In the name of God

Part 4. Decomposition Algorithms

4.1. Dantzig-Wolf Decomposition Algorithm

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Introduction

Introduction

- Real world linear programs having thousands of rows and columns.
- In such problems, **decomposition methods** must be applied to convert the large problems into one or more appropriately smaller problems of manageable sizes.
- Decomposition methods
  - **Dantzig-Wolf decomposition technique**
  - **Benders partitioning technique**
  - **Lagrangian relaxation technique**

Introduction

- The decomposition principle
  - is a systematic procedure for solving large-scale linear programs or
  - linear programs that contain constraints of special structure.
- The constraints are divided into two sets:
  - general constraints (or complicating constraints) and
  - constraints with special structure.

Introduction

- The strategy of the decomposition procedure is to operate on two separate linear programs:
  - one over the set of general constraints and
  - one over the set of special constraints.
- Information is passed back and forth between the two linear programs until a point is reached where the solution to the original problem is achieved.
- **Master problem vs. Subproblem**
  - The linear program over the general constraints is called the **master problem**, and
  - the linear program over the special constraints is called the subproblem.

Introduction

- The master problem passes down a new set of cost coefficients to the subproblem and receives a new column based on these cost coefficients.

Decomposition Algorithm

Decomposition Algorithm

- Consider the following linear program:

$$\begin{aligned} &\text{Minimize} \quad \mathbf{cx} \\ &\text{Subject to } \mathbf{Ax} = \mathbf{b} \\ &\qquad\qquad\qquad \mathbf{x} \in X \end{aligned}$$

- where
  - **A** is an $m \times n$ matrix
  - **b** is an m vector
  - X is a **polyhedral set** representing constraints of special structure (assume that X is bounded)
  - **x** is any point of X

Decomposition Algorithm

- Any point $\mathbf{x} \in X$ can be represented as a convex combination of the extreme points of X
- Denoting **extreme points of** X by $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_t$
  - Where t is the number of extreme points of the set X
- Any $\mathbf{x} \in X$ can be represented as:

$$\mathbf{x} = \sum_{j=1}^{t} \lambda_j \mathbf{x}_j$$

$$\sum_{j=1}^{t} \lambda_j = 1$$

$$\lambda_j \geqslant 0 \qquad j = 1, 2, \ldots, t$$

Decomposition Algorithm

- Substituting for **x**, the foregoing optimization problem can be transformed into the following so-called **master problem** in the variables $\lambda_1, \lambda_2, \ldots, \lambda_t$

$$\text{Minimize} \quad \sum_{j=1}^{t} (\mathbf{c}\mathbf{x}_j)\lambda_j$$

$$\text{Subject to} \quad \sum_{j=1}^{t} (\mathbf{A}\mathbf{x}_j)\lambda_j = \mathbf{b}$$

$$\sum_{j=1}^{t} \lambda_j = 1$$

$$\lambda_j \geqslant 0 \qquad j = 1, 2, \ldots, t$$

Decomposition Algorithm

- Since *t*, the number of extreme points of the set *X*, is usually very large,
- Attempting to explicitly enumerate all the extreme points $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_t$ and explicitly solving this problem is a very difficult task.
- We shall attempt to find an optimal solution of the problem (and hence the original problem) without explicitly enumerating all the extreme points.

Application of the Revised Simplex Method

Application of the Revised Simplex Method

- Suppose that we have a basic feasible solution

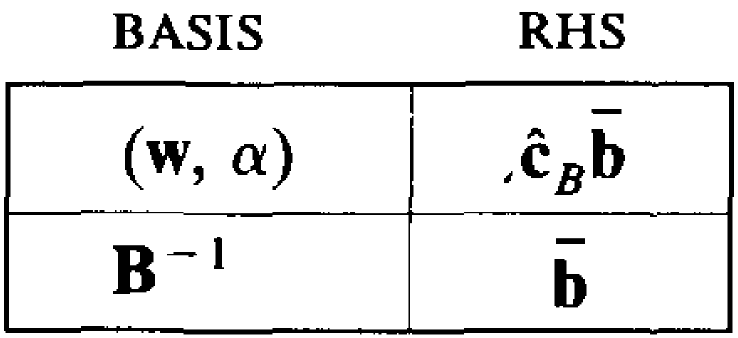
$$\lambda = (\lambda_{\mathbf{B}}, \lambda_{\mathbf{N}})$$

- Further suppose that the $(m + 1) \times (m + 1)$ basis inverse $\mathbf{B}^{-1}$ is known
- Denoting the dual variables **w** and α to constraints:

$$\begin{array}{llll}
\text{Minimize} & \displaystyle\sum_{j=1}^{t} (\mathbf{c}\mathbf{x}_j)\lambda_j & & \\
 & & & \text{dual variables} \\
\text{Subject to} & \displaystyle\sum_{j=1}^{t} (\mathbf{A}\mathbf{x}_j)\lambda_j = \mathbf{b} & & \mathbf{w} \\
 & \displaystyle\sum_{j=1}^{t} \lambda_j = 1 & & \alpha \\
 & \lambda_j \geqslant 0 & j = 1, 2, \ldots, t &
\end{array}$$

Application of the Revised Simplex Method

- $(\mathbf{w}, \alpha) = \hat{\mathbf{c}}_B \mathbf{B}^{-1}$
  - where $\hat{\mathbf{c}}_B$ is the cost of the basic variables with $\hat{c}_j = \mathbf{c}\mathbf{x}_j$ for each basic variable λ_j
  - The basis inverse, the dual variables, the values of the basic variables, and the objective function are:



BASIS	RHS
$(\mathbf{w}, \alpha)$	$\hat{\mathbf{c}}_B \bar{\mathbf{b}}$
$\mathbf{B}^{-1}$	$\bar{\mathbf{b}}$

  - Where $\bar{\mathbf{b}} = \mathbf{B}^{-1}\begin{pmatrix}\mathbf{b}\\1\end{pmatrix}$

Application of the Revised Simplex Method

- The revised simplex method proceeds by concluding that the current solution is optimal or else by deciding to increase a nonbasic variable.
- This is done by first calculating:

$$z_k - \hat{c}_k = \underset{1 \leq j \leq t}{\text{Maximum}}\ z_j - \hat{c}_j$$

$$= \underset{1 \leq j \leq t}{\text{maximum}}\ (\mathbf{w}, \alpha)\begin{bmatrix} \mathbf{Ax}_j \\ 1 \end{bmatrix} - \mathbf{cx}_j$$

$$= \underset{1 \leq j \leq t}{\text{maximum}}\ \mathbf{wAx}_j + \alpha - \mathbf{cx}_\mathbf{j}$$

Application of the Revised Simplex Method

- Since $z_j - \hat{c}_j = 0$ for basic variables, then the foregoing maximum is ≥ 0.
- Thus if $z_k - \hat{c}_k = 0$, then $z_j - \hat{c}_j \leq 0$ for all nonbasic variables and the optimal solution is at hand.
- On the other hand, if $z_k - \hat{c}_k > 0$, then the nonbasic variable λ_k is increased.

