Game Theory

Lecture 07

<u>Introduction to Computing Game-Theoretic Solution Concepts</u>

A matrix view of zero-sum games

A finite 2-person zero-sum (2p-zs) strategic game Γ, is a strategic game where:

- For players $i \in \{1, 2\}$, the *payoff functions* $u_i : S \mapsto \mathbb{R}$ are such that for all $s = (s_1, s_2) \in S$,

$$u_1(s) + u_2(s) = 0$$

I.e., $u_1(s) = -u_2(s)$.

$u_i(s_1, s_2)$ can conveniently be viewed as a $m_1 \times m_2$ payoff matrix A_i, where:

$$A_1 = \begin{bmatrix} u_1(1,1) & \ldots\ldots & u_1(1,m_2) \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ u_1(m_1,1) & \ldots\ldots & u_1(m_1,m_2) \end{bmatrix}$$

Note, $A_2 = -A_1$. Thus we may assume only one function $u(s_1, s_2)$ is given, as one matrix, $A = A_1$. Player 1 wants to maximize $u(s)$, while Player 2 wants to minimize it (i.e., to maximize its negative).

matrices and vectors

As just noted, a 2p-zs game can be described by an $m_1 \times m_2$ matrix:

$$A = \begin{bmatrix} a_{1,1} & \cdots\cdots & a_{1,m_2} \\ \vdots & \vdots & \vdots \\ \vdots & a_{i,j} & \vdots \\ \vdots & \vdots & \vdots \\ a_{m_1,1} & \cdots\cdots & a_{m_1,m_2} \end{bmatrix}$$

where $a_{i,j} = u(i,j)$.

For any $(n_1 \times n_2)$-matrix A we'll either use $a_{i,j}$ or $(A)_{i,j}$ to denote the entry in the i'th row and j'th column of A.

For $(n_1 \times n_2)$ matrices A and B, let

$$A \geq B$$

denote that for all i, j, $a_{i,j} \geq b_{i,j}$.
Let

$$A > B$$

denote that for all i, j, $a_{i,j} > b_{i,j}$.

For a matrix A, let $A \geq 0$ denote that every entry is ≥ 0. Likewise, let $A > 0$ mean every entry is > 0.

more review of matrices and vectors

Recall matrix multiplication: given $(n_1 \times n_2)$-matrix A and $(n_2 \times n_3)$-matrix B, the product AB is an $(n_1 \times n_3)$-matrix C, where

$$c_{i,j} = \sum_{k=1}^{n_2} a_{i,k} * b_{k,j}$$

Fact: matrix multiplication is "associative": i.e.,

$$(AB)C = A(BC)$$

(Note: for the multiplications to be defined, the dimensions of the matrices A, B, and C need to be "consistent": $(n_1 \times n_2)$, $(n_2 \times n_3)$, and $(n_3 \times n_4)$, respectively.)

Fact: For matrices A, B, C, of appropriate dimensions, if $A \geq B$, and $C \geq 0$, then

$AC \geq BC$, and likewise, $CA \geq CB$.

(C's dimensions might be different in each case.)

more on matrices and vectors

For a $(n_1 \times n_2)$ matrix B, let B^T denote the $(n_2 \times n_1)$ **transpose** matrix, where $(B^T)_{i,j} := (B)_{j,i}$.

We can view a column vector, $y = \begin{bmatrix} y(1) \\ \vdots \\ \vdots \\ y(m) \end{bmatrix}$, as a $(m \times 1)$-matrix. Then, y^T would be a $(1 \times m)$-matrix, i.e., a row vector.

Typically, we think of "vectors" as column vectors and explicitly transpose them if we need to. We'll call a length m vector an m-vector.

Multiplying a $(n_1 \times n_2)$-matrix A by a n_2-vector y is just a special case of matrix multiplication:
Ay is a n_1-vector.

Likewise, pre-multiplying A, by a n_1-row vector y^T, is also just a special case of matrix multiplication:
$y^T A$ is a n_2-row vector.

For a column (row) vector y, we use $(y)_j$ to denote the entry $(y)_{j,1}$ (respectively, $(y)_{1,j}$).

Example

Suppose we have a 2p-zs game given by a $(m_1 \times m_2)$-matrix, A.

