In the name of God

Network Flows

1. Introduction
1.2 Notation and Definitions

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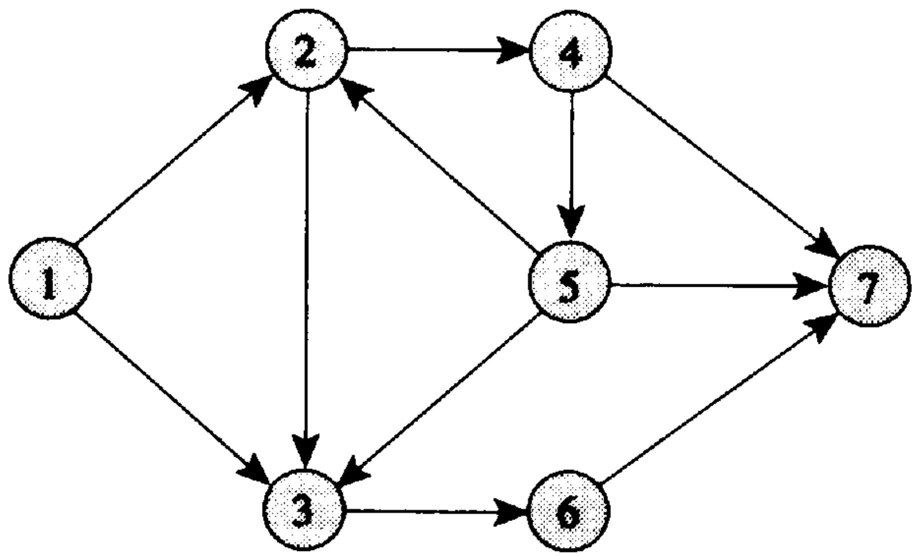
Notation and Definitions

- In this section we give several basic definitions from graph theory and present some basic notation.
- We also state some elementary properties of graphs.

Notation and Definitions

- ***Directed graph***
  - A directed graph $G = (N, A)$ consists of a set N of nodes and a set A of arcs whose elements are ordered pairs of distinct nodes.



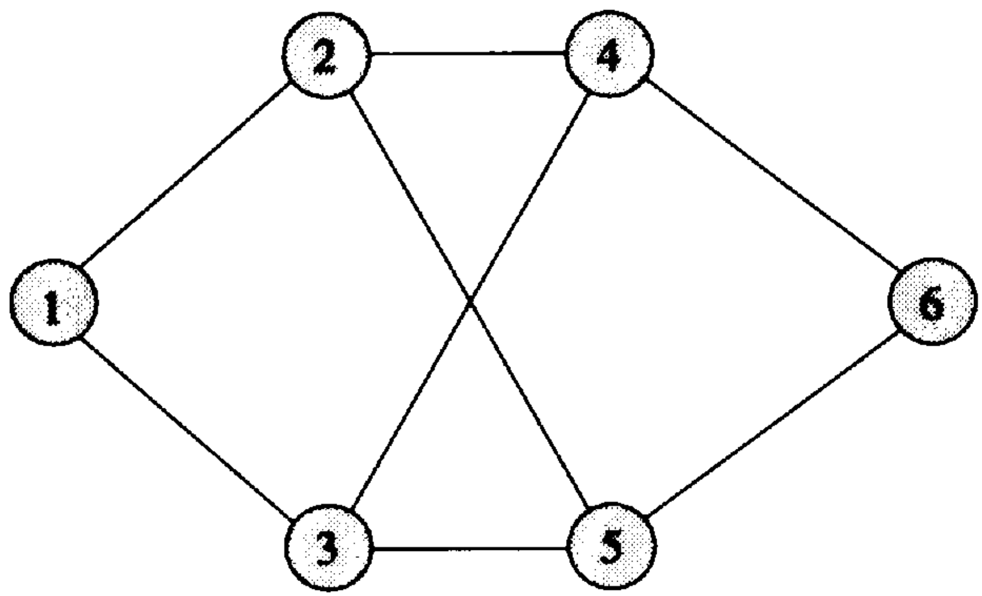
  - $N = \{1, 2, 3, 4, 5, 6, 7\}$
  - $A = \{(1, 2), (1, 3), (2, 3), (2,4), (3, 6), (4, 5), (4, 7), (5, 2), (5, 3), (5, 7), (6, 7)\}$.

Notation and Definitions

- ***Directed network***
  - A ***directed network*** is a **directed graph** whose nodes and/or arcs have associated:
    - costs
    - capacities
    - and/or supplies and demands
- We use the terms "graph" and "network" synonymously.
- We let n denote the number of nodes and m denote the number of arcs in G.

Notation and Definitions

- ***Undirected Graphs***
  - In an ***undirected graph*** arcs are unordered pairs of distinct nodes.



  - In an undirected graph, we can refer to an arc joining the node pair i and j as either (i, j) or (j, i).