Application of the Revised Simplex Method

- Determining the index *k* using

$$\underset{1 \leqslant j \leqslant t}{\text{maximum}} \; \mathbf{wAx}_j + \alpha - \mathbf{cx}_{\mathbf{j}}$$

- computationally infeasible because *t* is very large and the extreme points $\mathbf{x}_j$'s corresponding to the nonbasic λ_j's are not explicitly known.
- Therefore an alternative scheme must be devised.
- Since *X* is a bounded polyhedral set, the maximum of any linear objective can be achieved at one of the extreme points. Therefore

$$\underset{1 < j < t}{\text{Maximum}} \, (\mathbf{wA} - \mathbf{c})\mathbf{x}_j + \alpha = \underset{\mathbf{x} \in X}{\text{Maximum}} \, (\mathbf{wA} - \mathbf{c})\mathbf{x} + \alpha$$

Application of the Revised Simplex Method

- To summarize, given a basic feasible solution (λ_B, λ_N) with dual variables ($\mathbf{w}$, α),
- solve the following linear **subproblem**, which is "easy" because of the special structure of X.

$$\text{Maximize} \quad (\mathbf{wA} - \mathbf{c})\mathbf{x} + \alpha$$

$$\text{Subject to} \quad \mathbf{x} \in X$$

Application of the Revised Simplex Method

- Note that the objective function contains a constant.
- This is easily handled by initializing the RHS value for z to α instead of the normal value of 0.
- Let $\mathbf{x}_k$ be an optimal solution to the foregoing subproblem with objective value z_k - $\hat{c}_k$
- If z_k - $\hat{c}_k = 0$, then the basic feasible solution $(\boldsymbol{\lambda}_{\mathbf{B}}, \boldsymbol{\lambda}_{\mathbf{N}})$ is optimal.
- Otherwise if z_k - $\hat{c}_k > 0$, then the variable λ_k enters the basis.

Application of the Revised Simplex Method

- As in the revised simplex method the corresponding column $\begin{pmatrix} \mathbf{Ax}_k \\ 1 \end{pmatrix}$
- is updated by premultiplying it by $\mathbf{B}^{-1}$ giving

$$\mathbf{y}_k = \mathbf{B}^{-1}\begin{pmatrix} \mathbf{Ax}_k \\ 1 \end{pmatrix}$$

- Note that $\mathbf{y}_k \leq 0$ cannot occur since X was assumed bounded; producing a bounded master problem.

Application of the Revised Simplex Method

- The updated column

$$\begin{pmatrix} z_k - \hat{c}_k \\ \mathbf{y}_k \end{pmatrix}$$

- is adjoined to the foregoing above array.
- The variable λ_{Br} leaving the basis is determined by the usual minimum ratio test.
- The basis inverse, dual variables, and right RHS are updated by pivoting at y_{rk}
- After updating, the process is repeated.

Application of the Revised Simplex Method

- Note that the **master step** gives an improved feasible solution of the overall problem, and
- The **subproblem step** checks whether z_j - $\hat{c}_j \leq 0$ for all λ_j or else determines the most positive z_k - $\hat{c}_k$.

Summary of the Decomposition Algorithm

Summary of the Decomposition Algorithm

- **INITIALIZATION STEP**
  - Find an initial basic feasible solution of the system:

$$\text{Minimize} \quad \sum_{j=1}^{t} (\mathbf{c}\mathbf{x}_j)\lambda_j$$

$$\text{Subject to} \quad \sum_{j=1}^{t} (\mathbf{A}\mathbf{x}_j)\lambda_j = \mathbf{b}$$

$$\sum_{j=1}^{t} \lambda_j = 1$$

$$\lambda_j \geqslant 0 \qquad j = 1, 2, \ldots, t$$

Summary of the Decomposition Algorithm

- Let the basis be **B** and form the following **master array**

BASIS INVERSE	RHS
$(\mathbf{w}, \alpha)$	$\hat{\mathbf{c}}_B\bar{\mathbf{b}}$
$\mathbf{B}^{-1}$	$\bar{\mathbf{b}}$

- The master array includes basis inverse, the dual variables, the values of the basic variables, and the objective function
- Where $(\mathbf{w}, \alpha) = \hat{\mathbf{c}}_\mathbf{B}\,\mathbf{B}^{-1}$ with $\hat{c}_j = \mathbf{c}\mathbf{x}_\mathbf{j}$ for each basic variable λ_j
- and

$$\bar{\mathbf{b}} = \mathbf{B}^{-1}\begin{pmatrix}\mathbf{b}\\ 1\end{pmatrix}$$

Summary of the Decomposition Algorithm

- **MAIN STEP**
  - **1. Solve the following subproblem:**

$$\begin{array}{ll} \text{Maximize} & (\mathbf{wA} - \mathbf{c})\mathbf{x} + \alpha \\ \text{Subject to} & \mathbf{x} \in X \end{array}$$

  - Let $\mathbf{x}_k$ be an optimal basic feasible solution with objective value of z_k - $\hat{c}_k$
  - If z_k - $\hat{c}_k = 0$ stop, z_j - $\hat{c}_j \leq 0$ for all nonbasic variables and the optimal solution is at hand.
  - Otherwise (If z_k - $\hat{c}_k > 0$) go to step 2.
    - The nonbasic variable λ_k is increased

Summary of the Decomposition Algorithm

- **2. Let** $\mathbf{y}_k = \mathbf{B}^{-1}\begin{bmatrix} \mathbf{A}\mathbf{x}_k \\ 1 \end{bmatrix}$
- and adjoin the updated column to the master array

$$\begin{pmatrix} z_k - \hat{c}_k \\ \mathbf{y}_k \end{pmatrix}$$

- Pivot at y_{rk} where the index r is determined as follows:

$$\frac{\bar{b}_r}{y_{rk}} = \underset{1 \leqslant i \leqslant m+1}{\text{Minimum}} \left\{ \frac{\bar{b}_i}{y_{ik}} : y_{ik} > 0 \right\}$$

- This updates the dual variables, the basis inverse, and the right-hand side.
- After pivoting, delete the column of λ_k and go to step 1.

