In the name of God

Network Flows

1. Introduction
1.3 Network Representations

Fall 2010

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Network Representations

- The performance of a network algorithm depends:
  - the algorithm
  - the data structures
- **Data structure**
  - The manner used to represent the network within a computer and the storage scheme used for maintaining and updating the intermediate results.

Network Representations

- In representing a network, we need to store two types of information:
  - (1) the network topology, that is, the network's node and arc structure; and
  - (2) data such as costs, capacities, and supplies/demands associated with the network's nodes and arcs.
- Usually the scheme we use to store the network's topology will suggest a natural way for storing the associated node and arc information.

Network Representations

- Network Representations
  - Node-Arc Incidence Matrix
  - Node-Node Adjacency Matrix
  - Adjacency Lists
  - Forward and Reverse Star Representations
  - Compact Forward and Reverse Star Representation

Node-Arc Incidence Matrix

Node-Arc Incidence Matrix

- ***Node-arc incidence matrix / incidence matrix***
  - It represents a network as the constraint matrix of the minimum cost flow problem
  - It stores the network as an n x m matrix $\mathcal{N}$
  - It contains one row for each node of the network and one column for each arc.
  - The column corresponding to arc (i, j) has only two nonzero elements: It has a '+1' in the row corresponding to node i and a '-1' in the row corresponding to node j.

Minimum Cost Flow Problem

- **Minimum Cost Flow Problem**

Minimize $\sum_{(i,j) \in A} c_{ij} x_{ij}$

subject to

$$\sum_{\{j:(i,j)\in A\}} x_{ij} - \sum_{\{j:(j,i)\in A\}} x_{ji} = b(i) \qquad \text{for all } i \in N,$$

$$l_{ij} \le x_{ij} \le u_{ij} \qquad \text{for all } (i,j) \in A,$$

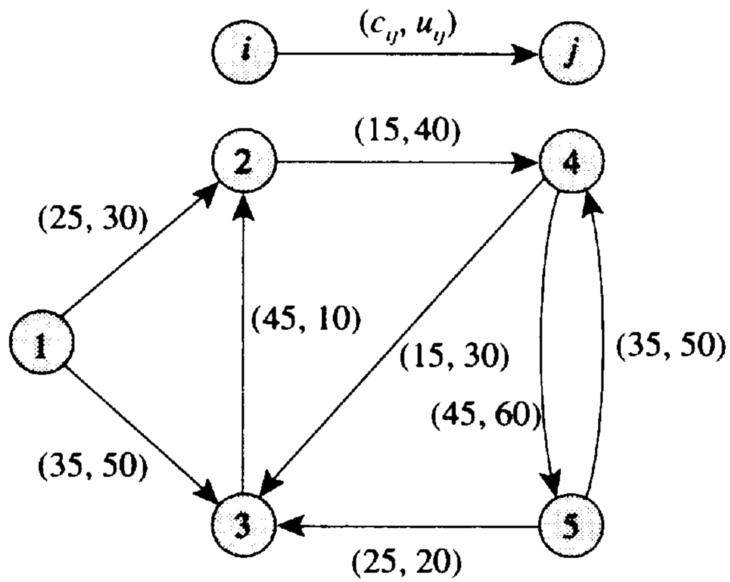
- **In matrix form**

Minimize cx

subject to $\mathcal{N}x = b,$

$$l \le x \le u.$$

Node-Arc Incidence Matrix



$$
\begin{array}{c}
 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5
\end{array}
\begin{array}{c}
\begin{array}{cccccccc} (1,2) & (1,3) & (2,4) & (3,2) & (4,3) & (4,5) & (5,3) & (5,4) \end{array} \\
\left[\begin{array}{cccccccc}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & -1 & 0 & -1 & 0 \\
0 & 0 & -1 & 0 & 1 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 & 1
\end{array}\right]
\end{array}
$$

Node-Arc Incidence Matrix

- The node-arc incidence matrix has a very special structure
  - Only $2m$ out of its nm entries are nonzero,
  - all of its nonzero entries are +1 or -1, and
  - each column has exactly one +1 and one -1.
  - the number of + l's in a row equals the outdegree of the corresponding node and the number of -1 's in the row equals the indegree of the node.

