>2/3 and r+s<=1

A

		[q] l	[1−q] r
[p]	U	9	0
[1−p]	D	0	0

B

		[q] l	[1−q] r
[p]	U	0	9
[1−p]	D	9	0

C

		[q] l	[1−q] r
[p]	U	0	0
[1−p]	D	0	9

D

		[q] l	[1−q] r
[p]	U	6	0
[1−p]	D	0	6

- We concluded that “D” is undominated for agent 3.
- Now, we show that there is no opponent profile against which “D” is a best-response for player 3.
  - Therefore, while “D” is undominated, it is never-a-BR.
- We plot the graph of player 3’s payoffs against all mixed strategies of its opponents:

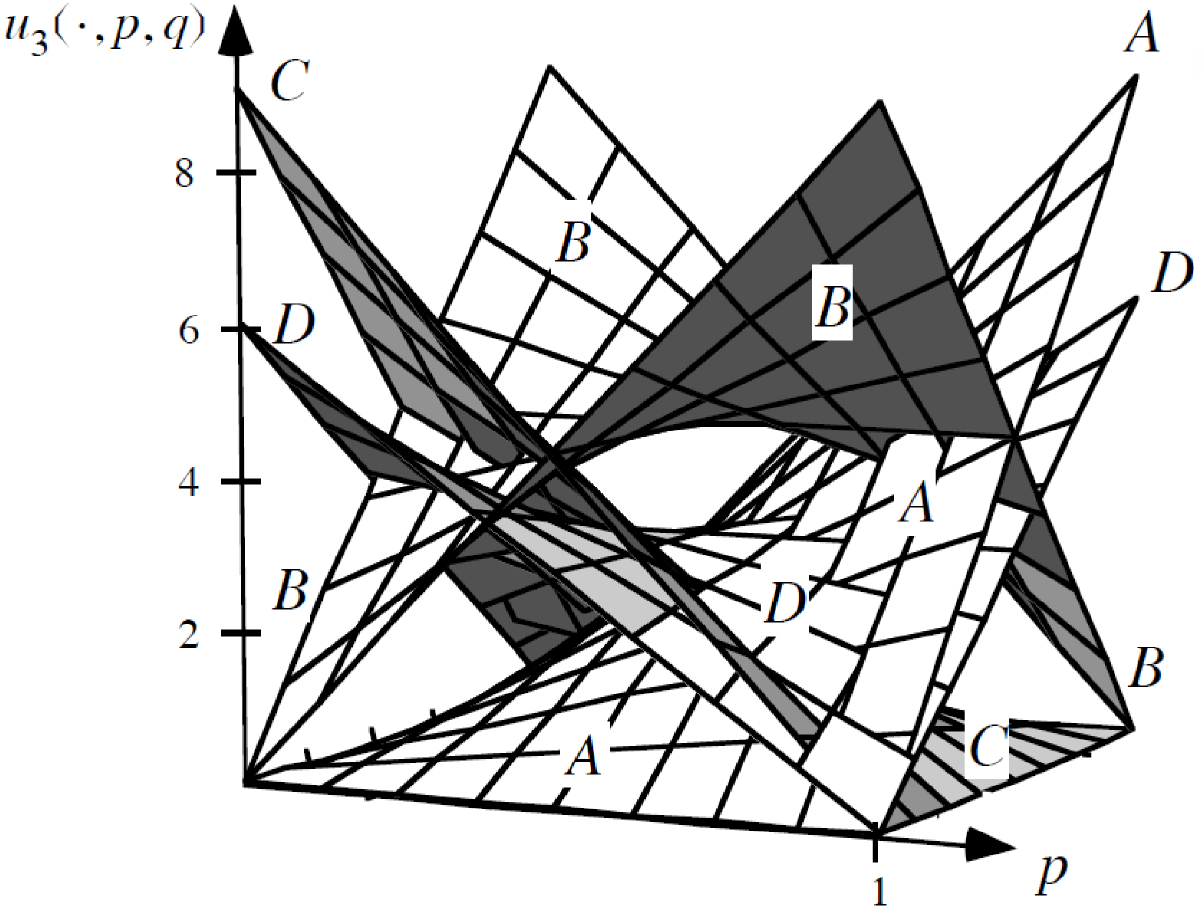
$$u_3(A;p,q)=9pq,$$

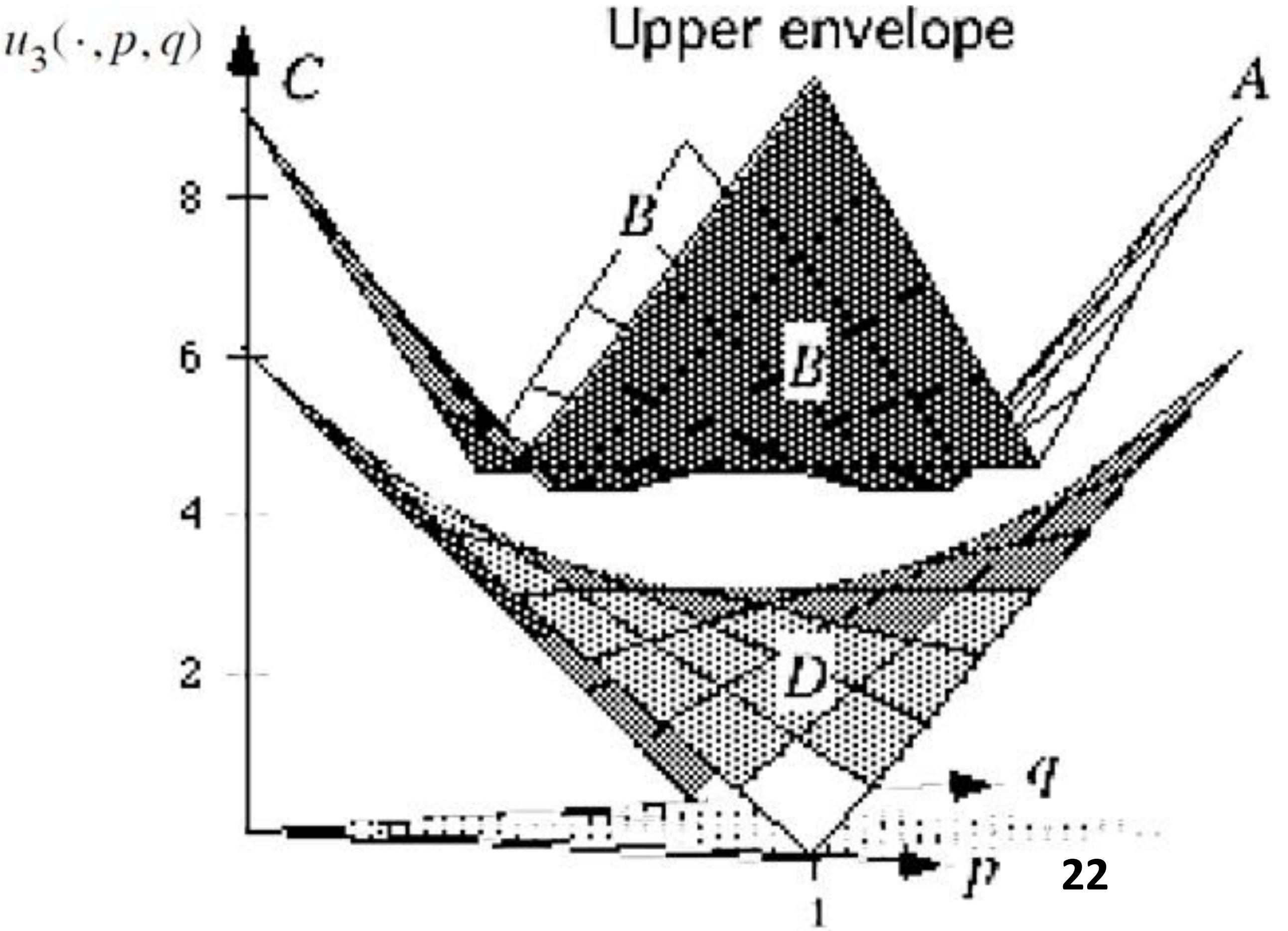
$$u_3(B;p,q)=9[p(1-q)+(1-p)q]=9(p+q-2pq),$$