Suppose Player 1 chooses a mixed strategy x_1, and Player 2 chooses mixed strategy x_2 (assume x_1 and x_2 are given by column vectors). Consider the product

$$x_1^T A x_2$$

If you do the calculation,

$$x_1^T A x_2 = \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} (x_1(i) * x_2(j)) * a_{i,j}$$

But note that $(x_1(i) * x_2(j))$ is precisely the probability of the pure combination $s = (i, j)$. Thus, for the mixed profile $x = (x_1, x_2)$

$$x_1^T A x_2 = U_1(x) = -U_2(x)$$

where $U_1(x)$ is the expected payoff (which Player 1 is trying to maximize, and Player 2 is trying to minimize).

minimax as an optimization problem

Consider the following "optimization problem":

Maximize v

Subject to constraints:

$(x_1^T A)_j \geq v$ for $j = 1, \ldots, m_2$,

$x_1(1) + \ldots + x_1(m_1) = 1$,

$x_1(j) \geq 0$ for $j = 1, \ldots, m_1$

It follows that an optimal solution (x_1^*, v^*) would give precisely the maxmin value v^*, and a minmaximizer x_1^* for Player 1.

We are optimizing a "linear objective", under "linear constraints" (or "linear inequalities").

That's what Linear Programming is.

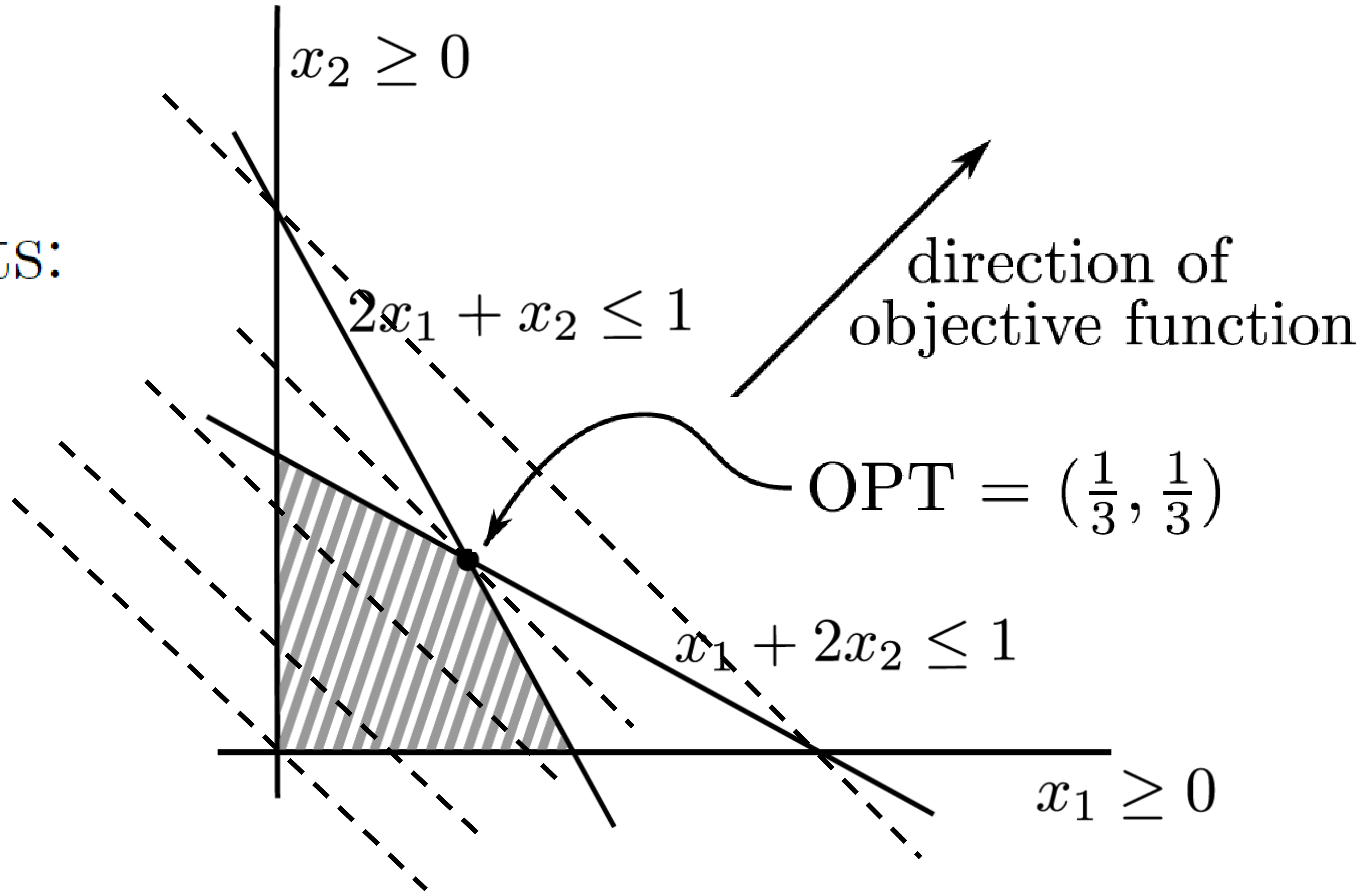
Fortunately, we have good algorithms for it.