Some Remarks

Some Remarks

- **Note 1:**
  - The foregoing algorithm is a direct implementation of the revised simplex method except that the calculation z_k - $\hat{c}_k$ is performed by solving a subproblem.
  - Therefore the algorithm converges in a finite number of iterations provided that a cycling prevention rule is used in both the master step and the subproblem in the presence of degeneracy.

Some Remarks

- **Note 2:**
  - At each iteration the master step provides a new improved basic feasible solution of the system given by introducing the nonbasic variable λ_k, which is generated by the subproblem.
  - At each iteration the subproblem provides an extreme point $\mathbf{x}_k$,which corresponds to an updated column

$$\begin{bmatrix} z_k - \hat{c}_k \\ \mathbf{y}_k \end{bmatrix}$$

  - and hence this procedure is sometimes referred to as a **column generation scheme**.

Some Remarks

- **Note 3:**
  - At each iteration a different **dual vector** is passed from the master step to the subproblem.
  - Rather than solving the subproblem anew at each iteration, the optimal basis of the last iteration could be utilized by modifying the cost row.

Some Remarks

- **Note 4:**
  - At each iteration, the subproblem need not be completely optimized.
  - It is only necessary that the current extreme point $\mathbf{x}_k$ satisfies

    $$z_k - \hat{c}_k = (\mathbf{wA} - \mathbf{c})\ \mathbf{x}_k + \alpha > 0$$

  - In this case λ_k is a candidate to enter the basis of the master problem.

Some Remarks

- **Note 5:**
  - If the master constraints are of the inequality type, then we must check the $z_j - \hat{c}_j$ for **nonbasic slack variables** in addition to solving the subproblem.
  - For a master constraint i of the $\leq$ type with associated slack variables s_i we get:

$$z_{s_i} - c_{s_i} = (\mathbf{w}, \alpha)\begin{pmatrix} \mathbf{e}_i \\ 0 \end{pmatrix} - 0 = w_i$$

  - Thus, for a minimization problem a slack variable associated with a $\leq$ constraint is eligible to enter the basis if $\boldsymbol{w_i > 0}$.
  - The entry criterion is $w_i < 0$ for constraints of the $\geq$ type.

Calculation and Use of Lower Bounds

Calculation and Use of Lower Bounds

- Recall that the decomposition algorithm stops when Maximum z_j - $\hat{c}_j = 0$.
- Because of the large number of variables λ_1 , λ_2, ..., λ_t continuing the computations until this condition is satisfied may be time-consuming for large problems.
- We shall develop a **lower bound** on the objective of any feasible solution of the overall problem, and hence a lower bound on the optimal objective.

Calculation and Use of Lower Bounds

- Since the decomposition algorithm generates feasible points with improving objective values, we may stop when the difference between the objective of the current feasible point and the lower bound is within an acceptable tolerance.
- This may not give the true optimal point, but will guarantee good feasible solutions, within any desirable accuracy from the optimal.

Calculation and Use of Lower Bounds

- Consider the following subproblem:

$$\begin{array}{ll} \text{Maximize} & (\mathbf{wA} - \mathbf{c})\mathbf{x} + \alpha \\ \text{Subject to} & \mathbf{x} \in X \end{array}$$

  - where **w** is the dual vector passed from the master step.
- Let the optimal objective of the foregoing subproblem be z_k - $\hat{c}_k$
- Now let **x** be any feasible solution of the overall problem, that is, $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{x} \in X$.

Calculation and Use of Lower Bounds

- We have

$$(\mathbf{wA} - \mathbf{c})\mathbf{x} + \alpha \leqslant (z_k - \hat{c}_k)$$

- Since **Ax = b**, then the above inequality implies that

$$\mathbf{cx} \geqslant \mathbf{wAx} - (z_k - \hat{c}_k) + \alpha = \mathbf{wb} + \alpha - (z_k - \hat{c}_k)$$
$$= \hat{\mathbf{c}}_B\bar{\mathbf{b}} - (z_k - \hat{c}_k)$$

- Since this is true for each $\mathbf{x} \in X$ with **Ax = b**, then

$$\underset{\substack{\mathbf{Ax}=\mathbf{b} \\ \mathbf{x}\in X}}{\text{Minimum}} \quad \mathbf{cx} \geqslant \hat{\mathbf{c}}_B\bar{\mathbf{b}} - (z_k - \hat{c}_k)$$

Calculation and Use of Lower Bounds

- In other words,

$$\hat{\mathbf{c}}_B\bar{\mathbf{b}} - (z_k - \hat{c}_k)$$

- is a lower bound on the optimal objective value of the overall problem.

Numeric Example

Numeric Example

- Consider the following problem:

$$\begin{array}{llllll}
\text{Minimize} & -2x_1 - & x_2 - & x_3 + & x_4 & \\
\text{Subject to} & x_1 & & + \; x_3 & & \leqslant 2 \\
 & x_1 + & x_2 & & +2x_4 & \leqslant 3 \\
 & x_1 & & & & \leqslant 2 \\
 & x_1 + 2x_2 & & & & \leqslant 5 \\
 & & & -\; x_3 + & x_4 & \leqslant 2 \\
 & & & 2x_3 + & x_4 & \leqslant 6 \\
 & x_1, & x_2, & x_3, & x_4 & \geqslant 0
\end{array}$$

Numeric Example

- Note that:
  - The third and fourth constraints involve only x_1 and x_2
  - The fifth and sixth constraints involve only x_3 and x_4
- Let
  - X consist of the last four constraints, in addition to the nonnegativity restrictions,
  - then minimizing a linear function over X becomes a simple process, because the subproblem can be decomposed into two subproblems.