$$u_3(C;p,q)=9(1-p)(1-q),$$

$$u_3(D;p,q)=6[pq+(1-p)(1-q)]=6(1+2pq-p-q)$$

- Note that there is no (p,q)-mixing of the opponents, for which player 3's payoff from "D" is part of the upper envelope of its payoffs ➔ There is no opponent profile against which D is a BR.





Iterated strict dominance

- We saw that in some games, e.g. the Prisoners' Dilemma, each player has a dominant strategy and we could therefore make a very precise prediction about the outcome of the game.
  - To achieve this conclusion we only needed to assume that each player was rational and knew her own payoffs.
- We also saw an example, viz. matching pennies, where dominance arguments got us nowhere—no player had any dominated strategies.
- There are games which lie between these two extremes: dominance analysis rejects some outcomes as impossible when the game is played by rational players but still leaves a multiplicity of outcomes.
- The technique we'll discuss now is called the ***iterated elimination of strictly dominated strategies***.
  - In order to employ it we will need to make stronger informational assumptions than we have up until now.

Iterated strict dominance

- Consider a two-player game between Row and Column, whose pure-strategy spaces are S_R and S_C, respectively.
- Prior to a dominance analysis of a game, we know only that the outcome will be one of the strategy profiles from the space of strategy profiles $\mathbf{S}=S_R \times S_C$.
- We reasoned that a rational player would never play a dominated strategy.
  - If Row has a dominated strategy, say $\tilde{s}_R$, but Column does not, then Row, being rational, would never play this strategy.
  - We could therefore confidently predict that the outcome of the game must be drawn from the smaller space of strategy profiles

$$S' = (S_R \setminus \{\tilde{s}_R\}) \times S_C.$$

Here is the interesting point and the key to the utility of the iterative process we're developing: Although Column had no dominated strategy in the original game, he may well have a dominated strategy $\tilde{s}_C$ in the new, smaller game S'.

Common Knowledge of Rationality

We had to make assumptions to justify the deletion of Column's dominated strategy $\tilde{s}_C$.

What assumptions are necessary for this step?

- First, Column must be rational.
- Additionally, in order for Column to see that $\tilde{s}_C$ is dominated for him, he must see that Row will never play $\tilde{s}_R$.
- Row will never play $\tilde{s}_R$ if she is rational; therefore we must assume that Column *knows* that Row is rational.

With these additional assumptions we can confidently predict that any outcome of the game must be drawn from:

$$S'' = (S_R \backslash \{\tilde{s}_R\}) \times (S_C \backslash \{\tilde{s}_C\}).$$

Common knowledge of rationality

- Let’s carry this out one more level:
- It may be the case that in the game defined by the strategy-profile space S” there is now a strategy of Row’s which is newly dominated, call it $\hat{s}_R$.
  - ➢ However, we can’t rule out that Row will play $\hat{s}_R$ unless we can assure that Row knows that the possible outcomes are indeed limited to S”, i.e. that Column will not choose $\tilde{s}_C$.
  - ➢ Column won’t choose $\tilde{s}_C$ if he is rational and knows that Row is rational.
  - Therefore we must assume that Row knows that Column is rational and knows that Column knows that Row is rational.

Common knowledge of rationality

- In any finite game this **chain of assumptions** can only be usefully carried out to a finite depth. To ensure that we can make such assumptions to an arbitrary depth we often make a convenient assumption: that it is ***common knowledge*** that all players are rational.
- *What does it mean for something to be common knowledge?*

Let $\mathscr{P}$ be a proposition, e.g. that "player 1 is rational."