Node-Arc Incidence Matrix

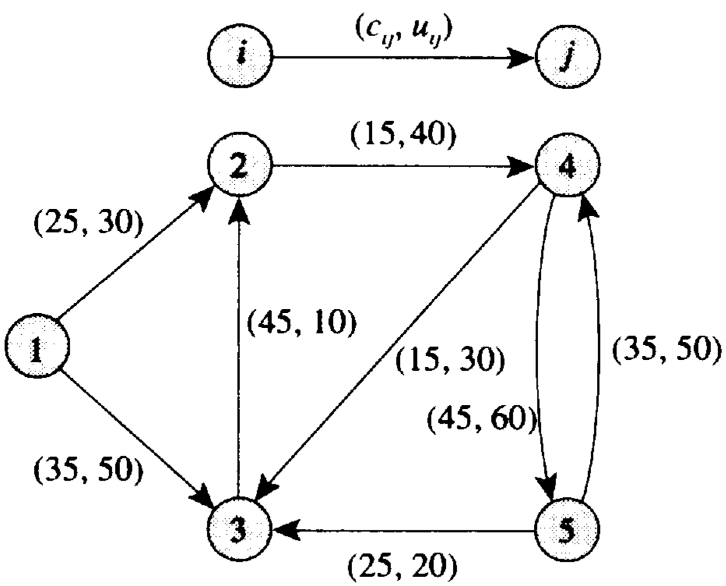
- Because the node-arc incidence matrix $\mathcal{N}$ contains **so few nonzero coefficients**, the incidence matrix representation of a network is not space efficient.
- More efficient schemes would merely keep track of the nonzero entries in the matrix.
- The node-arc incidence matrix rarely produces efficient algorithms.
- This representation is important because
  - it represents the constraint matrix of the minimum cost flow problem and
  - the node-arc incidence matrix possesses several interesting theoretical properties.

Node-Node Adjacency Matrix

Node-Node Adjacency Matrix

- ***Node-node adjacency matrix / adjacency matrix***
  - It stores the network as an $n \times n$ matrix $\mathcal{H} = \{h_{ij}\}$.
  - The matrix has a row and a column corresponding to every node
  - Its ijth entry h_{ij} equals 1 if $(i, j) \in A$ and equals 0 otherwise.

Node-Node Adjacency Matrix



$$
\begin{array}{c}
\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}
\begin{bmatrix}
0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 0
\end{bmatrix}
\end{array}
$$

Node-Node Adjacency Matrix

- If we wish to store **arc costs** and **capacities** as well as the network topology, we can store this information in two additional $n \times n$ matrices $\mathcal{L}$ and $\mathcal{U}$
- The adjacency matrix has n^2 elements, only m of which are nonzero.
- Consequently, this representation is space efficient only if the network is sufficiently dense; for sparse networks this representation wastes considerable space.
- The simplicity of the adjacency representation permits us to use it to implement most network algorithms rather easily.

Node-Node Adjacency Matrix

- We can determine the cost or capacity of any arc
- (i, j) simply by looking up the ijth element in the matrix $\mathcal{L}$ or $\mathcal{U}$.
- We can obtain the arcs emanating from node i by scanning row i
  - If the jth element in this row has a nonzero entry, (i, j) is an arc of the network.
- Similarly, we can obtain the arcs entering node j by scanning column j
  - If the ith element of this column has a nonzero entry, (i, j) is an arc of the network.

Node-Node Adjacency Matrix

- These steps permit us to identify all the outgoing or incoming arcs of a node in time proportional to n.
- For **dense networks** we can usually afford to spend this time to identify the incoming or outgoing arcs
- **For sparse networks** these steps might be the bottleneck operations for an algorithm.