A toy example of linear program

$$\max\ x_1 + x_2$$

four (linear) constraints:

$$\begin{aligned} x_1 &\geq 0 \\ x_2 &\geq 0 \\ 2x_1 + x_2 &\leq 1 \\ x_1 + 2x_2 &\leq 1. \end{aligned}$$



- Geometrically, the objective function asks for the feasible point furthest in the direction of the coefficient vector (1,1) - the most “northeastern“ feasible point.
- Put differently, the <u>level sets</u> of the objective function are parallel lines running southwest to northeast.
- Eyeballing the feasible region, the optimal point is (1/3,1/3) for an optimal objective function value of 2/3.
- This is the “<u>last point of intersection</u>" between a level set of the objective function and the feasible region (as one sweeps from southwest to northeast).
- The geometric intuition above remains valid for general linear programs, with an arbitrary number of dimensions (i.e., decision variables) and constraints.

The General Linear Program

Definition: A *Linear Programming* or *Linear Optimization* problem instance

$$(f, \texttt{Opt}, C)$$

consists of

1. A linear *objective function* $f : \mathbb{R}^n \mapsto \mathbb{R}$, given by:

   $$f(x_1, \ldots, x_n) = c_1\, x_1 + c_2\, x_2 + \ldots + c_n\, x_n + d$$

   where we assume the coefficients c_i and constant d are rational numbers.

2. An optimization criterion:
   $\texttt{Opt} \in \{\texttt{Maximize}, \texttt{Minimize}\}$.

3. A set (or "system") $C(x_1, \ldots, x_n)$ of m *linear constraints*, or linear inequalities/equalities,

   $C_i(x_1, \ldots, x_n), \quad i = 1, \ldots, m,$
   where each $C_i(x)$ has the form:

   $$a_{i,1}\, x_1 + a_{i,2}\, x_2 + \ldots + a_{i,n}\, x_n \;\Delta\; b_i$$

   where $\Delta \in \{\leq, \geq, =\}$,
   and where $a_{i,j}$'s and b_i's are rational numbers.

What does it mean to solve an LP?

For a constraint $C_i(x_1, \ldots, x_n)$, we say a vector $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$ *satisfies* $C_i(x)$ if, plugging in v for the variables $x = (x_1, \ldots, x_n)$, the constraint $C_i(v)$ holds true. E.g., $(3, 6)$ satisfies $-x_1 + x_2 \leq 7$.

A vector $v \in \mathbb{R}^n$ is called a *solution* to the system $C(x)$, if v satisfies every constraint $C_i \in C$. I.e., $C_1(v) \wedge \ldots \wedge C_m(v)$ holds true.

Let $K(C) \subseteq \mathbb{R}^n$ denote the set of all solutions to the system $C(x)$. We say C is **feasible** if $K(C)$ is not empty.

An *optimal solution*, for $\texttt{Opt} = \texttt{Maximize}$ ($\texttt{Minimize}$), is some $x^* \in K(C)$ such that

$$f(x^*) = \max_{x \in K(C)} f(x)$$

(respectively, $f(x^*) = \min_{x \in K(C)} f(x)$).

Given an LP problem $(f, \texttt{Opt}, C)$, our goal in principle is to find an "optimal solution".

Oops!! There may not be an optimal solution!

Things that can go wrong

At least two things can go wrong when looking for an optimal solution:

1. There may be no solutions at all! I.e., C is not feasible, i.e., $K(C)$ is empty. Consider:

   Maximize x
   Subject to:
   $x \leq 3$, and $x \geq 5$

2. $\max / \min_{x \in K(C)} f(x)$ may not exist! because $f(x)$ is unbounded above/below in $K(C)$. Consider:

   Maximize x
   Subject to:
   $x \geq 5$

Note: If we allowed strict inequalities, e.g., $x < 5$, there would have been yet another problem:

Maximize x
Subject to:
$x < 5$

The LP Problem Statement

Given an LP problem instance $(f, \texttt{Opt}, C)$ as input, output one of the following three:

1. "The problem is Infeasible."
2. "The problem is Feasible But Unbounded."
3. "An Optimal Feasible Solution (OFS) exists.
   One such optimal solution is $x^* \in \mathbb{R}^n$.
   The optimal objective value is $f(x^*) \in \mathbb{R}$."