Numeric Example

- We shall handle the first two constraints as:

$$\mathbf{Ax} \le \mathbf{b}$$

- where

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ 3 \end{bmatrix},$$

Initialization Step

- The problem is reformulated as follows:

$$\begin{aligned} \text{Minimize} \quad & \sum_{j=1}^{t} \hat{c}_j \lambda_j \\ \text{Subject to} \quad & \sum_{j=1}^{t} (\mathbf{A}\mathbf{x}_j)\, \lambda_j + \mathbf{s} = \mathbf{b} \\ & \sum_{j=1}^{t} \lambda_j = 1 \\ & \lambda_j \geqslant 0 \qquad j = 1, 2, \ldots, t \\ & \mathbf{s} \geqslant \mathbf{0} \end{aligned}$$

- where $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_t$ are the extreme points of X,
- $\hat{c}_j = \mathbf{c}\mathbf{x}_j$ for $j = 1, 2, \ldots, t$ and
- $\mathbf{s} \geq \mathbf{0}$ is the slack vector

Initialization Step

- We need a starting basis with known $\mathbf{B}^{-1}$.
- Let the starting basis consist of **s** and λ_1,
- Where $\mathbf{x}_1 = (0, 0, 0, 0)$ is an extreme point of X with $\mathbf{c}\mathbf{x}_1 = 0$.
- Therefore:

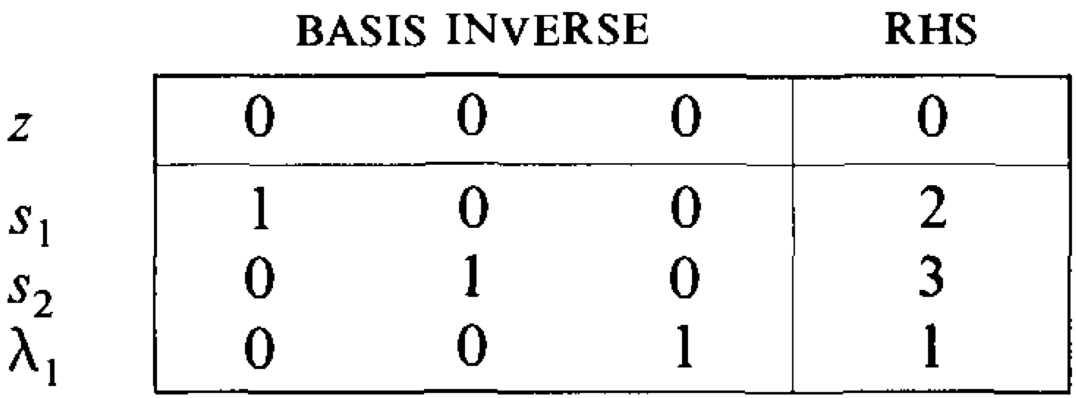
$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(\mathbf{w}, \alpha) = \hat{\mathbf{c}}_B \mathbf{B}^{-1} = \mathbf{0}\mathbf{B}^{-1} = \mathbf{0}$$

$$\bar{\mathbf{b}} = \mathbf{B}^{-1} \begin{bmatrix} \mathbf{b} \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ 1 \end{bmatrix}$$

Initialization Step

- This gives the following tableau.



	BASIS INVERSE			RHS
z	0	0	0	0
s_1	1	0	0	2
s_2	0	1	0	3
λ_1	0	0	1	1

- The first three columns in row 0 give (w_1, w_2, α) and $\mathbf{B}^{-1}$ in the remaining rows.

Iteration 1: SUBPROBLEM

- Solve the following subproblem

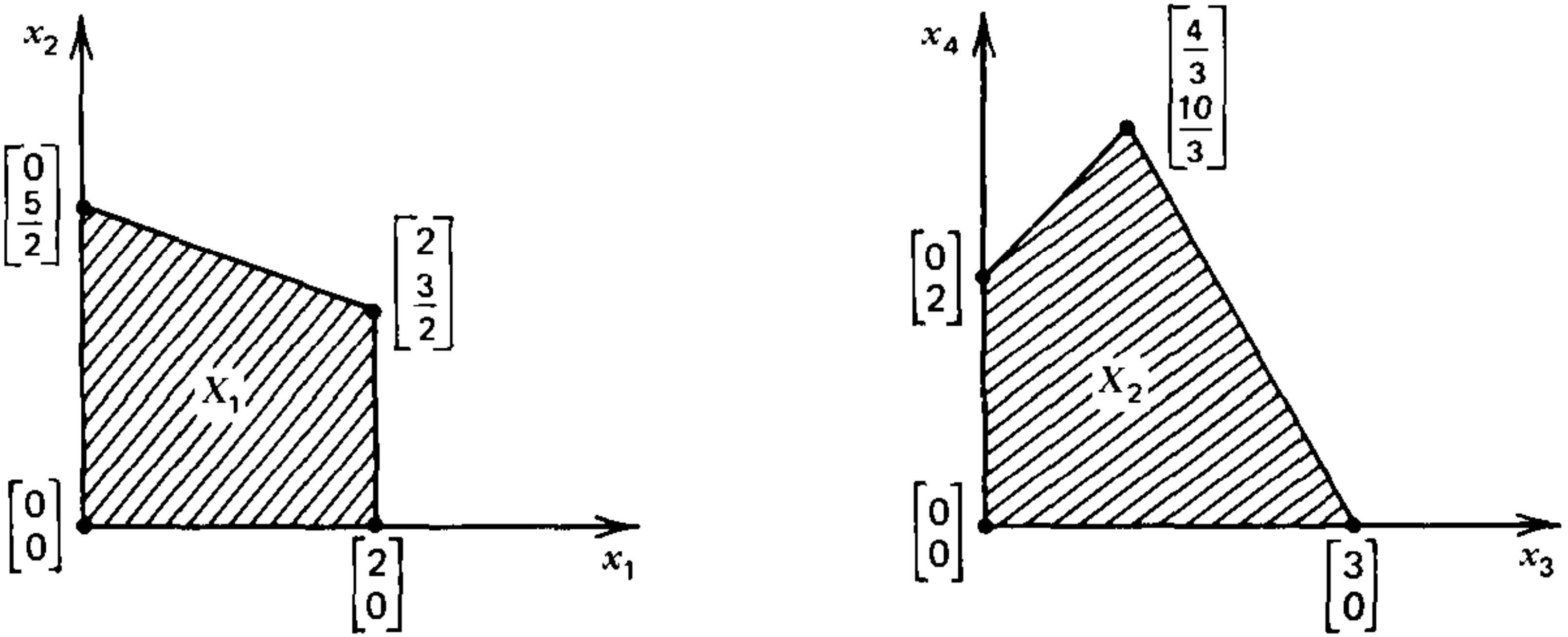
$$\begin{array}{ll} \text{Maximize} & (\mathbf{wA} - \mathbf{c})\mathbf{x} + \alpha \\ \text{Subject to} & \mathbf{x} \in X \end{array}$$