Adjacency Lists

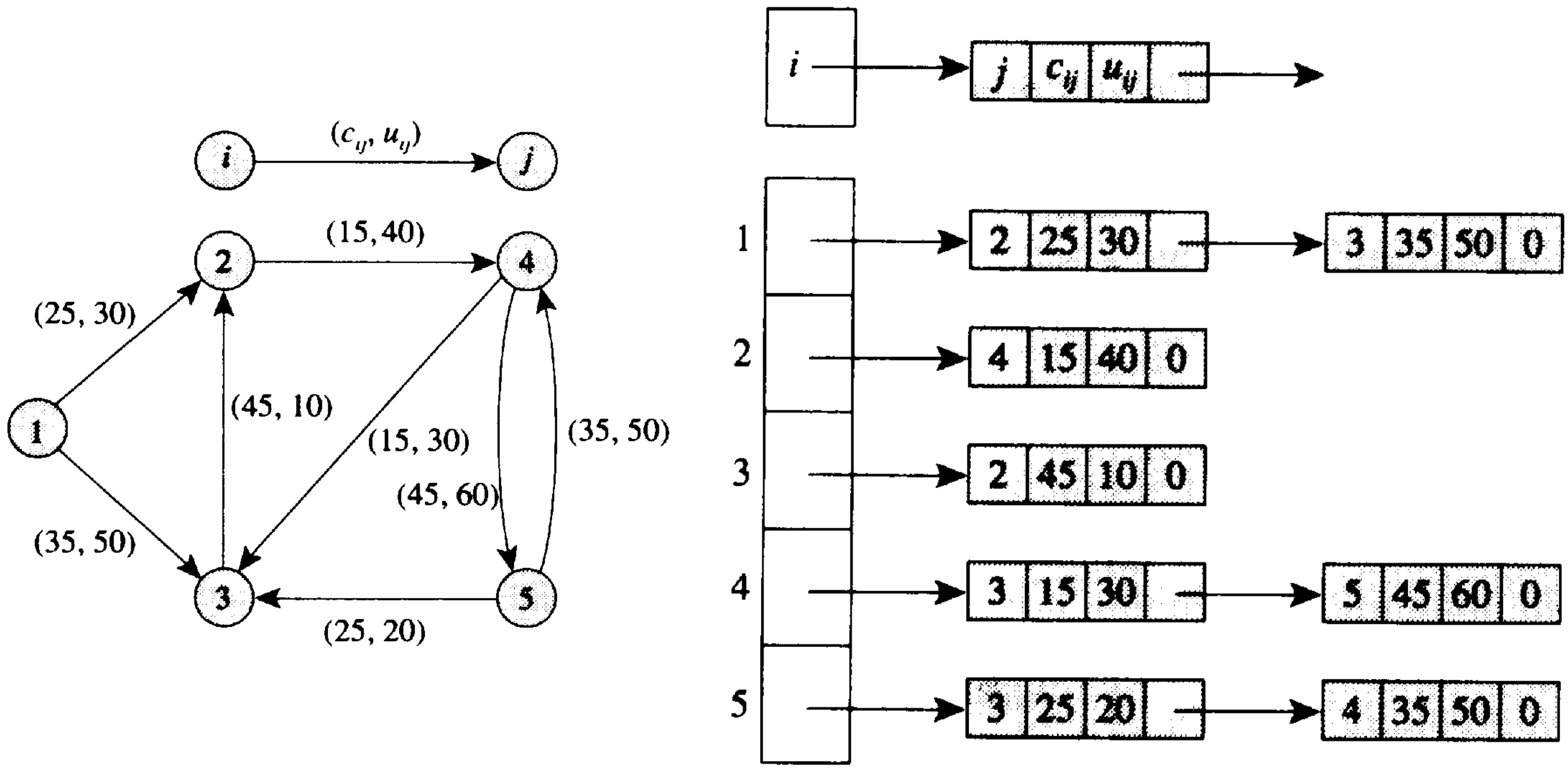
Adjacency Lists

- We defined before:
  - The ***arc adjacency list*** $A(i)$ of a node i as the set of arcs emanating from that node
  - The ***node adjacency list*** of a node i as the set of nodes j for which $(i, j) \in A$.
- ***Adjacency list***
  - It stores the ***node adjacency list*** of each node as a ***singly linked list***.
  - A ***linked list*** is a collection of cells each containing one or more fields.

Adjacency Lists

- The node adjacency list for node i will be a linked list having **| $A(i)$ | cells** and each cell will correspond to an arc $(i, j) \in A$.
- Each cell corresponding to the arc (i, j) will have:
  - One data field will store node j.
  - Two other data fields to store the ***arc cost*** c_{ij} and the ***arc capacity*** u_{ij}.
  - One additional ***link*** field, which stores a pointer to the next cell in the adjacency list.
    - If a cell happens to be the last cell in the adjacency list, by convention we set its link to value zero.
- We also need an array of pointers that point to the first cell in each linked list.