Notation and Definitions

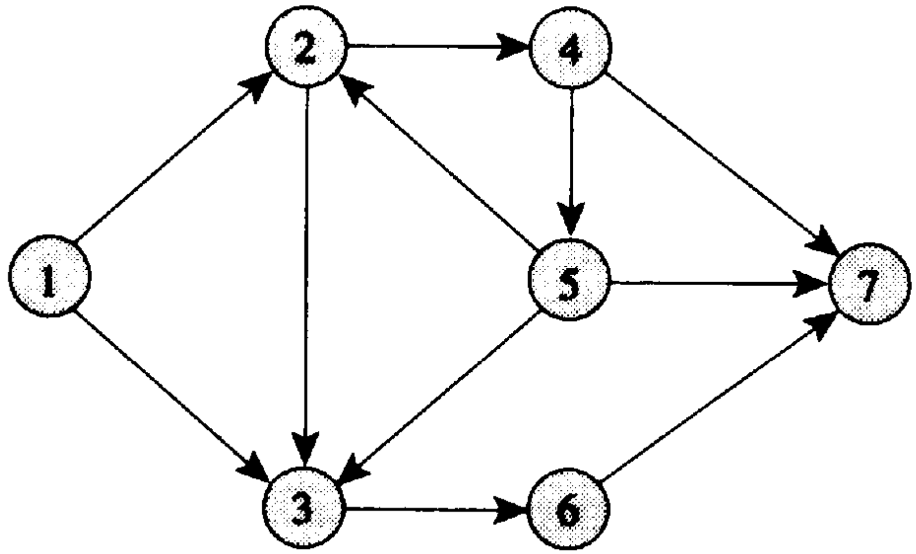
- ***Tails and Heads***
  - A directed arc (i, j) has two ***endpoints*** i and j.
  - Node i is the ***tail*** of arc (i, j) and node j is its ***head***.
  - The arc (i, j) ***emanates*** from node i and ***terminates*** at node j.
  - An arc (i, j) is ***incident to*** nodes i and j.
  - The arc (i, j) is an ***outgoing arc*** of node i and an ***incoming arc*** of node j.
  - Whenever an arc $(i, j) \in A$, we say that node j is ***adjacent*** to node i.

Notation and Definitions

- ***Degrees***
  - The ***indegree*** of a node is the number of **incoming arc**s of that node and its ***outdegree*** is the number of its **outgoing arcs**.
  - The ***degree*** of a node is the sum of its **indegree** and **outdegree**.
  - The sum of indegrees of all nodes equals the sum of outdegrees of all nodes and both are equal to the number of arcs m in the network.

Notation and Definitions

- **Example of degree:**



  – Node 3 has an indegree of 3, an outdegree of 1, and a degree of 4.

Notation and Definitions

- ***Adjacency List***
  - The ***arc adjacency list*** $A(i)$ of a node i is the set of arcs emanating from that node, $A(i) = \{(i, j) \in \text{A}: j \in N\}$.
  - The ***node adjacency list*** $A(i)$ is the set of nodes adjacent to that node; in this case, $A(i) = \{j \in N: (i, j) \in \text{A}\}$.
  - We assume that arcs in the adjacency list $A(i)$ are arranged so that the head nodes of arcs are in increasing order.

Notation and Definitions

- ***Multiarcs and Loops***
  - ***Multiarcs*** are two or more arcs with the same tail and head nodes.
  - A ***loop*** is an arc whose tail node is the same as its head node.

Notation and Definitions

- ***Subgraph***
  - A graph $G' = (N', A')$ is a ***subgraph*** of $G = (N, A)$ if $N' \subseteq N$ and $A' \subseteq A$.
  - We say that $G' = (N', A')$ is the subgraph of G ***induced*** by N' if A' contains each arc of A with both **endpoints** in N'.
  - A graph $G' = (N', A')$ is a ***spanning subgraph*** of $G = (N, A)$ if $N' = N$ and $A' \subseteq A$.

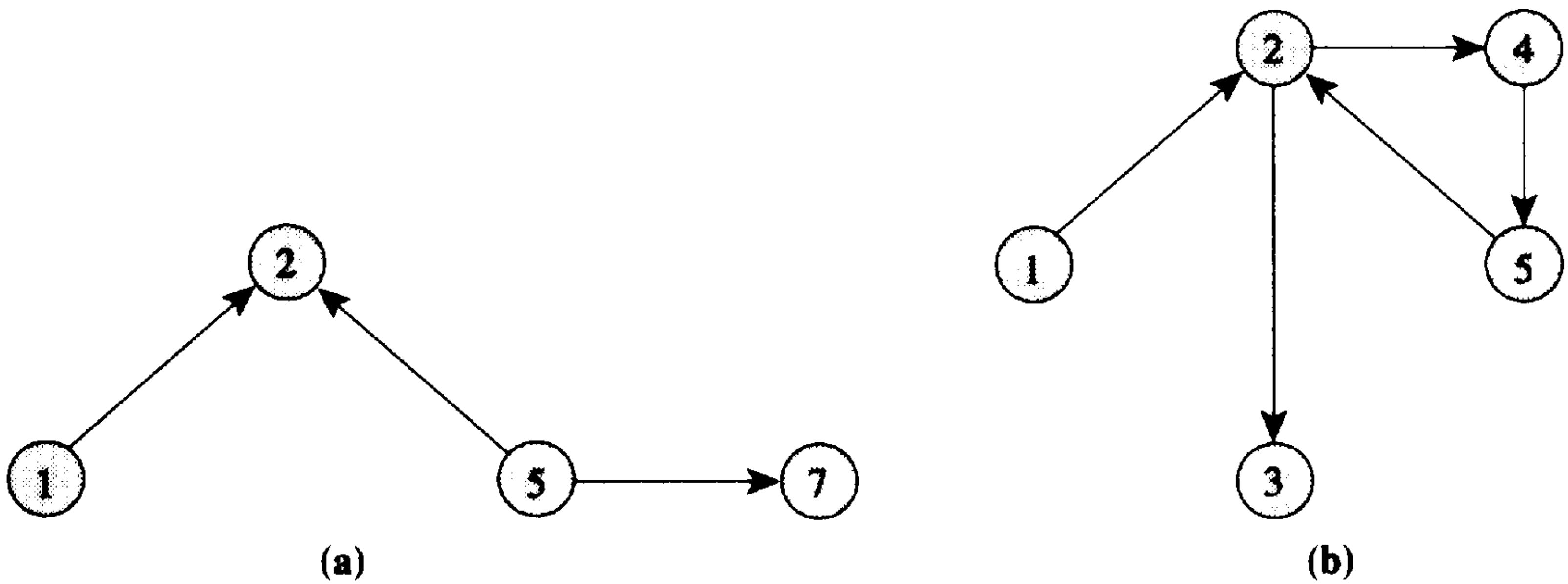
Notation and Definitions

- ***Walk***
  - A ***walk*** in a directed graph $G = (N, A)$ is a subgraph of G consisting of a sequence of nodes and arcs

$$i_1 - a_1 - i_2 - a_2 - \ldots - i_{r-1} - a_{r-1} - i_r$$

  - satisfying the property that for all $1 \leq k \leq r - 1$,
  - either $a_k = (i_k , i_{k+1}) \in \mathrm{A}$ or $a_k = (i_{k+1} , i_k) \in \mathrm{A}$.
  - Alternatively, we shall sometimes refer to a walk as a set of (sequence of) arcs (or of nodes) without any explicit mention of the nodes (without explicit mention of arcs).

