In the name of God

Network Flows

7. Multicommodity Flows Problems
7.1 Introduction

Fall 2010

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Introduction

Introduction

- In many application contexts, several physical commodities, vehicles, or messages, each governed by their own network flow constraints, share the same network.
- If the commodities do not interact in any way, then to solve problems with several commodities, we would solve each single-commodity problem separately.
- In other situations, however, because the commodities do share common facilities, the individual single commodity problems are not independent, so to find an optimal flow, we need to solve the problems in concert with each other.

Introduction

- One such model, known as the ***multicommodity flow problem***, in which the individual commodities share common arc capacities.
- That is, each arc has a capacity $\boldsymbol{u_{ij}}$ that restricts the total flow of all commodities on that arc.

Introduction

- **Notation:**
  - x_{ij}^{k} : the flow of commodity k on arc (i, j)
  - $\boldsymbol{x}^k$: the flow vector for commodity k.
  - $\boldsymbol{c}^k$: the per unit cost vector for commodity k.
  - Using this notation we Can formulate the multicommodity flow problem as follows:

Introduction

- ***The multicommodity flow problem formulation:***

$$\text{Minimize} \quad \sum_{1 \le k \le K} c^k x^k$$

subject to

$$\sum_{1 \le k \le K} x_{ij}^k \le u_{ij} \qquad \text{for all } (i, j) \in A,$$

$$\mathcal{N} x^k = b^k \qquad \text{for } k = 1, 2, \ldots, K,$$

$$0 \le x_{ij}^k \le u_{ij}^k \qquad \text{for all } (i, j) \in A \text{ and all } k = 1, 2, \ldots, K.$$

Introduction

- This formulation has a collection of K ordinary mass ***balance constraints***, modeling the flow of each commodity $k = 1, 2, \ldots, K$.

$$\mathcal{N}x^k = b^k \qquad \text{for } k = 1, 2, \ldots, K,$$

- The ***bundle constraints*** tie together the commodities by restricting the total flow of all the commodities on each arc (i, j) to at most u_{ij}.

$$\sum_{1 \le k \le K} x_{ij}^k \le u_{ij} \qquad \text{for all } (i, j) \in A,$$

Introduction

- Note that we also impose individual flow bounds on the flow of commodity k on arc (i, j).

$$0 \le x_{ij}^k \le u_{ij}^k \qquad \text{for all } (i, j) \in A \text{ and all } k = 1, 2, \ldots, K.$$

- Many applications do not impose these bounds, so for these applications we set each bound to $+\infty$.

Introduction

- In our discussion, it will be more convenient to state the bundle constraints as equalities instead of inequalities.
- In these instances we introduce nonnegative ***slack*** variables $\boldsymbol{s_{ij}}$ and write the bundle constraints as:

$$\sum_{1 \le k \le K} x_{ij}^k + s_{ij} = u_{ij} \qquad \text{for all } (i, j) \in A.$$

- The slack variable s_{ij} for the arc (i, j) measures the unused bundle capacity on that arc.

Assumptions

- The multicommodity flow model imposes capacities on the arcs but not on the nodes.
- This modeling assumption imposes no loss of generality, since by using the node splitting techniques, we can use this formulation to model situations with node capacities as well.
- Three assumptions are:
  - ***Homogeneous goods assumption***
  - ***No congestion assumption***
  - ***Indivisible goods assumption***

Assumptions

- ***Homogeneous goods assumption***
  - We are assuming that every unit flow of each commodity uses 1 unit of capacity of each arc.
  - A more general model would permit the unit flow of each commodity *k* to consume a given amount ρ_{ij}^k of the capacity associated with each arc (*i*, *j*), and replace the bundle constraint with a more general resource availability constraint:

$$\sum_{1 \le k \le K} \rho_{ij}^k x_{ij}^k \le u_{ij}.$$

Assumptions

- ***No congestion assumption***
  - We are assuming that we have a hard (i.e., fixed) capacity on each arc and that the cost on each arc is linear in the flow on that arc.
  - In some applications as the flow of any commodity increases on an arc, we incur an increasing and nonlinear cost on that arc.
  - **For example**, in traffic networks where the objective function is to find the flow pattern of all the commodities that minimizes overall system delay.
    - In this setting, because of queuing effects, the greater the flow on an arc, the greater is the queuing delay on that arc.
    - This make the ***nonlinear multicommodity flow problems***.

