In the name of God

Network Flows

1. Introduction
1.1 Introduction

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Outline

- **Introduction**
- **Minimum Cost Flow Problem**

Introduction

Introduction

- Everywhere we look in our daily lives, networks are apparent.
  - Electrical and power networks
  - Telephone networks
  - National highway systems,
  - Rail networks,
  - Airline service networks
  - Manufacturing and distribution networks
  - Computer networks, such as airline reservation systems

Introduction

- In all of these problem domains,
  - we wish to move some entity (electricity, a consumer product, a person or a vehicle, a message) from one point to another in an underlying network,
  - and to do so as efficiently as possible,
  - both to provide good service to the users of the network and to use the underlying (and typically expensive) transmission facilities effectively.

Introduction

- We want to learn:
  - how to model application settings as mathematical objects known as network flow problems
  - Various ways (algorithms) to solve the resulting models.

Introduction

- Network flows is a problem domain that lies at the several fields including:
  - applied mathematics
  - computer science
  - engineering,
  - management
  - operations research

Minimum Cost Flow Problem

Minimum Cost Flow Problem

- We wish to determine a least cost shipment of a commodity through a network in order to satisfy demands at certain nodes from available supplies at other nodes.
- It is the most fundamental of all network flow problems.
- Some applications:
  - the distribution of a product from manufacturing plants to warehouses, or from warehouses to retailers;
  - the flow of raw material and intermediate goods through the various machining stations in a production line;
  - the routing of automobiles through an urban street network; and the routing of calls through the telephone system.

Minimum Cost Flow Problem

- Notation:
  - $\boldsymbol{G} = (\boldsymbol{N}, \boldsymbol{A})$ be a directed network defined by a set N of n nodes and a set A of *m directed arcs*.
  - $\boldsymbol{c_{ij}}$: cost c_{ij} that denotes the cost per unit flow on that arc
  - We assume that the flow cost varies linearly with the amount of flow.
  - $\boldsymbol{u_{ij}}$: a capacity that denotes the maximum amount that can flow on the arc
  - $\boldsymbol{l_{ij}}$: a lower bound l_{ij} that denotes the minimum amount that must flow on the arc.

Minimum Cost Flow Problem

- Notation:
  - ***b*(*i*)** : an integer number $b(i)$ representing its supply/demand for each node $i \in N$
    - If $b(i) > 0$, node i is a supply node;
    - if $b(i) < 0$, node i is a demand node with a demand of - $b(i)$;
    - if $b(i) = 0$, node i is a transshipment node.
  - $\boldsymbol{x_{ij}}$: arc flows as decision variables

Minimum Cost Flow Problem

- The minimum cost flow problem is an optimization model formulated as follows:

$$\text{Minimize} \sum_{(i,j)\in A} c_{ij}x_{ij}$$

subject to

$$\sum_{\{j:(i,j)\in A\}} x_{ij} - \sum_{\{j:(j,i)\in A\}} x_{ji} = b(i) \qquad \text{for all } i \in N,$$

$$l_{ij} \le x_{ij} \le u_{ij} \qquad \text{for all } (i,j) \in A,$$

where $\sum_{i=1}^{n} b(i) = 0$

Minimum Cost Flow Problem

- In matrix form, we represent the minimum cost flow problem as follows:

$$\text{Minimize } cx$$

$$\text{subject to} \qquad \mathcal{N}x = b,$$

$$l \leq x \leq u.$$

  - $\mathcal{N}$ is an $n \times m$ matrix, called the ***node-arc incidence matrix***
  - Each column $\mathcal{N}_{ij}$ in the matrix corresponds to the variable x_{ij}.
  - The column $\mathcal{N}_{ij}$ has a +1 in the ith row, a -1 in the jth row; the rest of its entries are zero.

Minimum Cost Flow Problem

- ***Mass balance constraints***

$$\sum_{\{j:(i,j)\in A\}} x_{ij} - \sum_{\{j:(j,i)\in A\}} x_{ji} = b(i) \qquad \text{for all } i \in N,$$

  - The first term in this constraint for a node represents the total ***outflow*** of the node
  - the second term represents the total ***inflow*** of the node
  - The mass balance constraint states that the outflow minus inflow must equal the supply/demand of the node.
  - If the node is a supply node, its outflow exceeds its inflow; if the node is a demand node, its inflow exceeds its outflow; and if the node is a transshipment node, its outflow equals its inflow.

Minimum Cost Flow Problem

- ***Flow bound constraints***

$$l_{ij} \le x_{ij} \le u_{ij} \qquad \text{for all } (i,j) \in A,$$

  - The flow must also satisfy the lower bound and capacity constraints
  - The flow bounds typically model physical capacities or restrictions imposed on the flows' operating ranges.
  - In most applications, the lower bounds on arc flows are zero; therefore, if we do not state lower bounds for any problem, we assume that they have value zero.