If $\mathscr{P}$ is common knowledge, then

Everyone knows $\mathscr{P}$;
Everyone knows that (Everyone knows $\mathscr{P}$);
Everyone knows that [Everyone knows that (Everyone knows $\mathscr{P}$)];
Etc.

In other words, if $\mathscr{P}$ is common knowledge, then every statement of the form

$$\text{(Everyone knows that)}^k \text{ everyone knows } \mathscr{P},$$

is true for all $k \in \{0, 1, 2, \ldots\}$.

Example: Iterated strict dominance

- "step-by-step" presentation of the application of IDSDS

		P_2	
		Left	Right
P_1	Up	2,2	0,1
	Middle	1,2	1,0
	~~Down~~	~~0,-1~~	~~0,0~~

1st step

- First, player 1's utility satisfies:
  - u_1 (Middle,s_2) $>$ u_1 (Down,s_2) for any strategy s_2 that player 2 selects.
  - Hence, "DOWN" is strictly dominated for player 1, and we can delete it since he will never use it.
- Next step$\longrightarrow$

Example: Iterated strict dominance

- Hence, the remaining matrix after the first step of deleting a strictly dominated strategies is the following 2×2 matrix:

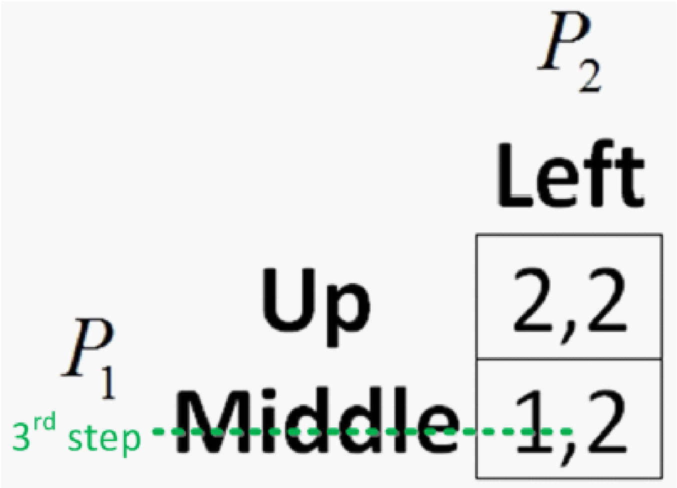
		P_2	
		Left	**Right**
P_1	**Up**	2,2	0,1
	Middle	1,2	1,0

2nd step

- Secondly, player 2's utility satisfies:
  - $u_2(\text{Left}, s_1) > u_2(\text{Right}, s_1)$ for any s_1 chosen by player 1.
  - Hence, "Right" is a strictly dominated strategy for player 2, and we can delete is since he will never select it
- Next step$\longrightarrow$

Example: Iterated strict dominance

- The remaining matrix after two steps of applying IDSDS is:



- In particular, player 1's utility satisfies:
  - u_1 (Up,s_2) $>$ u_1 (Middle,s_2) , i.e., $2 > 1$, s_2: only "Left".
  - Hence, "Middle" is a strategy dominated strategy for player 1, and we can delete it.
- Therefore, the only cell surviving IDSDS is that corresponding to strategy profile (Up,Left) with corresponding payoff $(2, 2)$.

Example: Iterated strict dominance

- There are no pure-strategy dominance relationships in the original game.
- However, the mixed strategy $\frac{1}{2} \circ U \oplus \frac{1}{2} \circ M$

  dominates Down.
- After deleting Down, left dominates right for Column.
- After deleting right, Up dominates Middle.
- Therefore the only possible outcome under common knowledge of rationality is **(U,l).**

	l	r
U	6,4	0,2
M	0,3	6,1
D	2,1	2,4

- **Definition of weakly dominated strategy**:
  - A strategy s_i^* is WEAKLY dominated by another strategy s_i' if the latter does at least as well as s_i^* against every strategy of one of the other players, and against some strategy it does strictly better.

$$u_i\left(s_i', s_{-i}\right) \geqslant u_i\left(s_i^*, s_{-i}\right) \text{ for all } s_{-i} \in S_{-i}$$
$$u_i\left(s_i', s_{-i}\right) > u_i\left(s_i^*, s_{-i}\right) \text{ for at least one } s_{-i} \in S_{-i}$$

IDWDS

Order of elimination matters: if we eliminate *weakly*(rather than strictly) dominated strategies.