- Here $(w_1, w_2) = (0, 0)$, Therefore the subproblem is as follows:

$$\begin{array}{ll} \text{Maximize} & 2x_1 + x_2 + x_3 - x_4 + 0 \\ \text{Subject to} & \mathbf{x} \in X \end{array}$$

Iteration 1: SUBPROBLEM

- This problem is separable in the vectors (x_1, x_2) and (x_3, x_4) and can be solved geometrically.



- **Representation of *X* by two sets.**

Iteration 1: SUBPROBLEM

- It is easily verified that the optimal solution is

$$\mathbf{x}_2 = (2,\ \tfrac{3}{2},\ 3,\ 0)$$

- with objective

$$z_2 - \hat{c}_2 = \frac{17}{2}$$

- Since

$$z_2 - \hat{c}_2 = \frac{17}{2} > 0$$

- Then λ_2 corresponding to $\mathbf{x}_2$ is introduced.
- The lower bound: $\hat{\mathbf{c}}_B\bar{\mathbf{b}} - (z_2 - \hat{c}_2) = 0 - \frac{17}{2}$
- Recall that the best objective so far is 0.

Iteration 1: MASTER STEP

- MASTER STEP

$$z_2 - \hat{c}_2 = \frac{17}{2}$$

$$\mathbf{A}\mathbf{x}_2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ \frac{3}{2} \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ \frac{7}{2} \end{bmatrix}$$

- Then

$$\begin{bmatrix} \mathbf{A}\mathbf{x}_2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ \frac{7}{2} \\ 1 \end{bmatrix}$$

Iteration 1: MASTER STEP

- is updated by premultiplying by $\mathbf{B}^{-1}$, So

$$\mathbf{y}_2 = \mathbf{B}^{-1}\begin{bmatrix} 5 \\ \frac{7}{2} \\ 1 \end{bmatrix} = \mathbf{I}\begin{bmatrix} 5 \\ \frac{7}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ \frac{7}{2} \\ 1 \end{bmatrix}$$

- Insert the column

$$\begin{bmatrix} z_2 - \hat{c}_2 \\ \mathbf{y}_2 \end{bmatrix} = \begin{bmatrix} \frac{17}{2} \\ 5 \\ \frac{7}{2} \\ 1 \end{bmatrix}$$

- into the foregoing array and pivot.

Iteration 1: MASTER STEP

- This leads to the following two tableaux

	BASIS INVERSE			RHS	λ_2
z	0	0	0	0	$\frac{17}{2}$
s_1	1	0	0	2	⑤
s_2	0	1	0	3	$\frac{7}{2}$
λ_1	0	0	1	1	1

	BASIS INVERSE			RHS
z	$\frac{17}{10}$	0	0	$-\frac{17}{5}$
λ_2	$\frac{1}{5}$	0	0	$\frac{2}{5}$
s_2	$-\frac{7}{10}$	1	0	$\frac{8}{5}$
λ_1	$-\frac{1}{5}$	0	1	$\frac{3}{5}$

Iteration 1: MASTER STEP

- The best-known feasible solution of the overall problem is given

$$\mathbf{x} = \lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2$$

$$= \tfrac{3}{5}(0, 0, 0, 0) + \tfrac{2}{5}(2, \tfrac{3}{2}, 3, 0) = (\tfrac{4}{5}, \tfrac{3}{5}, \tfrac{6}{5}, 0)$$

- The objective is $-\frac{17}{5}$
- Also

$$(w_1, w_2, \alpha) = (-\tfrac{17}{10}, 0, 0)$$

Iteration 2

- **Iteration 2**
  - Because the master constraints are of the inequality type, then we must check the z_j - $\hat{c}_j$ for **nonbasic slack variables**.
  - $\mathbf{s_1}$ is a nonbasic slack variavble
  - Since $w_1 < 0$, s_1 is not qualified to enter the basis at this time.

Iteration 2: SUBPROBLEM

- **SUBPROBLEM**
  - Solve the following problem:

$$\begin{array}{ll} \text{Maximize} & (\mathbf{wA} - \mathbf{c})\mathbf{x} + \alpha \\ \text{Subject to} & \mathbf{x} \in X \end{array}$$

$$\mathbf{wA} - \mathbf{c} = \left(-\tfrac{17}{10}, 0\right)\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 2 \end{bmatrix} - (-2, -1, -1, 1)$$

$$= \left(\tfrac{3}{10}, 1, -\tfrac{7}{10}, -1\right)$$

  - Therefore the subproblem is:

$$\begin{array}{ll} \text{Maximize} & \tfrac{3}{10}x_1 + x_2 - \tfrac{7}{10}x_3 - x_4 + 0 \\ \text{Subject to} & \mathbf{x} \in X \end{array}$$

Iteration 2: SUBPROBLEM

- The problem decomposes into two problems involving (x_1, x_2) & (x_3, x_4).
- Using the figure, the optimal solution is

$$\mathbf{x}_3 = (0, \tfrac{5}{2}, 0, 0)$$

- with objective

$$z_3 - \hat{c}_3 = \frac{5}{2}$$

- Since $z_3 - \hat{c}_3 > 0$, then λ_3 is introduced.
- The lower bound is:

$$\hat{\mathbf{c}}_B\bar{\mathbf{b}} - (z_3 - \hat{c}_3) = -\frac{17}{5} - \frac{5}{2} = -5.9.$$

Iteration 2: MASTER STEP

- **MASTER STEP**

$$z_3 - \hat{c}_3 = \frac{5}{2}$$

$$\mathbf{A}\mathbf{x}_3 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{5}{2} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{5}{2} \end{bmatrix}$$

$$\mathbf{y}_3 = \mathbf{B}^{-1} \begin{bmatrix} \mathbf{A}\mathbf{x}_3 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & 0 & 0 \\ -\frac{7}{10} & 1 & 0 \\ -\frac{1}{5} & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{5}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{5}{2} \\ 1 \end{bmatrix}$$

Iteration 2: MASTER STEP

- Insert the column

$$\begin{bmatrix} z_3 - \hat{c}_3 \\ \mathbf{y}_3 \end{bmatrix}$$

  - into the foregoing array and pivot.
- This leads to the following two tableaux
- The λ_3 column is deleted after pivoting.