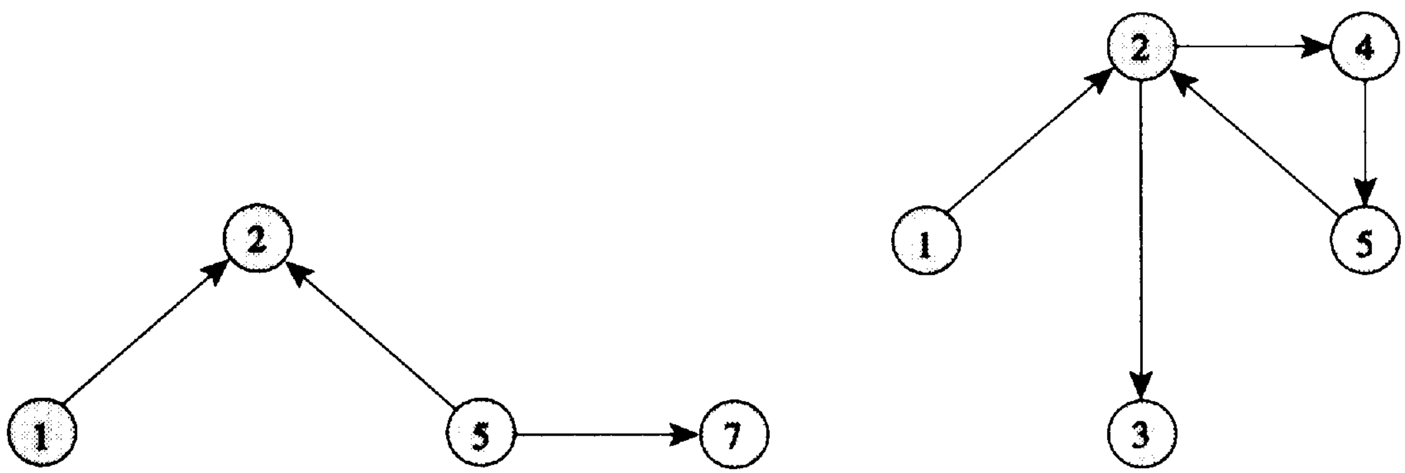
Notation and Definitions



- (a) 1-2-5-7
- (b) 1-2-4-5-2-3

Notation and Definitions

- ***Directed Walk***
  - A ***directed walk*** is an oriented version of a walk in the sense that for any two consecutive nodes i_k and i_{k+1} on the walk, $(i_k , i_{k+1}) \in A$.

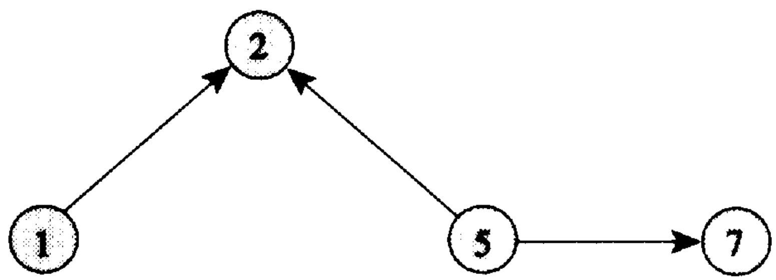


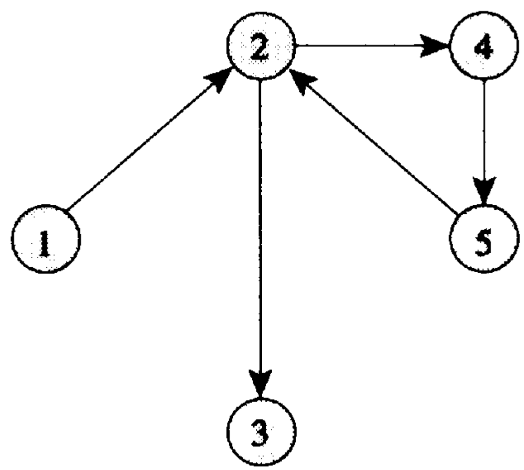
(a) (b)

  - (a) is not a directed walk
  - (b) is a directed walk

Notation and Definitions

- ***Path***
  - A ***path*** is a walk without any repetition of nodes.
  - Directions are ignored.

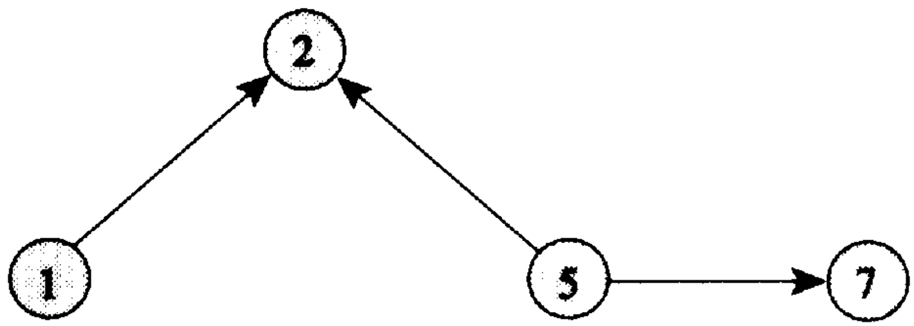




  - (a) is a path, 1-2-5-7
  - (b) is not a path because it repeats node 2 twice., 1-2-4-5-2-3

Notation and Definitions

- We can partition the arcs of a path into two groups: ***forward arcs*** and ***backward arcs***.