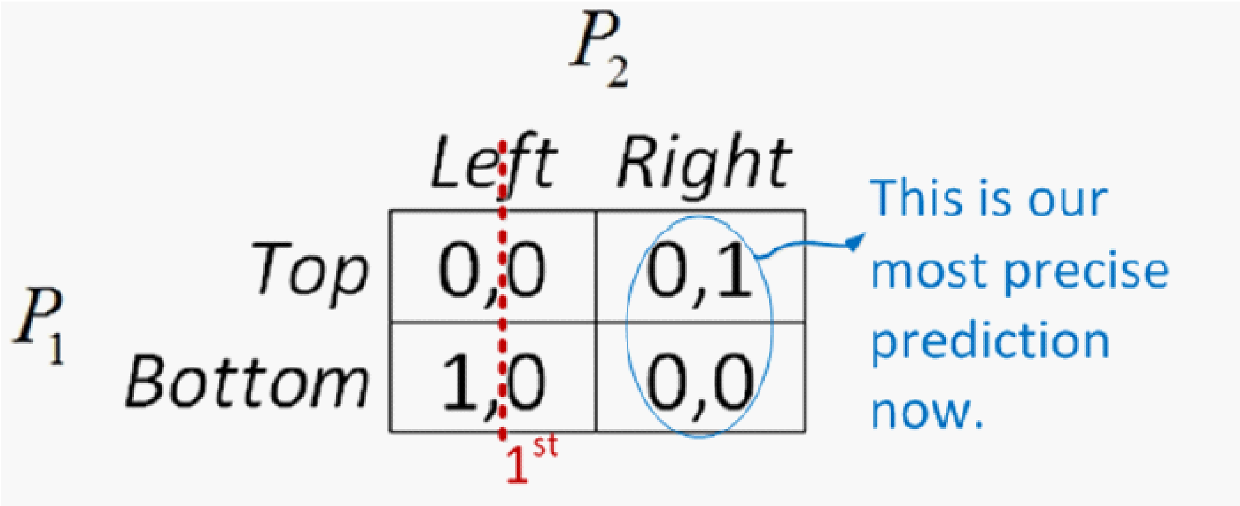
		P_2	
		Left	*Right*
P_1	1 st ~~*Top*~~	~~0,0~~	~~0,1~~
	Bottom	1,0	0,0

This is our most precise prediction.

- First, we eliminate Top as being weakly dominated by Bottom
- No further deletions for player 2 since he is indifferent between *Left* and *Right*.

IDWDS

- But what if we start by eliminating Left from Player 2 (it is a weakly dominated strategy for him).



- No further dominated strategies to delete since player 1 is indifferent between *Top* and *Bottom*.
- Bottom line: the set of strategies surviving IDWDS (NOT for IDSDS) depends on the order of deletion.

Rationalizability

- Common knowledge of rationality implies that the game's outcome must survive the IDSDS procedure.
  - We did *not* show that every surviving strategy could be reasonably chosen by a rational player.
  - A rational player must choose a **best response to her beliefs** about the actions of the other players.

- The ***rationalizable*** outcomes are those which survive the iterated elimination of strategies which are never best responses.

- Recall that in two-player games the rationalizable outcomes are exactly those which survive the IDSDS.