Iteration 2: MASTER STEP

	BASIS INVERSE			RHS	λ_3
z	$-\frac{17}{10}$	0	0	$-\frac{17}{5}$	$\frac{5}{2}$
λ_2	$\frac{1}{5}$	0	0	$\frac{2}{5}$	0
s_2	$-\frac{7}{10}$	1	0	$\frac{8}{5}$	$\frac{5}{2}$
λ_1	$-\frac{1}{5}$	0	1	$\frac{3}{5}$	①

	BASIS INVERSE			RHS
z	$-\frac{6}{5}$	0	$-\frac{5}{2}$	$-\frac{49}{10}$
λ_2	$\frac{1}{5}$	0	0	$\frac{2}{5}$
s_2	$-\frac{1}{5}$	1	$-\frac{5}{2}$	$\frac{1}{10}$
λ_3	$-\frac{1}{5}$	0	1	$\frac{3}{5}$

Iteration 2: MASTER STEP

- The best-known feasible solution of the overall problem is given by:

$$\mathbf{x} = \lambda_2 \mathbf{x}_2 + \lambda_3 \mathbf{x}_3$$

$$= \tfrac{2}{5}(2,\ \tfrac{3}{2},\ 3,\ 0) + \tfrac{3}{5}(0,\ \tfrac{5}{2},\ 0,\ 0) = (\tfrac{4}{5},\ \tfrac{21}{10},\ \tfrac{6}{5},\ 0)$$

- The objective is: -4.9
- The dual variables are:

$$(w_1,\ w_2,\ \alpha) = (-\tfrac{6}{5},\ 0,\ -\ \tfrac{5}{2}).$$

Iteration 3

- **Iteration 3:**
  - Since $w_1 < 0$, s_1 is not qualified to enter the basis at this time.

Iteration 3: SUBPROBLEM

- **SUBPROBLEM**
  - Solve the following problem:

$$\begin{array}{ll} \text{Maximize} & (\mathbf{wA} - \mathbf{c})\mathbf{x} + \alpha \\ \text{Subject to} & \mathbf{x} \in X \end{array}$$

$$\mathbf{wA} - \mathbf{c} = (-\tfrac{6}{5}, 0)\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 2 \end{bmatrix} - (-2, -1, -1, 1)$$

$$= (\tfrac{4}{5}, 1, -\tfrac{1}{5}, -1)$$

  - Therefore the subproblem is:

$$\begin{array}{ll} \text{Maximize} & \tfrac{4}{5}x_1 + x_2 - \tfrac{1}{5}x_3 - x_4 - \tfrac{5}{2} \\ \text{Subject to} & \mathbf{x} \in X \end{array}$$

Iteration 3: SUBPROBLEM

- Using the figure, the optimal solution is

$$\mathbf{x}_4 = (2, \tfrac{3}{2}, 0, 0)$$

- with objective

$$z_4 - \hat{c}_4 = \tfrac{3}{5}$$

- Since $z_4 - \hat{c}_4 > 0$, then λ_4 is introduced.
- The lower bound is:

$$\hat{\mathbf{c}}_B\bar{\mathbf{b}} - (z_4 - \hat{c}_4) = -\tfrac{49}{10} - \tfrac{3}{5} = -5.5.$$

Iteration 3: SUBPROBLEM

- Recall that the best-known objective so far is -4.9.
- If we are interested only in an approximate solution, we could have stopped here with the feasible solution

$$\mathbf{x} = \left(\frac{4}{5}, \frac{21}{10}, \frac{6}{5}, 0\right)$$

  - whose objective is -4.9.
- Because the difference between the lower bound (-5.5) and the current objective value (-4.9) is small (-0.6).

Iteration 3: MASTER STEP

- **MASTER STEP**

$$z_4 - \hat{c}_4 = \frac{3}{5}$$

$$\mathbf{Ax}_4 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ \frac{3}{2} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ \frac{7}{2} \end{bmatrix}$$

$$\mathbf{y}_4 = \mathbf{B}^{-1} \begin{bmatrix} \mathbf{Ax}_4 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & 0 & 0 \\ -\frac{1}{5} & 1 & -\frac{5}{2} \\ -\frac{1}{5} & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ \frac{7}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} \\ \frac{3}{5} \\ \frac{3}{5} \end{bmatrix}$$

Iteration 3: MASTER STEP

- Insert the column

$$\begin{bmatrix} z_4 - \hat{c}_4 \\ \mathbf{y}_4 \end{bmatrix}$$

  - into the foregoing array and pivot.
- This leads to the following two tableaux
- The λ_3 column is deleted after pivoting.