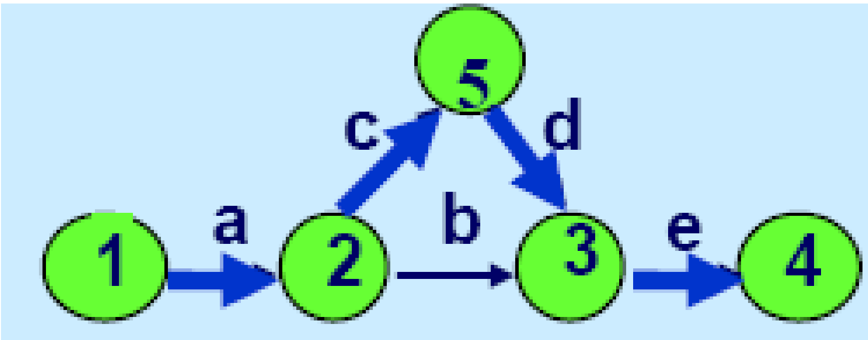


(a)

- The arcs (1, 2) and (5, 7) are forward arcs and the arc (5, 2) is a backward arc.

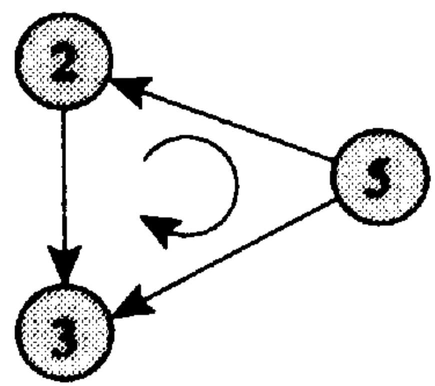
Notation and Definitions

- ***Directed Path***
  - A directed path is a directed walk without any repetition of nodes.
  - In other words, a directed path has no backward arcs.
  - Directions are important.



Notation and Definitions

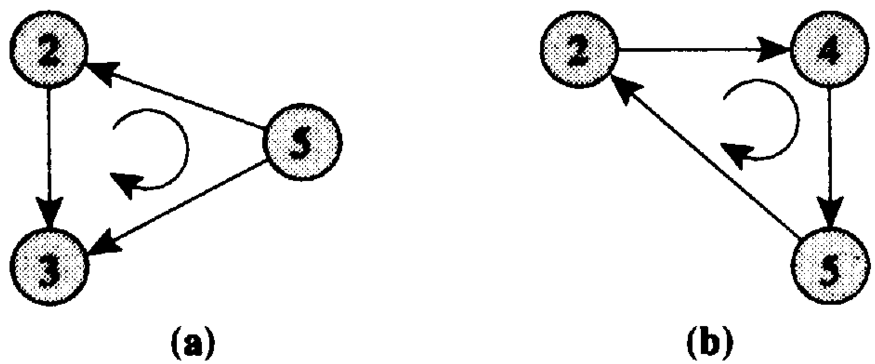
- ***Cycle***
  - A ***cycle*** is a path i_1 - i_2 - ... - i_r together with the arc (i_r, i_1) or (i_1, i_r).
  - We refer to a cycle using the notation i_1- i_2 - ... - $i_r - i_1$.
  - Just as we did for paths, we can define forward and backward arcs in a cycle.
  - Directions are ignored.



  - The arcs (5, 3) and (3, 2) are forward arcs and the arc (5, 2) is a backward arc of the cycle 2-5-3.

Notation and Definitions

- ***Directed Cycle***
  - A directed cycle is a directed path i_1 - i_2 - … - i_r together with the arc (i_r, i_1).
  - Directions are important.



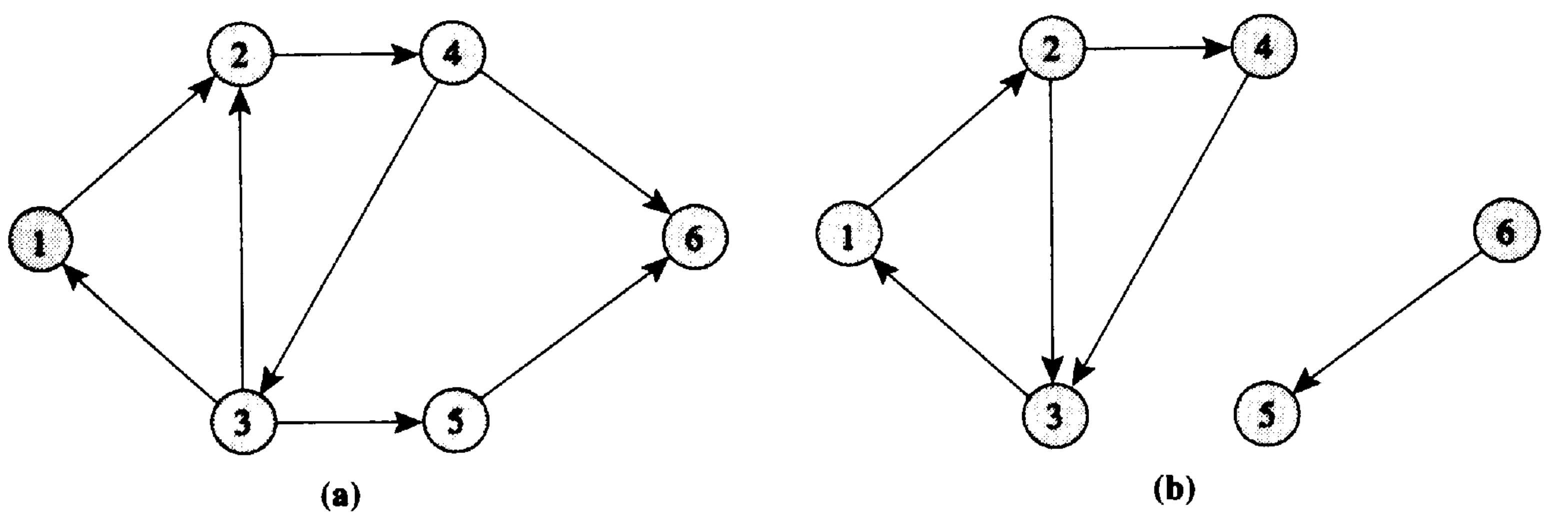
  - (a) is a cycle, but not a directed cycle;
  - (b) is a directed cycle.