Adjacency Lists



Forward and Reverse Star Representations

Forward and Reverse Star Representations

- ***Forward star representation***
  - It stores the node adjacency list of each node.
  - It is similar to the ***adjacency list*** representation, but instead of maintaining these lists as linked lists, it stores them in a ***single array***.
  - It provides us with an efficient means for determining the set of outgoing arcs of any node.

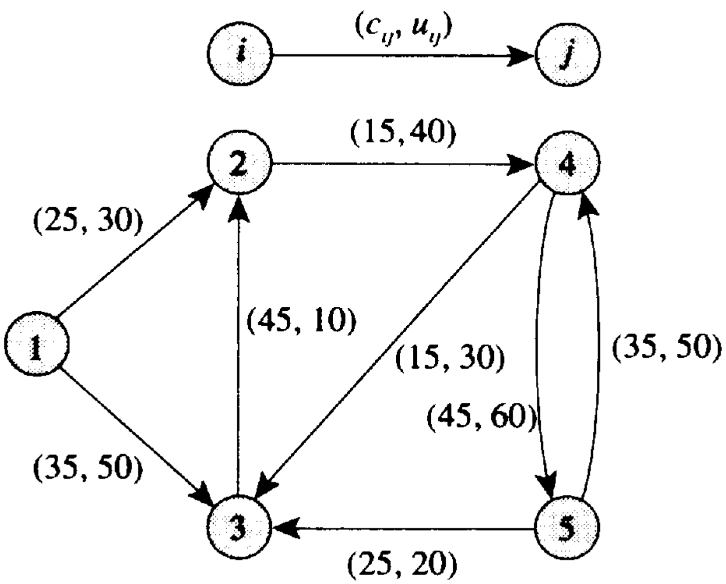
Forward and Reverse Star Representations

- To develop this representation,
  - We first associate a unique sequence number with each arc, thus defining an ordering of the arc list.
  - We number the arcs in a specific order:
    - first those emanating from node 1, then
    - those emanating from node 2, and so on.
  - We number the arcs emanating from the same node in an arbitrary fashion.
  - We then sequentially store information about each arc in the arc list.
  - We store the tails, heads, costs, and capacities of the arcs in four arrays: ***tail***, ***head***, ***cost***, and ***capacity***.
  - If arc (i, j) is arc number 20, we store the tail, head, cost, and capacity data for this arc in the array positions tail(20), head(20), cost(20), and capacity(20).

Forward and Reverse Star Representations

- ***A pointer***
  - We also maintain **a pointer** with each node i, denoted by $point(i)$, that indicates the smallest-numbered arc in the arc list that emanates from node i.
  - If node i has no outgoing arcs, we set $point(i)$ equal to $point(i + 1)$.
  - Therefore, the outgoing arcs of node i at positions $point(i)$ to $(point(i + 1) - 1)$ in the arc list.
  - If $point(i) > point(i + 1) - 1$, node i has no outgoing arc.
  - For consistency, we set point(1) = 1 and $point(n + 1) = m + 1$.

Forward and Reverse Star Representations



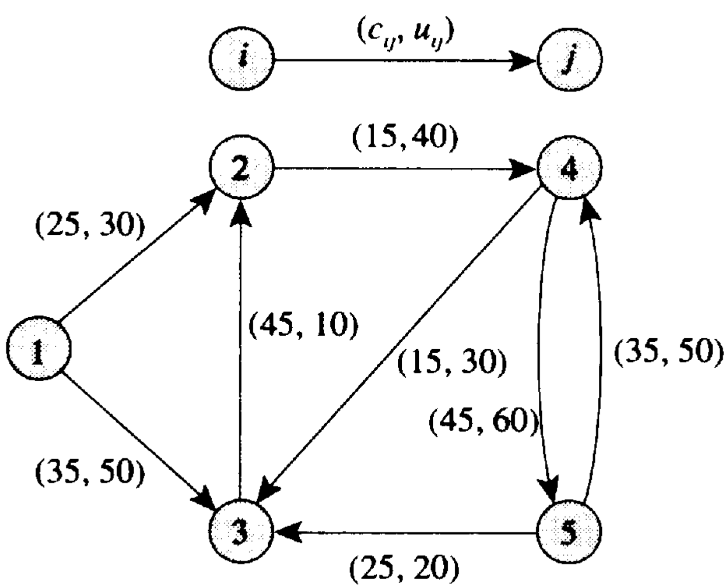
	point
1	1
2	3
3	4
4	5
5	7
6	9

	tail	head	cost	capacity
1	1	2	25	30
2	1	3	35	50
3	2	4	15	40
4	3	2	45	10
5	4	3	15	30
6	4	5	45	60
7	5	3	25	20
8	5	4	35	50

