In the name of God

Network Flows

6. Lagrangian Relaxation
6.1 Introduction

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Outline

- Introduction
- Lagrangian Relaxation Technique

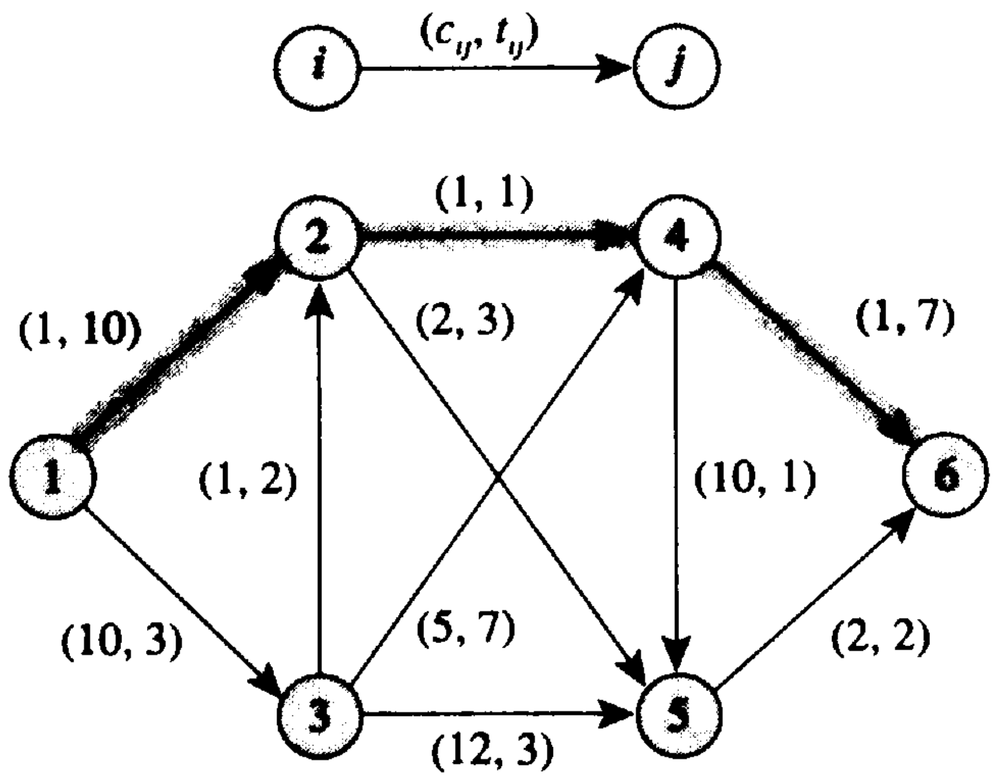
Introduction

Introduction

- ***Lagrangian relaxation*** has become one of the very few solution methods in optimization that cuts across the domains of linear and integer programming, combinatorial optimization, and nonlinear programming.
- The basic idea of Lagrangian relaxation is described via an example.

Introduction

- ***Constrained Shortest Paths***
  - Consider the following network



Introduction

- The network has two attributes associated with each arc (i, j): a cost c_{ij} and a traversal time t_{ij}.
- Suppose that we wish to find the shortest path from the source node 1 to the sink node 6, but we wish to restrict our choice of paths to those that require no more than $T = 10$ time units to traverse.
- The constrained shortest path problem is an NP-hard problem

Introduction

- The constrained shortest path problem from node 1 to node n can be stated as the following integer programming problem:

$$\text{Minimize} \quad \sum_{(i,j)\in A} c_{ij}x_{ij}$$

subject to

$$\sum_{\{j:(i,j)\in A\}} x_{ij} - \sum_{\{j:(j,i)\in A\}} x_{ji} = \begin{cases} 1 & \text{for } i = 1 \\ 0 & \text{for } i \in N - \{1, n\}, \\ -1 & \text{for } i = n \end{cases}$$

$$\sum_{(i,j)\in A} t_{ij}x_{ij} \leq T,$$

$$x_{ij} = 0 \text{ or } 1 \qquad \text{for all } (i, j) \in A.$$

Introduction

- The problem is not a shortest path problem because of the timing restriction.
- Rather, it is a shortest path problem with an additional side constraint.
- Instead of solving this problem directly, suppose that we adopt an indirect approach by combining time and cost into a single ***modified cost***; that is, we place a dollar equivalent on time.
- So instead of setting a limit on the total time we can take on the chosen path, we set a "toll charge" on each arc proportional to the time that it takes to traverse that arc.

Introduction

- For example,
  - we might charge $2 for each hour that it takes to traverse any arc.
- Note that
  - if the toll charge is zero, we are ignoring time altogether and the problem becomes a usual shortest path problem with respect to the given costs.
  - if the toll charge is very large, these charges become the dominant cost and we will be seeking the quickest path from the source to the sink.