Iteration 3: MASTER STEP

	BASIS INVERSE			RHS
z	$-\frac{6}{5}$	0	$-\frac{5}{2}$	$\frac{49}{10}$
λ_2	$\frac{1}{5}$	0	0	$\frac{2}{5}$
s_2	$-\frac{1}{5}$	1	$-\frac{5}{2}$	$\frac{1}{10}$
λ_3	$-\frac{1}{5}$	0	1	$\frac{3}{5}$

λ_4
$\frac{3}{5}$
$\frac{2}{5}$
$\enclose{circle}{\frac{3}{5}}$
$\frac{3}{5}$

	BASIS INVERSE			RHS
z	-1	-1	0	-5
λ_2	$\frac{1}{3}$	$-\frac{2}{3}$	$\frac{5}{3}$	$\frac{1}{3}$
λ_4	$-\frac{1}{3}$	$\frac{5}{3}$	$-\frac{25}{6}$	$\frac{1}{6}$
λ_3	0	-1	$\frac{7}{2}$	$\frac{1}{2}$

Iteration 3: MASTER STEP

- The best-known feasible solution of the overall problem is given by:

$$\mathbf{x} = \lambda_2\mathbf{x}_2 + \lambda_3\mathbf{x}_3 + \lambda_4\mathbf{x}_4$$

$$= \tfrac{1}{3}(2,\ \tfrac{3}{2},\ 3,\ 0) + \tfrac{1}{2}(0,\ \tfrac{5}{2},\ 0,\ 0) + \tfrac{1}{6}(2,\ \tfrac{3}{2},\ 0,\ 0) = (1,\ 2,\ 1,\ 0)$$

- The objective is: -5
- The dual variables are:

$$(w_1, w_2, \alpha) = (-1,\ -\ 1,\ 0).$$

Iteration 4

- **Iteration 4:**
  - Since $w_1 < 0$ and $w_2 < 0$, s_1 and s_2 is not qualified to enter the basis at this time.

Iteration 4: SUBPROBLEM

- **SUBPROBLEM**
  - Solve the following problem:

$$\begin{array}{ll} \text{Maximize} & (\mathbf{wA} - \mathbf{c})\mathbf{x} + \alpha \\ \text{Subject to} & \mathbf{x} \in X \end{array}$$

$$\mathbf{wA} - \mathbf{c} = (-1, -1)\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 2 \end{bmatrix} - (-2, -1, -1, 1)$$

$$= (0, 0, 0, -3)$$

  - Therefore the subproblem is:

$$\begin{array}{ll} \text{Maximize} & 0x_1 + 0x_2 + 0x_3 - 3x_4 + 0 \\ \text{Subject to} & \mathbf{x} \in X \end{array}$$

Iteration 4: SUBPROBLEM

- Using the figure, the optimal solution is

$$\mathbf{x}_5 = (0, 0, 0, 0)$$

- with objective

$$z_5 - \hat{c}_5 = 0$$

- Since $z_5 - \hat{c}_5 = 0$, which is the termination criterion.
- Also note that the lower bound is

$$\hat{\mathbf{c}}_B\bar{\mathbf{b}} - (z_5 - \hat{c}_5) = -5 - 0 = -5$$

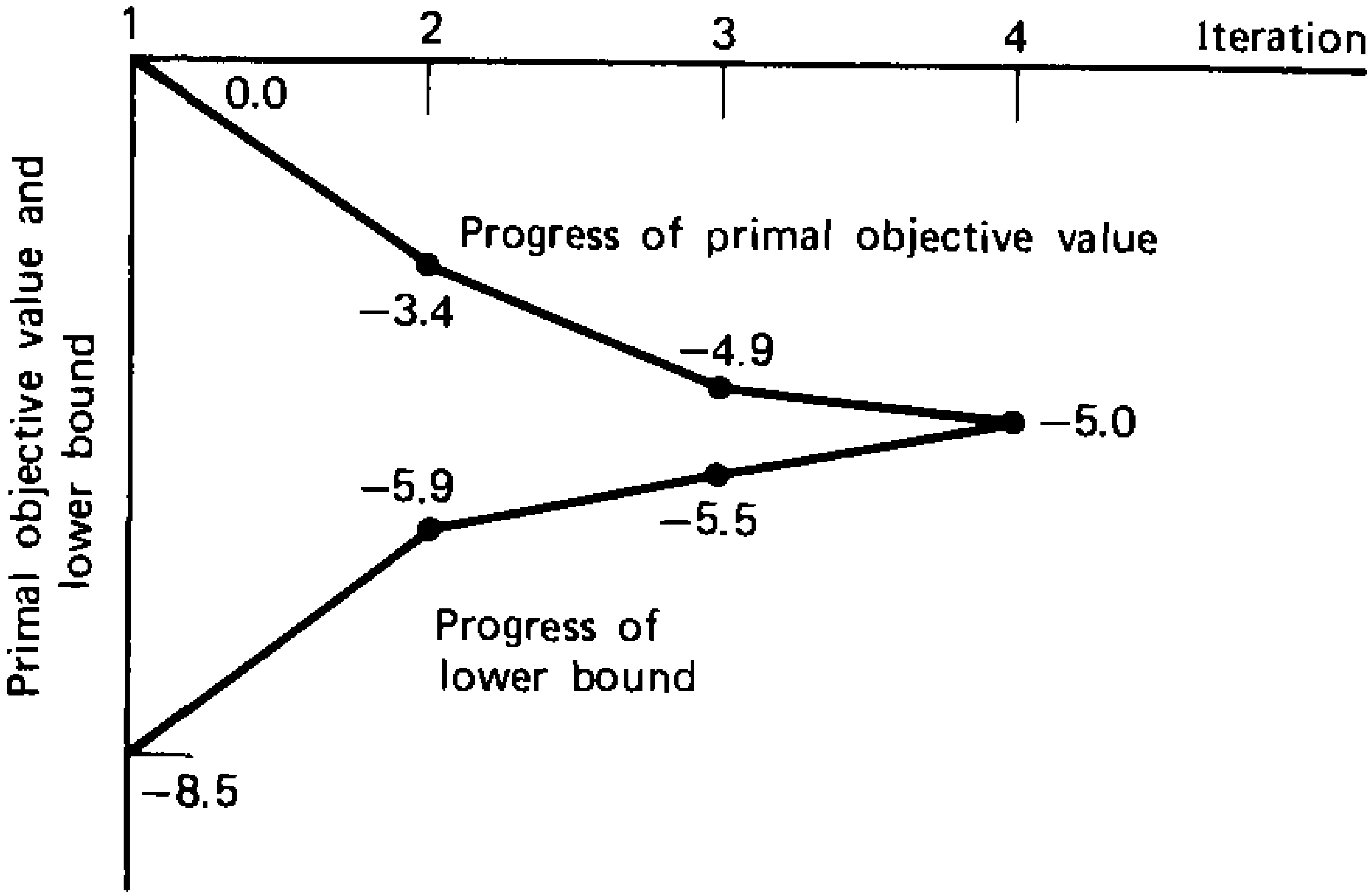
- which is equal to the best (and therefore optimal) solution known so far.