Forward and Reverse Star Representations

- ***Reverse star representation***
  - To determine the set of incoming arcs of any node efficiently
- To develop a reverse star representation
  - We examine the nodes $i = 1$ to n in order and sequentially store the heads, tails, costs, and capacities of the incoming arcs at node i.
  - We maintain a reverse pointer with each node i, denoted by *rpoint*(i), which denotes the first position in these arrays that contains information about an incoming arc at node i.
  - If node i has no incoming arc, we set *rpoint*(i) equal to *rpoint*($i + 1$).
  - For consistency, we set rpoint(1) = 1 and rpoint(n +. 1) = m + 1.
  - As before, we store the incoming arcs at node i at positions rpoint(i) to (rpoint(i + 1) - 1).

Forward and Reverse Star Representations



	cost	capacity	tail	head
1	45	10	3	2
2	25	30	1	2
3	35	50	1	3
4	15	30	4	3
5	25	20	5	3
6	35	50	5	4
7	15	40	2	4
8	45	60	4	5

	rpoint
1	1
2	1
3	3
4	6
5	8
6	9

Compact Forward and Reverse Star Representation

Forward and Reverse Star Representations

- Observe that by storing both the forward and reverse star representations
  - We will maintain a significant amount of duplicate information.
  - We can avoid this duplication by storing arc numbers in the reverse star instead of the tails, heads, costs, and capacities of the arcs.
- So instead of storing the tails, costs, and capacities of the arcs, we simply store arc numbers;
  - and once we know the arc numbers, we can always retrieve the associated information from the forward star representation.
  - We store arc numbers in an array ***trace*** of size m.

Forward and Reverse Star Representations

- **Compact forward and reverse star representation**

	point
1	1
2	3
3	4
4	5
5	7
6	9

	tail	head	cost	capacity	trace	
1	1	2	25	30	4	1
2	1	3	35	50	1	2
3	2	4	15	40	2	3
4	3	2	45	10	5	4
5	4	3	15	30	7	5
6	4	5	45	60	8	6
7	5	3	25	20	3	7
8	5	4	35	50	6	8

rpoint	
1	1
1	2
3	3
6	4
8	5
9	6

Forward and Reverse Star Representations

- As an illustration,
  - arc (3, 2) has arc number 4 in the forward star representation and
  - arc (1, 2) has an arc number 1 in the forward star representation

Forward Star vs. Adjacency List

- **Forward star representation**
  - The major advantage is its space efficiency.
  - It requires less storage than does the **adjacency list** representation.
  - It is easier to implement in languages such as FORTRAN that have no natural provisions for using linked lists.

Node-Node Adjacency Matrix

- **Adjacency list**
  - The major advantage its ease of implementation in languages such as C or Java that are able to manipulate linked lists efficiently.
  - Further, using an adjacency list representation, we can add or delete arcs (as well as nodes) in constant time.
  - On the other hand, in the forward star representation these steps require time proportional to m, which can be too time consuming.

Summary

Network representations	Storage space	Features
Node–arc incidence matrix	nm	1. Space inefficient 2. Too expensive to manipulate 3. Important because it represents the constraint matrix of the minimum cost flow problem
Node–node adjacency matrix	kn^2 for some constant k	1. Suited for dense networks 2. Easy to implement
Adjacency list	$k_1n + k_2m$ for some constants k_1 and k_2	1. Space efficient 2. Efficient to manipulate 3. Suited for dense as well as sparse networks
Forward and reverse star	$k_3n + k_4m$ for some constants k_3 and k_4	1. Space efficient 2. Efficient to manipulate 3. Suited for dense as well as sparse networks

The End