- In three-or-more–player games the set of rationalizable outcomes is a weakly smaller set than those survivors of IDSDS.

	w	*x*	*y*	*z*
A	7,5	-8,4	0,4	99,3
B	5,0	4,1	15,9	100,8
C	6,0	5,8	20,4	10,2
D	2,6	7,-10	3,9	10,8
E	1,6	2,-10	1,7	8,6

IDSDS

	w	*x*	*y*
A	7,5	-8,4	0,4
C	6,0	5,8	20,4
D	2,6	7,-10	3,9

Rationalizability as a consistent system of beliefs

- We defined the rationalizable outcomes as those which survived the iterated elimination of strategies which were never best responses.
- In order to focus explicitly on the constraints which common knowledge of rationality imposes upon players' beliefs, we will now discuss rationalizability from a different perspective:
  - Consider the strategy profile (C,x) in this game:

	w	*x*	*y*
A	**7,5**	**-8,4**	**0,4**
C	**6,0**	**5,8**	**20,4**
D	**2,6**	**7,-10**	**3,9**

- We will show that there exists a *consistent system of beliefs* for the players which justifies their choices—i.e. which shows that these choices do not conflict with the common knowledge of rationality assumption.

Rationalizability as a consistent system of beliefs

Let's establish some notation so that we can tractably talk about beliefs about beliefs about beliefs about….

- Let $\mathcal{R}$ and $\mathcal{C}$ stand for the Row and Column players, respectively.
- If Row chooses A, we write $\mathcal{R}$(A), and similarly for other choices by either player.
- If Column believes that Row will choose A, we
- write $\mathcal{C}\mathcal{R}$(A).
- If Column believes that Row believes that Column will choose y, we write $\mathcal{C}\mathcal{R}\mathcal{C}$(A), etc.

	w	*x*	*y*
A	**7,5**	**-8,4**	**0,4**
C	**6,0**	**5,8**	**20,4**
D	**2,6**	**7,-10**	**3,9**

Rationalizability as a consistent system of beliefs

	w	*x*	*y*
A	7,5	-8,4	0,4
C	6,0	5,8	20,4
D	2,6	7,-10	3,9

$\mathscr{R}\ (C)$ $\mathscr{R}$ plays C,

$\mathscr{R}\ \mathscr{C}\ \ (y)$ $\mathscr{R}$ believes $\mathscr{C}$ will play y,

$\mathscr{R}\ \mathscr{C}\ \ \mathscr{R}\ \ (D)$ $\mathscr{R}$ believes $\mathscr{C}$ believes $\mathscr{R}$ will play D,

$\mathscr{R}\ \mathscr{C}\ \ \mathscr{R}\ \ \mathscr{C}\ \ (x)$ $\mathscr{R}$ believes $\mathscr{C}$ believes $\mathscr{R}$ believes $\mathscr{C}$ will play x,

$\mathscr{R}\ \mathscr{C}\ \ \mathscr{R}\ \ \mathscr{C}\ \ \mathscr{R}\ \ (C)$ $\mathscr{R}$ believes $\mathscr{C}$ believes $\mathscr{R}$ believes $\mathscr{C}$ believes $\mathscr{R}$ will play C.

Rationalizability as a consistent system of beliefs

	w	*x*	*y*
A	7,5	-8,4	0,4
C	6,0	5,8	20,4
D	2,6	7,-10	3,9

$$\mathscr{C}\ (x)$$

$$\mathscr{C}\ \mathscr{R}\ (C)$$

$$\mathscr{C}\ \mathscr{R}\ \mathscr{C}\ (y)$$

$$\mathscr{C}\ \mathscr{R}\ \mathscr{C}\ \mathscr{R}\ (D)$$

$$\mathscr{C}\ \mathscr{R}\ \mathscr{C}\ \mathscr{R}\ \mathscr{C}\ (x)$$

Example Problem Discussion

Problem Discussion: Voting Game I

- Assume that there are 100 voters.
- They choose one of the three candidates: *A*, *B*, or *C*.
- The candidate is chosen with the probability proportional to the # of votes.
  - So, if there are 35 votes for *A*,
  - 65 votes for *B* and
  - 0 for *C*,
    - ✓ **then *A* is chosen with 35% probability, and *B* is chosen with 65% probability.**
- Assume that each voter *i* has preferences over candidates given by utilities: $u_i(A)$, $u_i(B)$, and $u_i(C)$ and that the preferences are strict.