Notation and Definitions

- ***Acyclic Graph***
  - A graph is a ***acyclic*** if it contains no directed cycle.

Notation and Definitions

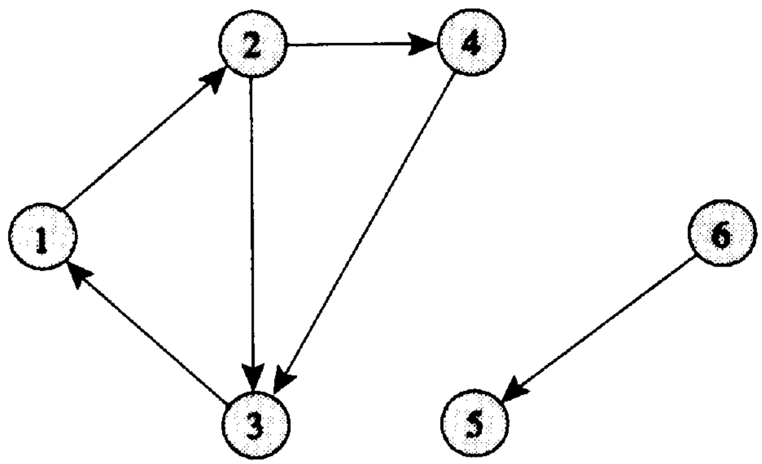
- ***Connectivity***
  - We will say that two nodes *i* and *j* are ***connected*** if the graph contains at least **one path** from node *i* to node *j*.
  - A graph is connected if every pair of its nodes is connected; otherwise, the graph is ***disconnected***.



  - (a) is connected graph, and (b) is disconnected graph

Notation and Definitions

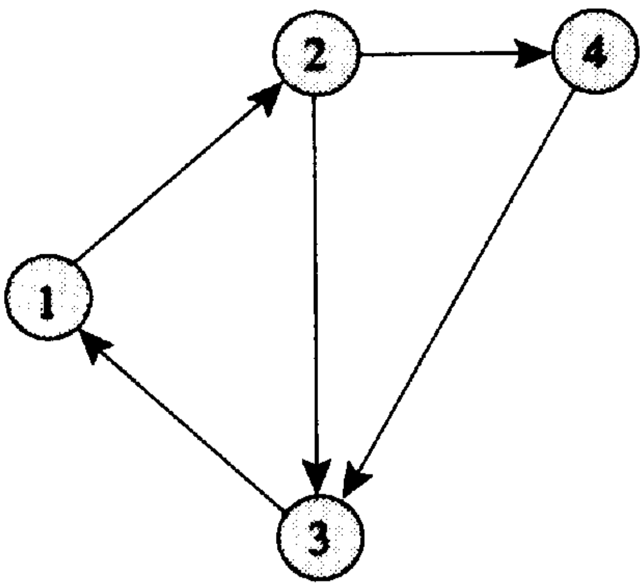
- ***Components***
  - We refer to the maximal connected subgraphs of a disconnected network as its ***components***.



  - This graph has two components consisting of the node sets {1, 2, 3, 4} and {5, 6}.

Notation and Definitions

- ***Strong Connectivity***
  - A connected graph is ***strongly connected*** if it contains at least **one directed path** from every node to every other node.

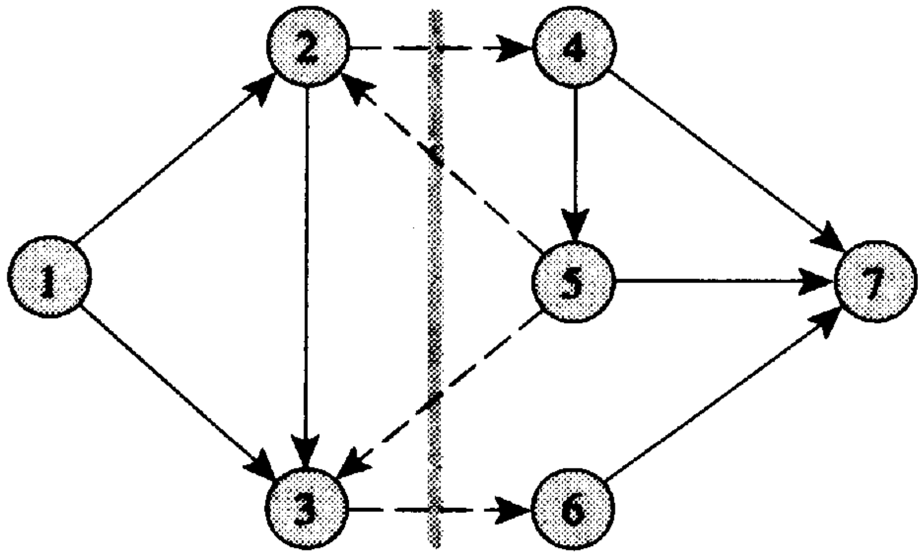


  - A strongly connected graph

Notation and Definitions

- ***Cut***
  - A ***cut*** is a partition of the node set N into two parts, S and $\bar{S} = N - S$.
  - Each cut defines a set of arcs consisting of those arcs that have one endpoint in S and another endpoint in $\bar{S}$.
  - We refer to this set of arcs as a cut and represent it by the notation $[S, \bar{S}]$.