Introduction

- Can we find a toll charge somewhere in between these values so that by solving the shortest path problem with the combined costs (the toll charges and the original costs), we solve the constrained shortest path problem as a single shortest path problem?

Introduction

- Let P, with cost c_p and traversal time t_P be any feasible path to the constrained shortest path problem

$$c_P = \sum_{(i,j)\in P} c_{ij}$$
$$t_P = \sum_{(i,j)\in P} t_{ij},$$

- Let $l(\mu)$ denote the optimal length of the shortest path with the modified costs when we impose a toll of μ units.
- Since the path P is feasible for the constrained shortest path problem, the time t_P required to traverse this path is at most $T = 10$ units.

Introduction

- the modified costs is

$$c_{ij} + \mu t_{ij},$$

- the modified cost of the path P is

$$c_p + \mu t_P$$

- Because the path P is feasible, so

$$\mu t_P \leq \mu T$$

- Therefore, if we subtract μT from the modified cost $c_P + \mu t_P$ of this path, we obtain a lower bound:

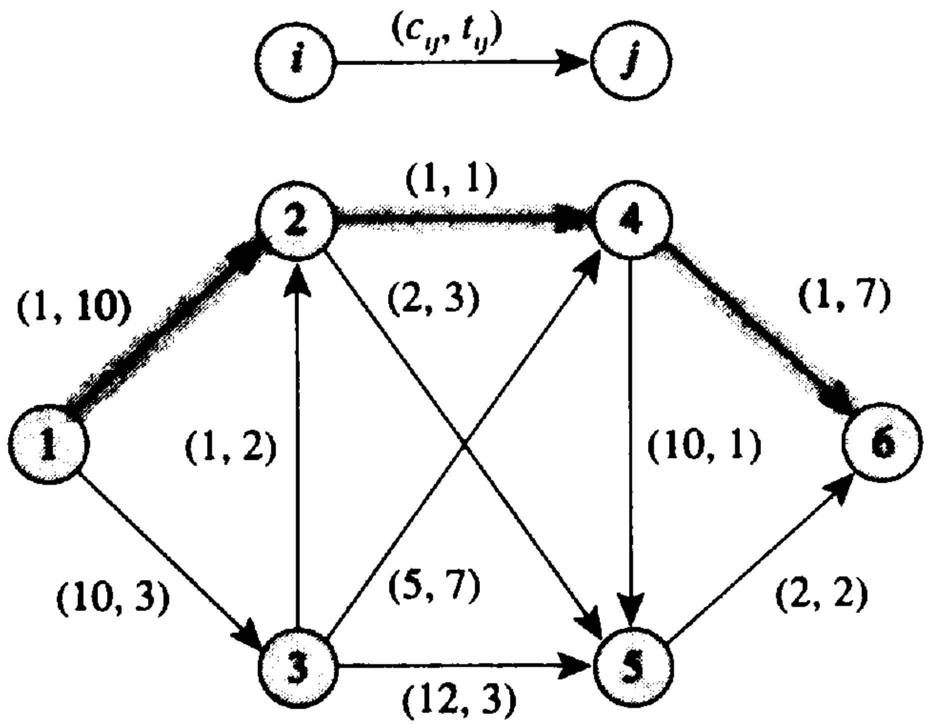
$$c_P + \mu t_P - \mu T$$

$$= c_P + \mu(t_P - T) \leq c_P$$

Introduction

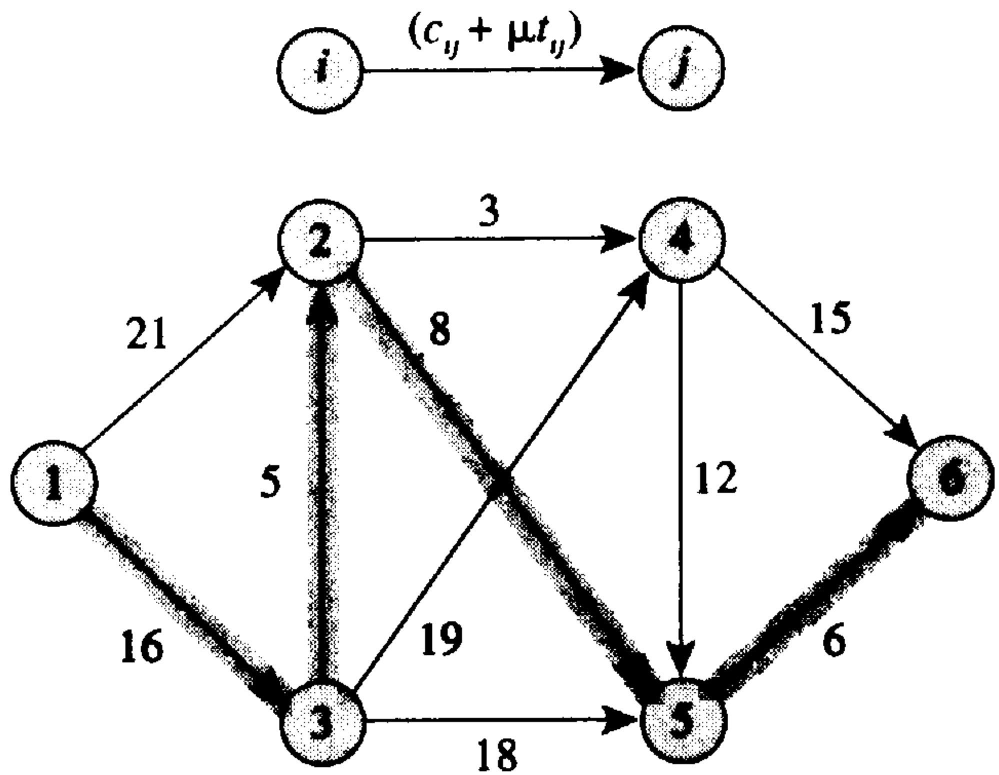
- ***Bounding Principle.***
  - For any $\mu \geq 0$, the length $l(\mu)$ of the modified shortest path with costs $c_{ij} + \mu t_{ij} - \mu T$ is a lower bound on the length of the constrained shortest path.