Prove that voting for your favorite candidate is a strictly dominant strategy.

Solution

- To prove that a strategy is strictly dominant, we need to prove that that it brings about the highest utility *irrespective of* what strategies are chosen by other agents.

- We fix a player *i* and assume that (without loss of generality):

$$u_i(A) > u_i(B) > u_i(C)$$

- We will show that voting for *A* is a strictly dominant strategy for this player.

Take an arbitrary action profile of other agents and assume that there are:

- ✓ n_A other agents voting for *A*,
- ✓ n_B other agents choosing *B*, and
- ✓ n_C other agents voting for *C*.

(It holds that: $n_A + n_B + n_C = 99$).

Solution (Cont'd)

- The payoff of agent *i* is the expected value corresponding to the candidate selected from the voting procedure:
- The utility from strategy A is:

$$\frac{n_A + 1}{100} u_i(A) + \frac{n_B}{100} u_i(B) + \frac{n_C}{100} u_i(C).$$

- The utility from strategy B is:

$$\frac{n_A}{100} u_i(A) + \frac{n_B + 1}{100} u_i(B) + \frac{n_C}{100} u_i(C),$$

- The utility from strategy C is:

$$\frac{n_A}{100} u_i(A) + \frac{n_B}{100} u_i(B) + \frac{n_C + 1}{100} u_i(C).$$

Solution (Cont'd)

- Now contrast the utilities obtained from the three strategies:

  - The utility from strategy *A* minus the utility from *B* is:

$$\frac{n_A + 1}{100} u_i(A) + \frac{n_B}{100} u_i(B) + \frac{n_C}{100} u_i(C)$$

$$- \left(\frac{n_A}{100} u_i(A) + \frac{n_B + 1}{100} u_i(B) + \frac{n_C}{100} u_i(C) \right)$$

$$= \frac{1}{100}(u_i(A) - u_i(B)) > 0.$$

  - The last inequality is due to: A being strictly better than B.

  - Likewise, we argue that the utility from *A* is strictly better than the payoff from *C*…

Problem Discussion: Voting Game II

- There are *N* individuals.
- Three items: *A*, *B*, and *C*.
- Each person casts one vote.
- The item with the **least** # of votes wins.
- Ties are resolved by selecting the item with equal probability among all the items with the lest # of votes.

1. Assume that for person *i*, we have: $u_i(A) > u_i(B) > u_i(C)$. Does he have a strictly dominant strategy?
2. Does he have a weakly dominant strategy?
3. Does he have a weakly dominated strategy?

Solution

- *Part 1*. No. We will prove that agent *i* does not have a weakly dominant strategy, which implies that there is no strictly dominant strategy!

- *Part 2*. No. First, we show that voting *C* is not weakly dominant:
  - Let n_A= # of votes cast by other agents for item A; likewise define n_B and n_C.
  - Assume that $n_A = n_B < n_C$.
    - Now, if person *i* votes for *B*, then *A* will be chosen.
    - But if *i* votes for *C*, then the voting machine selects equi-probably between *A* and *B*.
    - As $u_i(A)$ is strictly better, *i* would strictly rather vote for *B*.

Solution (Cont’d)

- Next, we will prove that casting vote for *B* is not weakly dominant.
  - Assume that: $n_B = n_C < n_A$.
    - Then, voting for *B* results in *C* being selected;
    - While, voting for *C* leads to *B* being chosen.
    - Hence, in this case, voting for *C* results in a strictly better utility.
- A similar reasoning will prove that casting vote for *A* is not weakly dominant!

Solution (Cont'd)

- *Part 3*. Yes. Casting vote for A is weakly dominated by C.
  - Assume that: $n_A = n_C$ (both having the smallest # of votes):
    - Voting for C results in a strictly higher utility.
  - But, in general, *i*'s utility might either get higher or remain unchanged if *i* changes her vote from A to C; e.g.,

When $n_C \ll n_B < n_A$, *i* would be indifferent between voting for A and C.

($\ll$: much less than)

❑