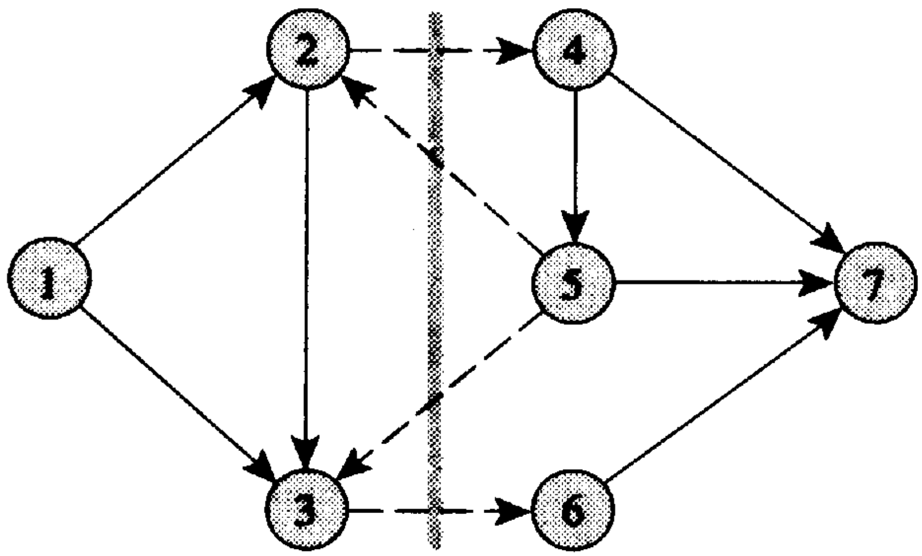
Notation and Definitions



- A cut with S = {1, 2, 3} and $\overline{S}$ = {4, 5, 6, 7}.
- The set of arcs in this cut are {(2, 4), (5, 2), (5, 3), (3, 6)}.

Notation and Definitions

- ***s-t Cut***
  - An *s*-*t* cut is defined with respect to two distinguished nodes *s* and *t*, and is a cut $[S, \bar{S}]$ satisfying the property that $s \in S$ and $t \in \bar{S}$.



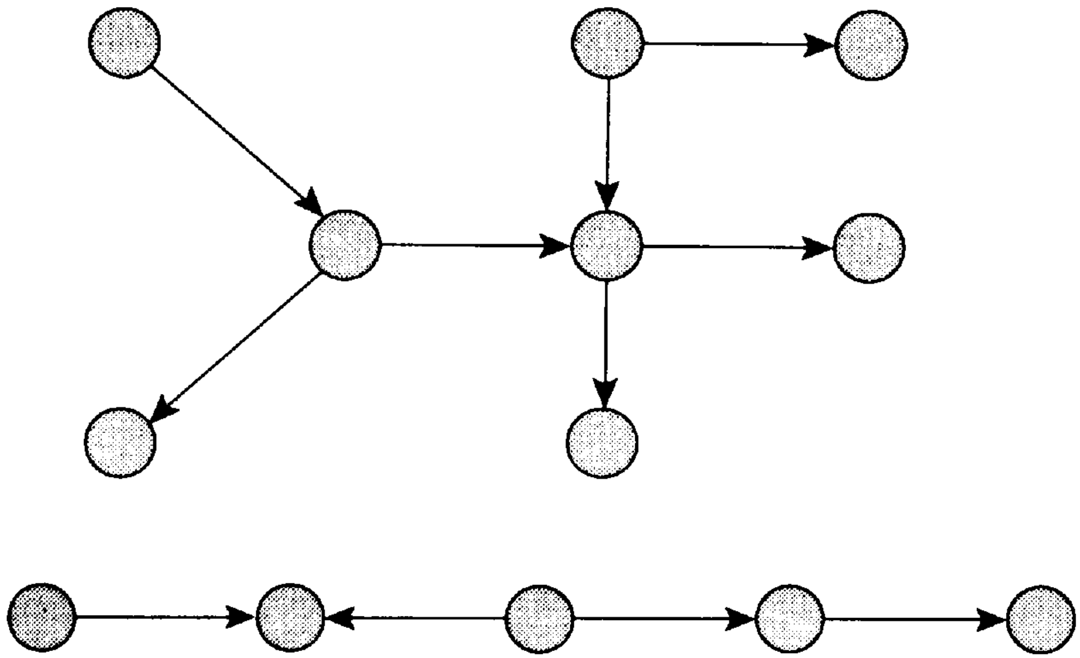
  - If $s = 1$ and $t = 6$, this cut is an *s*-*t* cut;
  - but if s = 1 and t = 3, this cut is not an *s*-*t* cut.

Notation and Definitions

- ***Tree***
  - A ***tree*** is a connected graph that contains no cycle.
- Property
  - (a) A tree on n nodes contains exactly n - 1 arcs.
  - (b) A tree has at least two leaf nodes (i.e., nodes with degree 1).
  - (c) Every two nodes of a tree are connected by a unique path.