Oops!! It seems yet another thing could go wrong:

"What if every optimal solution $x^* \in \mathbb{R}^n$ is irrational?
How can we "output" irrational numbers?
Likewise, what if the Opt value $f(x^*)$ is irrational?"

Fact

As we will soon see, this problem never arises. The above three answers cover all possibilities, and furthermore, as long as all our coefficients and constants are rational, if an OFS exists, there will be a rational OFS x^* and the optimal value $f(x^*)$ will also be rational.

Simplified forms for LP problems

1. In principle, we only need to consider Maximization, because
$$\min_{x \in K} f(x) = -\max_{x \in K} -f(x)$$

2. In principle, we only need an objective function $f(x_1, \ldots, x_n) = x_i$, for some x_i, because we can
   - Introduce new variable x_0. Add constraint $f(x) = x_0$ to the constraint set C.
   - Make the new objective "Optimize x_0".

3. We don't need equality constraints, because $\alpha = \beta$ if and only if ($\alpha \leq \beta$ and $\alpha \geq \beta$).

4. We don't need "$\alpha \geq b$", where $b \in \mathbb{R}$, because $\alpha \geq b$ if and only if $-\alpha \leq -b$.

5. We can constrain every variable x_i to be $x_i \geq 0$: Introduce two variables x_i^+, x_i^- for each variable x_i. Replace each occurence of x_i by $(x_i^+ - x_i^-)$, and add the constraints $x_i^+ \geq 0$, $x_i^- \geq 0$.

A lovely but terribly inefficient algorithm for LP

Input: LP instance $(x_0, \texttt{Opt}, C(x_0, x_1, \ldots, x_n))$.

1. **For** $i = n$ downto 1

   (a) Rewrite every constraint involving x_i as either:
   $$\alpha \leq x_i \text{ or as } x_i \leq \beta$$
   Let these be:
   $\alpha_1 \leq x_i, \ldots, \alpha_k \leq x_i \;\; ; \;\; x_i \leq \beta_1, \ldots, x_i \leq \beta_r$
   (Retain these constraints, H_i, for later.)

   (b) Remove H_i, i.e., all constraints involving x_i.
   Replace them with all constraints:
   $\alpha_j \leq \beta_l$, $j = 1, \ldots, k$, and $l = 1, \ldots, r$.

2. Only x_0 (or no variable) remains. All constraints have the forms $a_j \leq x_0$, $x_0 \leq b_l$, or $a_j \leq b_l$, where a_j's and b_l's are constants. It's easy to check "feasibility" & "boundedness" for this one(or zero)-variable LP, and to find an optimal x_0^* if it exists.

3. Once you have x_0^*, plug it into H_1. Solve for x_1^*.
   Then use x_0^*, x_1^* in H_2 to solve for x_2^*,
   $\ldots$, use $x_0^*, \ldots, x_{i-1}^*$ in H_i to solve for x_i^*. $\ldots$
   $x^* = (x_0^*, \ldots, x_n^*)$ is an optimal feasible solution.

Example

As an example, let us consider the following system of 5 inequalities in 3 variables:

$$\begin{array}{rcrcrcr} 2x & - & 5y & + & 4z & \leq & 10 \\ 3x & - & 6y & + & 3z & \leq & 9 \\ 5x & + & 10y & - & z & \leq & 15 \\ -x & + & 5y & - & 2z & \leq & -7 \\ -3x & + & 2y & + & 6z & \leq & 12. \end{array} \qquad (1)$$

In the first step we would like to eliminate x.

For a moment let us imagine that y and z are some fixed real numbers, and let us ask under what conditions we can choose a value of x such that together with the given values y and z it satisfies (1)

The first three inequalities impose an upper bound on x, while the remaining two impose a lower bound.

To make this clearer, we rewrite the system as follows:

$$\begin{array}{rcrcrcr} x & \leq & 5 & + & \frac{5}{2}y & - & 2z \\ x & \leq & 3 & + & 2y & - & z \\ x & \leq & 3 & - & 2y & + & \frac{1}{5}z \\ x & \geq & 7 & + & 5y & - & 2z \\ x & \geq & -4 & + & \frac{2}{3}y & + & 2z. \end{array}$$

Example (Cont'd)

$$
\begin{array}{rcrcrcr}
x & \leq & 5 & + & \frac{5}{2}y & - & 2z \\
x & \leq & 3 & + & 2y & - & z \\
x & \leq & 3 & - & 2y & + & \frac{1}{5}z \\
x & \geq & 7 & + & 5y & - & 2z \\
x & \geq & -4 & + & \frac{2}{3}y & + & 2z.
\end{array}
$$

So given y and z, the admissible values of x are exactly those in the interval from

$\boxed{\max(7+5y-2z, -4+\frac{2}{3}y+2z)}$

to $\boxed{\min(5+\frac{5}{2}y-2z, 3+2y-z, 3-2y+\frac{1}{5}z).}$

If this interval happens to be empty, there is no admissible x.