Introduction



- bold lines denote the shortest path with ***Lagrange multiplier*** $\mu = 0$

Introduction



- modified cost $c + \mu t$ with ***Lagrange multiplier*** $\mu = 2$ (bold lines denote the shortest path).

Introduction

- The example, for $\mu = 2$, the cost of the modified shortest path problem is 35 units and so $35 - 2(T) = 35 - 2(10) = 15$ is a lower bound on the length of the optimal constrained shortest path.
- But since the path 1-3-2-5-6 is a feasible solution to the constrained shortest path problem and its cost equals the lower bound of 15 units, we can be assured that it is an optimal constrained shortest path.

Introduction

- Rather than solving the difficult optimization problem directly, we combined the complicating **timing constraint** with the original objective function, via the toll μ, so that we could then solve a resulting embedded shortest path problem.
- This general solution approach has become known as ***Lagrangian relaxation***.

Introduction

- The lower bounding mechanism of Lagrangian relaxation frequently provides valuable information that we can exploit algorithmically.
- In many instances in the context of integer programming, the bounds provided by Lagrangian relaxation methods are much better than those generated by solving the LP relaxation of the problems, and as a consequence, Lagrangian relaxation is often an attractive alternative to linear programming as a bounding mechanism in ***branch-and-bound methods*** for solving ***integer programs***.

Lagrangian Relaxation Technique

Lagrangian Relaxation Technique

- Consider the following generic optimization model formulated in terms of a vector x of decision variables:
- **Model (P)**

$$
\begin{aligned}
z^* &= \min cx \\
\text{subject to} \quad \mathcal{A}x &= b, \\
x &\in X.
\end{aligned}
$$

Lagrangian Relaxation Technique

- ***Lagrangian relaxation procedure***
  - uses the idea of relaxing the explicit linear constraints by bringing them into the objective function with associated ***Lagrange multipliers*** μ.

Lagrangian Relaxation Technique

- ***Lagrangian relaxation subproblem***
  - or ***Lagrangian subproblem*** of the original problem
  - is a problem that its linear constraints are relaxed by bringing them into the objective function with associated ***Lagrange multipliers*** μ

$$\text{Minimize} \quad cx + \mu(\mathcal{A}x - b)$$

subject to

$$x \in X,$$

  - ***Lagrangian function***

$$L(\mu) = \min\{cx + \mu(\mathcal{A}x - b) : x \in X\},$$

Lagrangian Relaxation Technique

- Note that since in forming the Lagrangian relaxation, we have eliminated the constraints $\boldsymbol{Ax} = \boldsymbol{b}$ from the problem formulation, the solution of the Lagrangian subproblem need not be feasible for the original problem (P).
- Can we obtain any useful information about the original problem even when the solution to the Lagrangian subproblem is not feasible in the original problem (P)?

Lagrangian Relaxation Technique

- ***Lagrangian Bounding Principle***
  - For any vector μ of the Lagrangian multipliers, the value $L(\mu)$ of the Lagrangian function is a lower bound on the optimal objective function value z^* of the original optimization problem (P).