Minimum Cost Flow Problem

- **Minimum Cost Flow Problem:**
  - Shortest path problem
  - Maximum flow problem
  - Assignment problem
  - Transportation problem
  - Circulation problem
  - Convex cost flow problems
  - Generalized flow problems
  - Multicommodity flow problems

Shortest path problem

- ***Shortest path problem*** is the simplest of all network flow problems.
- We wish to find a path of minimum cost (or length) from a specified ***source node s*** to another specified ***sink node t***,
- assuming that each arc $(i, j) \in A$ has an associated cost (or length) c_{ij}.
- Some of the simplest applications:
  - to determine a path between two specified nodes of a network that has minimum length, or
  - a path that takes least time to traverse, or
  - a path that has the maximum reliability.

Shortest path problem

- If we set $b(s) = 1$, $b(t) = -1$, and $b(i) = 0$ for all other nodes in the minimum cost flow problem,
  – the solution to the problem will send 1 unit of flow from node s to node t along the shortest path.
- The shortest path problem also models situations in which we wish to send flow from a single-source node to a single-sink node in an uncapacitated network.
  – If we wish to send v units of flow from node s to node t and the capacity of each arc of the network is at least v, we would send the flow along a shortest path from node s to node t.

Shortest path problem

- If we want to determine shortest paths from the source node s to every other node in the network, then in the minimum cost flow problem we set $b(s) = (n - 1)$ and $b(i) = -1$ for all other nodes.
  - We can set each arc capacity u_{ij} to any number larger than (n - 1).
  - The minimum cost flow solution would then send unit flow from node s to every other node i along a shortest path.

Maximum flow problem

- The ***maximum flow problem*** seeks a feasible solution that sends the maximum amount of flow from a specified source node s to another specified sink node t.
- In the maximum flow problem flow incurs no costs but is restricted by flow bounds.
- If we interpret u_{ij} as the maximum flow rate of arc (i, j), the maximum flow problem identifies the maximum steady-state flow that the network can send from node s to node t per unit time.

Maximum flow problem

- Examples of the maximum flow problem include determining the maximum steady-state flow of :
  - petroleum products in a pipeline network
  - cars in a road network
  - messages in a telecommunication network
  - electricity in an electrical network

Assignment problem

- The data of the ***assignment problem*** consist of two equally sized sets N_1 and N_2 (i.e., $|N_1| = |N_2|$),
- A cost c_{ij} associated with each element $(i, j) \in A$.
- In the assignment problem we wish to pair, at minimum possible cost, each object in N_1 with exactly one object in N_2 .
- Examples of the assignment problem include:
  - assigning people to projects,
  - jobs to machines,
  - tenants to apartments,
  - swimmers to events in a swimming meet, and
  - medical school graduates to available in-

Maximum flow problem

- The assignment problem is a minimum cost flow problem in a network:
  - $G = (N_1 \cup N_2, A)$ with $b(i) = 1$ for all $i \in N_1$
  - $b(i) = -1$ for all $i \in N_2$
  - and $u_{ij} = 1$ for all $(i, j) \in$ A.

Transportation problem

- In the ***transportation problem*** the node set N is partitioned into two subsets N_1 and N_2 (of possibly unequal cardinality) so that
  - (1) each node in N_1 is a supply node,
  - (2) each node N_2 is a demand node, and
  - (3) for each arc (i, j) in A, $i \in N_1$ and $j \in N_2$.

Transportation problem

- The classical example:
  - the distribution of goods from warehouses to customers.
  - In this context the nodes in N_1 represent the warehouses,
  - the nodes in N_2 represent customers (or, more typically, customer zones), and
  - an arc (i, j) in A represents a distribution channel from warehouse i to customer j.

Multicommodity flow problems

- The minimum cost flow problem models the flow of a single commodity over a network.
- ***Multicommodity flow problems*** arise when several commodities use the same underlying network.
- The commodities may either be differentiated by their physical characteristics or simply by their origin-destination pairs.
- Different commodities have different origins and destinations, and commodities have separate mass balance constraints at each node.

Multicommodity flow problems

- The sharing of the common arc capacities binds the different commodities together.
- In fact, the essential issue addressed by the multicommodity flow problem is the allocation of the capacity of each arc to the individual commodities in a way that minimizes overall flow costs.

Multicommodity flow problems

- **Some applications:**
  - The transportation of passengers from different origins to different destinations within a city;
  - The routing of nonhomogeneous tankers (non-homogeneous- in terms of speed, carrying capability, and operating costs);
  - The worldwide shipment of different varieties of grains (such as corn, wheat, rice, and soybeans) from countries that produce grains to those that consume it;
  - The transmission of messages in a communication network between different origin-destination pairs.

The End