So the inequality

$$
\begin{aligned}
&\max(7 + 5y - 2z, -4 + \tfrac{2}{3}y + 2z) \\
&\quad \leq \min(5 + \tfrac{5}{2}y - 2z, 3 + 2y - z, 3 - 2y + \tfrac{1}{5}z)
\end{aligned} \tag{2}
$$

is *equivalent* to the existence of x that together with the the considered y and z solves (1)

Example (Cont'd)

$$\max(7 + 5y - 2z, -4 + \tfrac{2}{3}y + 2z) \qquad (2)$$
$$\leq \min(5 + \tfrac{5}{2}y - 2z, 3 + 2y - z, 3 - 2y + \tfrac{1}{5}z)$$

The key observation in the Fourier–Motzkin elimination is that (2) can be rewritten as a system of linear inequalities in the variables y and z.

The inequalities simply say that *each of the lower bounds is less than or equal to each of the upper bounds*:

$$\begin{aligned}
7 + 5y - 2z &\leq 5 + \tfrac{5}{2}y - 2z \\
7 + 5y - 2z &\leq 3 + 2y - z \\
7 + 5y - 2z &\leq 3 - 2y + \tfrac{1}{5}z \\
-4 + \tfrac{2}{3}y + 2z &\leq 5 + \tfrac{5}{2}y - 2z \\
-4 + \tfrac{2}{3}y + 2z &\leq 3 + 2y - z \\
-4 + \tfrac{2}{3}y + 2z &\leq 3 - 2y + \tfrac{1}{5}z.
\end{aligned}$$

Example (Cont'd)

If we rewrite this system in the usual form $A\mathbf{x} \leq \mathbf{b}$, we arrive at

$$\begin{array}{rcrcr}
\frac{5}{2}y & & & \leq & -2 \\
3y & - & z & \leq & -4 \\
7y & - & \frac{11}{5}z & \leq & -4 \\
-\frac{11}{6}y & + & 4z & \leq & 9 \\
-\frac{4}{3}y & + & 3z & \leq & 7 \\
\frac{8}{3}y & + & \frac{9}{5}z & \leq & 7.
\end{array} \qquad (3)$$

This system has a solution exactly if the original system (1) has one, but it has one variable fewer.
you may continue with this example, eliminating y and then z.

We note that (3) gives 4 upper bounds for y and 2 lower bounds, and hence we obtain 8 inequalities after eliminating y.

For larger systems the number of inequalities generated by the Fourier– Motzkin elimination tends to explode.

This wasn't so apparent for our small example, but if we have m inequalities and, say, half of them impose upper bounds on the first variable and half impose lower bounds, then we get about $m^2/4$ inequalities after eliminating the first variable, about $m^4/64$ after eliminating the second variable (again, provided that about half of the inequalities give upper bounds for the second variable and half lower bounds), etc.

remarks on the lovely algorithm

- This algorithm was first discovered by Fourier (1826). It was rediscovered in the 1900's, by Motzkin (1936) among others.

- It is called *Fourier-Motzkin Elimination*, and can be viewed as a generalization of *Gaussian Elimination*, used for solving systems of linear equalities.

- Why is Fourier-Motzkin so inefficient? In the worst case, if every variable x_i is involved in every constraint, each iteration of the "For loop" squares the number of constraints. So, toward the end we could have roughly m^{2^n} constraints!!

- In 1947, Dantzig invented the celebrated Simplex Algorithm for LP. It can be viewed as a much more refined generalization of Gaussian Elimination. Next time, Simplex!

further remarks

- Immediate Corollary of Fourier-Motzkin:

  If an LP has an OFS, then it has a rational OFS, x^*, and $f(x^*)$ is also rational.

  Proof: We used only addition, multiplication, & division by rationals to arrive at the solution. ∎

- Although Fourier-Motzkin is bad in the worst case, it can still be quite useful.
  It can be used to remove redundant variables. Redundant constraints could also be removed, and sometimes the worst-case may not arise.

- Generalizations of Fourier-Motzkin are actually used in competitive tools (e.g., [Pugh,'92]) to solve "Integer Linear Programming", where we seek an optimal solution x^* not in $\mathbb{R}^n$, but in $\mathbb{Z}^n$. ILP is a much harder problem! (NP-complete.)

- For ordinary LP however, Fourier-Motzkin can't compete with Simplex.