Assumptions

- ***Indivisible goods assumption***
  - The model assumes that the flow variables can be fractional.
  - In some applications the variables must be integer valued.
  - In these instances the model that we are considering might still prove to be useful,
    - the linear programming model might either be a good approximation of the integer programming model
    - or we can use the linear programming model as a linear programming relaxation of the integer program and embed it within branch-and-bound approach.

Solution Approaches

- Researchers have developed several approaches for solving the multicommodity flow problem, including:
  - ***Price-directive decomposition methods***
    - ***Lagrangian Relaxation method***
    - ***Dantzig-Wolfe decomposition method***
  - ***Resource-directive decomposition methods***
  - ***Partitioning methods***

Solution Approaches

- ***Lagrangian Relaxation method***
  - bring bundle constraints into the objective function and place ***Lagrangian multipliers*** or ***prices*** on them
  - the resulting problem decomposes into a separate minimum cost flow problem for each commodity k.
  - These methods remove the capacity constraints and instead ***charge*** each commodity for the use of the capacity of each arc.
  - These methods attempt to find appropriate ***prices*** so that some optimal solution to the resulting ***pricing problem*** or ***Lagrangian subproblem*** also solves the overall multicommodity flow problem.
  - Several methods are available for finding appropriate prices.

Solution Approaches

- ***Dantzig-Wolfe decomposition method***
  - This is another approach for finding the correct prices;
  - this method is a general-purpose approach for decomposing problems that have a set of ***easy constraints*** and also a set of ***hard constraints*** (that is, constraints that make the problem much more difficult to solve).
  - For multicommodity flow problems, the ***network flow constraints*** are the easy constraints and the ***bundle constraints*** are the hard constraints.
  - The approach begins by ignoring or imposing prices on the bundle constraints and solving subproblems with only the single-commodity network flow constraints.

Solution Approaches

- ***Dantzig-Wolfe decomposition method* (cont.)**
  - The method uses linear programming to update the ***prices*** so that the solutions generated from the subproblems satisfy the bundle constraints.
  - The method iteratively solves two different problems:
    - A subproblem and
    - A price-setting linear program.

Solution Approaches

- ***Resource-directive decomposition methods***
  - These methods view the multicommodity flow problem as a ***capacity allocation problem***.
  - All the commodities are competing for the fixed capacity $\boldsymbol{u_{ij}}$ of every arc (i, j) of the network.
  - ***Resource-directive methods*** begin by allocating the capacities to the commodities, and
  - Then use information collected from the solution to the resulting single-commodity problems to reallocate the capacities in a way that improves the overall system cost.

Solution Approaches

- ***Partitioning methods***
  - These methods exploit the fact that the multicommodity flow problem is a specially structured linear program with embedded network flow problems.
  - We can use the ***network simplex method*** to solve any single-commodity flow problem, which works by generating a sequence of improving ***spanning tree solutions***.
  - The partitioning method maintains a linear programming basis that is composed of bases (spanning trees) of the individual single-commodity flow problems as well as additional arcs that are required to "tie" these solutions together to accommodate the bundle constraints.

Optimality Conditions

Optimality Conditions

- **Optimality conditions** for the multicommodity flow problem is for characterizing when a given feasible solution was optimal.
- It permitted us to assess whether or not we have found an optimal solution to the problem.
- Since the multicommodity flow problem is a linear program, we can use linear programming optimality conditions to characterize optimal solutions to the problem.

Optimality Conditions

- ***The multicommodity flow problem formulation:***

$$\text{Minimize} \quad \sum_{1 \le k \le K} c^k x^k$$

subject to

$$\sum_{1 \le k \le K} x_{ij}^k \le u_{ij} \qquad \text{for all } (i, j) \in A,$$

$$\mathcal{N} x^k = b^k \qquad \text{for } k = 1, 2, \ldots, K,$$

$$0 \le x_{ij}^k \quad \text{for all } (i, j) \in A \text{ and all } k = 1, 2, \ldots, K.$$

Optimality Conditions

- The multicommodity flow formulation has one bundle constraint for every arc (i, j) of the network and one mass balance constraint for each node-commodity combination
- Types of dual variables of dual program :
  - a **price** w_{ij} on each arc (i, j)
  - a **node potential** $\boldsymbol{\pi}^k(i)$ for each combination of commodity k and node i.