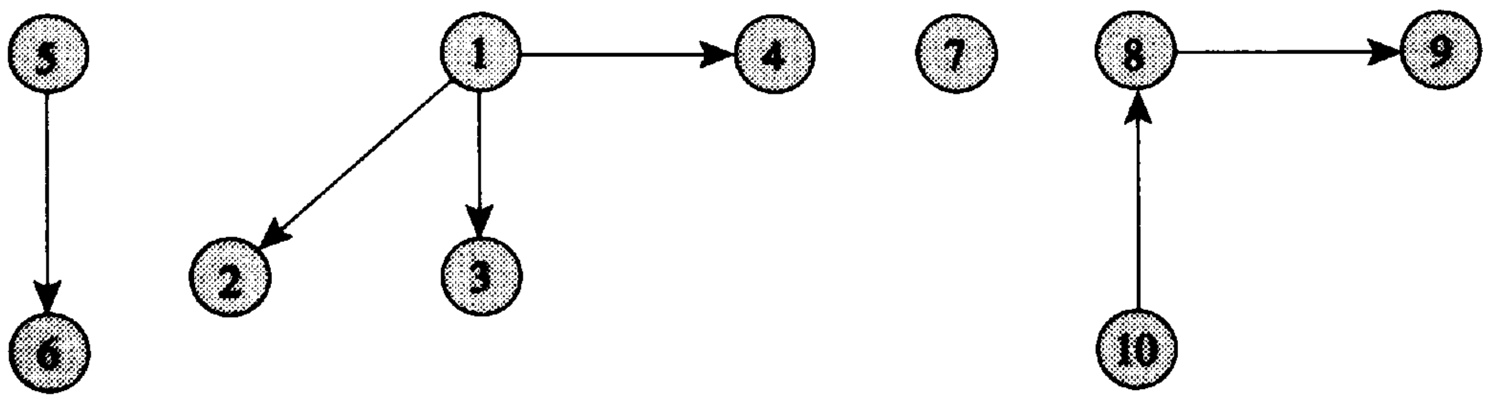
Notation and Definitions

- Two examples of trees



Notation and Definitions

- ***Forest***
  - A graph that contains no cycle is a ***forest***.
  - Alternatively, a forest is a collection of trees.



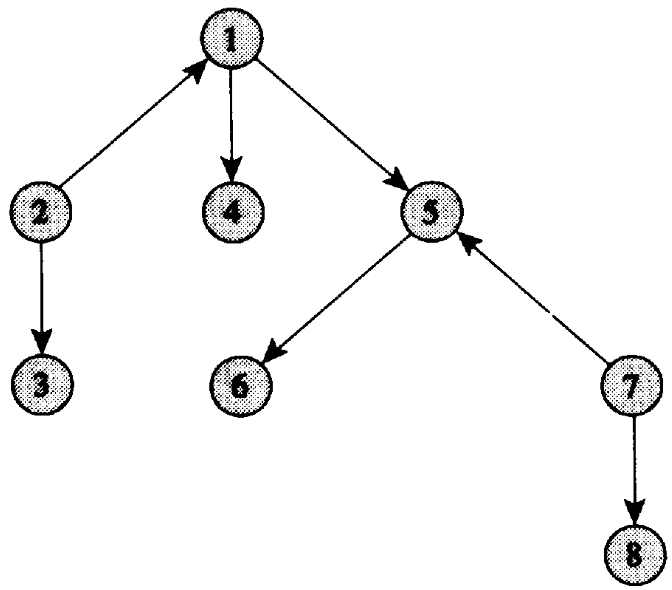
  - an example of a forest.

Notation and Definitions

- ***Subtree***
  - A connected subgraph of a tree is a ***subtree***.

Notation and Definitions

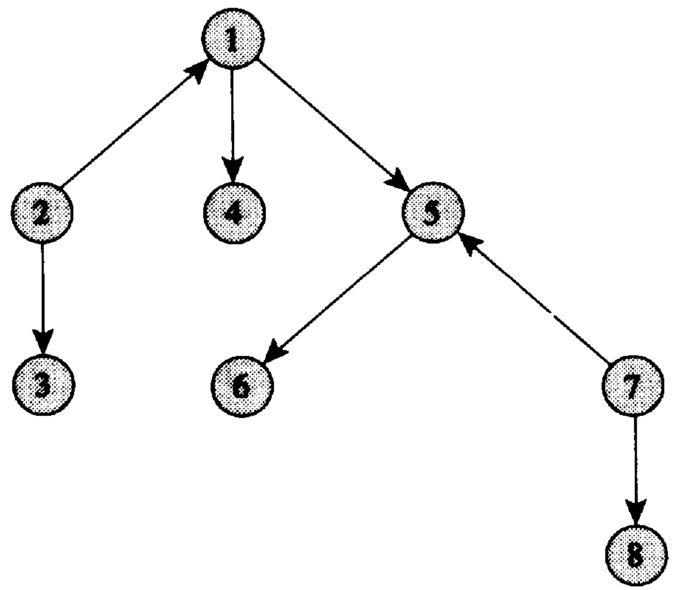
- ***Rooted Tree***
  - A rooted tree is a tree with a specially designated node, called its ***root***; we regard a rooted tree as though it were hanging from its root.



  - an instance of a rooted tree; node 1 is the root node.

Notation and Definitions

- We often view the arcs in a rooted tree as defining predecessor-successor (or parent-child) relationships.



  - Node 5 is the predecessor of nodes 6 and 7,
  - Node 1 is the predecessor of nodes 2, 4, and 5.

Notation and Definitions

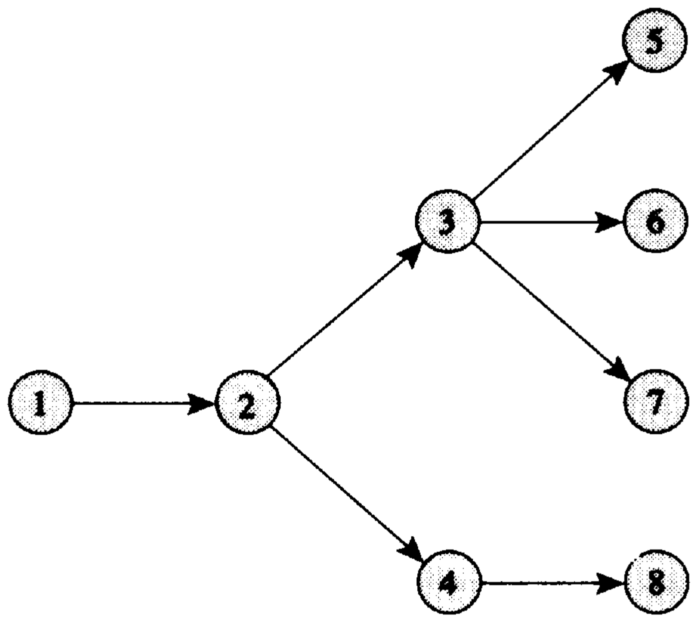
- Each node i (except the root node) has a unique predecessor, which is the next node on the unique path in the tree from that node to the root;
- We store the predecessor of node i using a predecessor index $pred(i)$.
- If $j = pred(i)$, we say that node j is the predecessor of node i and node i is a successor of node j.
- These predecessor indices uniquely define a rooted tree and also allow us to trace out the unique path from any node back to the root.