Lagrangian Relaxation Technique

$$z^* = \min \quad \sum_{(i,j)\in A} c_{ij} x_{ij}$$

$$\text{s.t.} \quad \sum_j x_{ij} - \sum_j x_{ji} = \begin{cases} 1 & \text{if } i = s \\ -1 & \text{if } i = t \\ 0 & \text{otherwise} \end{cases}$$

$$\sum_{(i,j)\in A} t_{ij} x_{ij} \le T$$

$$x_{ij} = 0 \text{ or } 1 \quad \text{for all } (i,j) \in A$$

For each $\mu \ge 0$, replace the objective by:

$$z(\mu) = \min \quad \sum_{(i,j)\in A} c_{ij} x_{ij} + \mu\left(\sum_{(i,j)\in A} t_{ij} x_{ij} - T\right)$$

$$= \sum_{(i,j)\in A} (c_{ij} + \mu t_{ij}) x_{ij} - \mu T$$

Then $z(\mu) \le z^*$ for $\mu \ge 0$ because $\sum_{(i,j)\in A} t_{ij} x_{ij} \le T$

Lagrangian Relaxation Technique

- Proof.
  - Since $Ax = b$ for every feasible solution to (P), for any vector μ of Lagrangian multipliers,

$$z^* = \min\{cx : Ax = b, x \in X\}$$
$$= \min\{cx + \mu(Ax - b) : Ax = b, x \in \mathrm{X}\}.$$

  - Since removing the constraints $Ax = b$ from the second formulation cannot lead to an increase in the value of the objective function

Lagrangian Relaxation Technique

- Therefore, for any value of the Lagrangian multiplier μ, $L(\mu)$ is a lower bound on the optimal objective function value of the original problem.

$$z^* \geq \min\{cx + \mu(\mathcal{A}x - b) : x \in X\} = L(\mu).$$

Lagrangian Relaxation Technique

- ***Lagrangian multiplier problem***
  - To obtain the sharpest possible lower bound, we would need to solve the following optimization problem:

$$L^* = \max_{\mu} L(\mu)$$

- ***Property (Weak Duality)***.
  - The optimal objective function value L^* of the Lagrangian multiplier problem is always a lower bound on the optimal objective function value of the problem (P)

$$L^* \leq z^*$$

Lagrangian Relaxation Technique

- The valid bounds for comparing objective function values of
  - L(μ) : the ***Lagrange multiplier problem*** for any choices of the Lagrange multipliers μ
  - L*(μ) : optimal objective function value of the Lagrangian multiplier problem
  - z^* optimal objective value of model (P)
  - cx: any feasible solution x of (P)

$$L(\mu) \leq L^* \leq z^* \leq cx.$$

Lagrangian Relaxation Technique

- **Property (a) (Optimality Test)**
  - Suppose that μ is a vector of Lagrangian multipliers and x is a feasible solution to the optimization problem (P) satisfying the condition $L(\mu) = cx$.
  - Then $L(\mu)$ is an optimal solution of the Lagrangian multiplier problem [i.e., $L^* = L(\mu)$] and x is an optimal solution to the optimization problem (P).

Lagrangian Relaxation Technique

- **Property (b) (Optimality Test)**
  - If for some choice of the Lagrangian multiplier vector μ, the solution x* of the Lagrangian relaxation is feasible in the optimization problem (P), then x* is an optimal solution to the optimization problem (P) and μ is an optimal solution to the Lagrangian multiplier problem.

$$L(\mu) = cx^* + \mu(Ax^* - b) \text{ and } Ax^* = b,$$

therefore,

$$L(\mu) = cx^*$$

Lagrangian Relaxation Technique

- ***One advantage of the Lagrangian relaxation approach***
  - the method can give us a certificate [in the form of the equality $L(\mu) = cx$ for some Lagrange multiplier μ] for guaranteeing that a given feasible solution x to the optimization problem (P) is an optimal solution.
  - Even if $L(\mu) < cx$, having the lower bound permits us to state a bound on how far a given solution is from optimality
  - If $[cx - L(\mu)]/L(\mu) \leq 0.05$, for example, we know that the objective function value of the feasible solution x is no more than 5% from optimality.

Lagrangian Relaxation Technique

- Lagrangian Relaxation and Inequality Constraints
  - The Lagrangian multiplier problem when we encounter models, that are formulated in inequality form $Ax \leq b$ becomes

$$L^* = \max_{\mu \geq 0} L(\mu).$$

  - the Lagrangian multipliers now are restricted to be nonnegative.
  - By introducing "slack variables" to formulate the inequality problem as an equivalent equality problem

Lagrangian Relaxation Technique

- Property.
  - Suppose that we apply Lagrangian relaxation to the optimization problem ($P^{\leq}$) defined as minimize {cx: Ax≤b and $x \in X$} by relaxing the inequalities $Ax \leq$ b.
  - Suppose, further, that for some choice of the Lagrangian multiplier vector μ, the solution x^* of the Lagrangian relaxation (1) is feasible in the optimization problem ($P^{\leq}$), and (2) satisfies the complementary slackness condition $\mu(Ax^* - b) = 0$.
  - Then x^* is an optimal solution to the optimization problem ($P^{\leq}$).

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