Optimality Conditions

- ***The dual of the multicommodity flow problem***:

$$\text{Maximize} \quad -\sum_{(i,j)\in A} u_{ij}w_{ij} + \sum_{k=1}^{K} b^k \pi^k$$

subject to

$$c_{ij}^{\pi,k} = c_{ij}^k + w_{ij} - \pi^k(i) + \pi^k(j) \geq 0 \qquad \text{for all } (i, j) \in A \text{ and all } k = 1, \ldots, K,$$

$$w_{ij} \geq 0 \qquad \text{for all } (i, j) \in A.$$

Optimality Conditions

- The reduced cost of arc (i, j) with respect to commodity k as follows:

$$c_{ij}^{\pi,k} = c_{ij}^{k} + w_{ij} - \pi^{k}(i) + \pi^{k}(j).$$

- In matrix notation, this definition is:

$$c^{\pi,k} = c^{k} + w - \pi^{k}\mathcal{N}.$$

Optimality Conditions

- **Complementary slackness (optimality) conditions**
  - The optimality conditions for a linear programming, called the complementary slackness (optimality) conditions,
  - It states that a primal feasible solution x and a dual feasible solution $(\boldsymbol{w}, \boldsymbol{\pi}^k)$ are optimal to the respective problems if and only if the product of each primal (dual) variable and the slack in the corresponding dual (primal) constraint is zero.

Optimality Conditions

- ***Multicommodity flow complementary slackness conditions***
  - Let y_{ij}^k denote a specific value of the flow variable x_{ij}^k
  - The commodity flows y_{ij}^k are optimal in the multicommodity flow problem if and only if they are feasible and for some choice of (nonnegative) arc prices w_{ij} and (unrestricted in sign) node potentials $\pi^k(i)$, the reduced costs and arc flows satisfy the complementary slackness conditions

Optimality Conditions

- ***Multicommodity flow complementary slackness conditions***

(a) $w_{ij} \left(\sum_{1 \le k \le K} y_{ij}^k - u_{ij} \right) = 0$ *for all arcs* $(i, j) \in A$.

(b) $c_{ij}^{\pi,k} \ge 0$ *for all arcs* $(i, j) \in A$ *and*

all commodities $k = 1, 2, \ldots, K$.

(c) $c_{ij}^{\pi,k} y_{ij}^k = 0$ *for all arcs* $(i, j) \in A$ *and*

all commodities $k = 1, 2, \ldots, K$.

Optimality Conditions

- Condition (a)
  - states that the price w_{ij} of arc (i, j) is zero if the optimal solution does not use all of the capacity u_{ij} of the arc.
  - That is, if the optimal solution does not fully use the capacity of that arc, we could ignore the constraint (place no price on it).

- ***Optimal arc prices and optimal node potentials***
  - We refer to any set of arc prices and node potentials that satisfy the complementary slackness conditions as **optimal arc prices** and **optimal node potentials**.

Optimality Conditions

- The connection between the multicommodity and single-commodity flow problems.
- **Theorem:** ***Partial Dualization***
  - Let y_{ij}^k be optimal flows and let w_{ij} be optimal arc prices for the multicommodity flow problem. Then for each commodity k, the flow variables y_{ij}^k for $(i, j) \in A$ solve the following (uncapacitated) minimum cost flow problem:

$$\min\{ \sum_{(i,j)\in A} (c_{ij}^k + w_{ij})x_{ij}^k : \mathcal{N}x^k = b,\ x_{ij}^k \geq 0 \textit{ for all } (i, j) \in A\}.$$

Optimality Conditions

- **Proof.**
  - Since y_{ij}^k are optimal flows and w_{ij} are optimal arc prices for the multicommodity flow problem, these variables together with some set of node potentials $\pi^k(i)$ satisfy the complementary slackness condition.
  - The following conditions are the optimality conditions for the uncapacitated minimum cost flow problem for commodity k with arc costs $(c_{ij}^k + w_{ij})$

$$\text{(b)} \quad c_{ij}^{\pi,k} \geq 0 \quad \textit{for all arcs } (i, j) \in A \textit{ and all commodities } k = 1, 2, \ldots, K.$$

$$\text{(c)} \quad c_{ij}^{\pi,k} y_{ij}^k = 0 \quad \textit{for all arcs } (i, j) \in A \textit{ and all commodities } k = 1, 2, \ldots, K.$$

  - This observation implies that the flows y_{ij}^k solve the corresponding minimum cost flow problems. .

The End