Notation and Definitions

- There are two special type of rooted trees, called a ***directed in-tree*** and a ***directed out-tree***.

Notation and Definitions

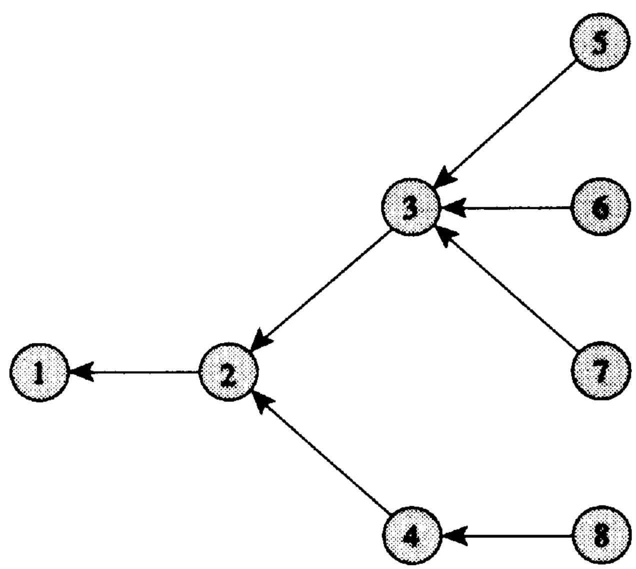
- ***Directed-Out- Tree***
  - A tree is a ***directed out-tree*** routed at node *s* if the unique path in the tree from node *s* to every other node is a directed path.



  - an instance of a directed out-tree rooted at node 1.
  - Observe that every node in the directed outtree (except node 1) has indegree 1.

Notation and Definitions

- ***Directed-In-Tree***
  - A tree is a ***directed in-tree*** routed at node *s* if the unique path in the tree from any node to node *s* is a directed path.



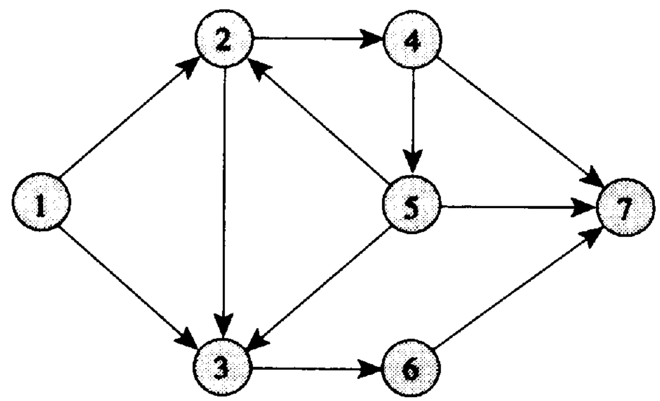
  - an instance of a directed in-tree rooted at node 1.
  - Observe that every node in the directed in-tree (except node 1) has outdegree 1.

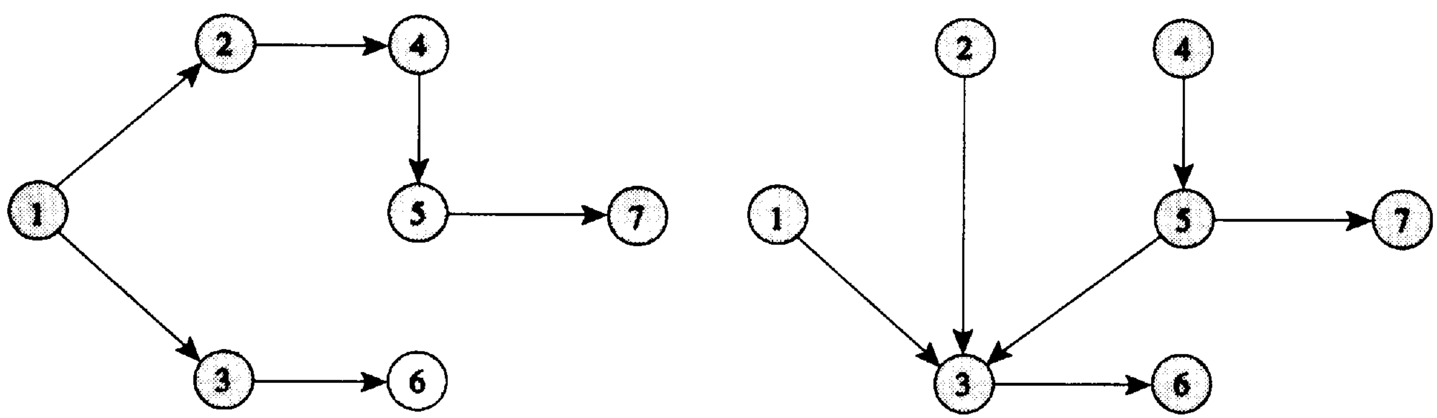
Notation and Definitions

- ***Spanning Tree***
  - A tree *T* is a spanning tree of *G* if *T* is a spanning subgraph of *G*.
  - Every spanning tree of a connected *n*-node graph G has (*n* - 1) arcs.
  - We refer to the arcs belonging to a spanning tree T as tree arcs and arcs not belonging to *T* as nontree arcs.

Notation and Definitions

– two spanning trees of the graph





(a) (b)

Notation and Definitions

- ***Undirected networks***
  - An undirected arc (i, j) has two endpoints, i and j, but its tail and head nodes are undefined.
  - If the network contains the arc (i, j), node i is adjacent to node j, and node j is adjacent to node i.

The End