Numeric Example

- **Summary**:
  - the optimal solution

    $(x_1, x_2, x_3, x_4) = (1, 2, 1, 0)$
  - With objective -5 is at hand.

Numeric Example

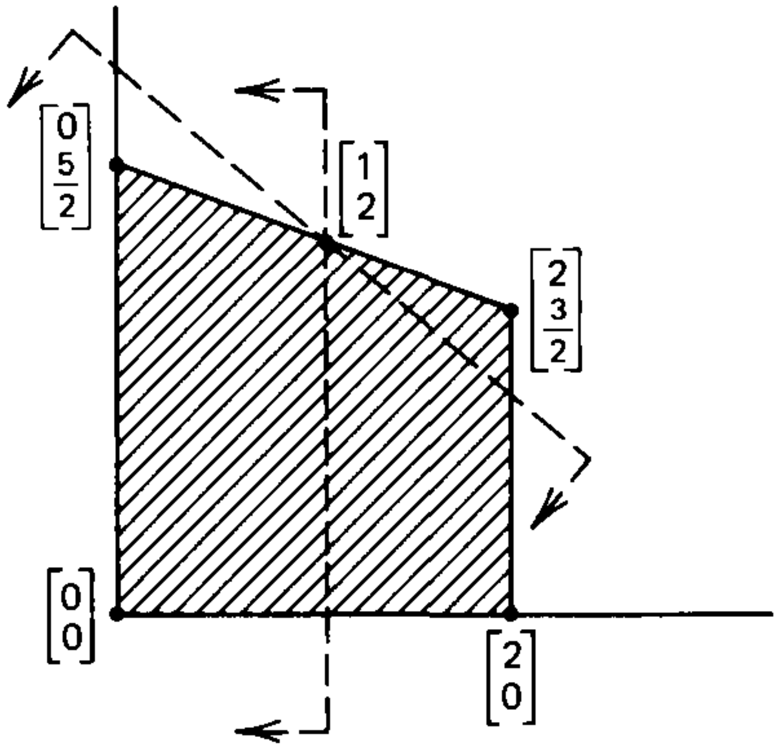


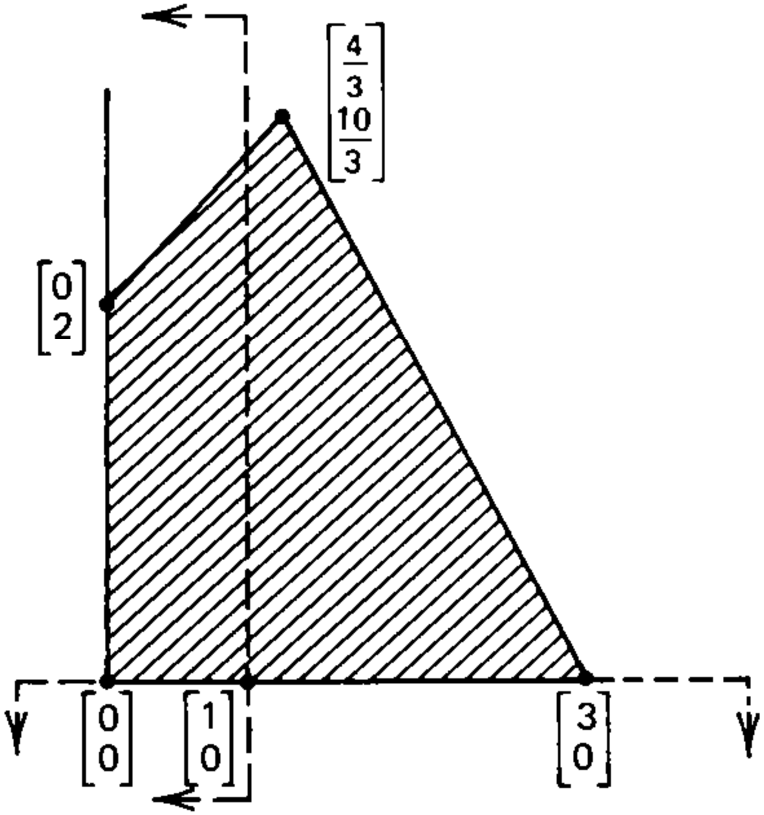
Numeric Example

- The progress of the lower bounds and the objective values of the primal feasible solutions generated by the decomposition algorithm is shown
- Optimality is reached at iteration 4.
- If we were interested in an approximate solution, we could have stopped at iteration 3,
- Since we have a feasible solution with an objective value equal to -4.9, and meanwhile are assured (by the lower bound) that there exist no feasible solutions with an objective less than -5.5.

Numeric Example

- The optimal point is shown, $(x_1, x_2, x_3, x_4) = (1, 2, 1, 0)$





The Case of Unbounded Region X

The Case of Unbounded Region X

- For an unbounded set *X*, the decomposition algorithm must be slightly modified.
- For an unbounded set *X*, the points in *X* can be represented as:
  - a convex combination of the **extreme points** plus
  - a nonnegative combination of the **extreme directions**

The Case of Unbounded Region *X*

- If set *X* is unbounded, x $\in$ *X* if and only if:

$$\mathbf{x} = \sum_{j=1}^{t} \lambda_j \mathbf{x}_j + \sum_{j=1}^{l} \mu_j \mathbf{d}_j$$

$$\sum_{j=1}^{t} \lambda_j = 1$$

$$\lambda_j \geqslant 0 \qquad j = 1, 2, \ldots, t$$

$$\mu_j \geqslant 0 \qquad j = 1, 2, \ldots, l$$

  - $\mathbf{x}_1$, $\mathbf{x}_2$, ..., $\mathbf{x}_t$ are the **extreme points** of *X*
  - $\mathbf{d}_1$, $\mathbf{d}_2$, ..., $\mathbf{d}_t$ are the **extreme directions** of *X*

The Case of Unbounded Region *X*

- The primal problem can be transformed as:

$$\text{Minimize} \quad \sum_{j=1}^{t} (\mathbf{cx}_j)\lambda_j + \sum_{j=1}^{l} (\mathbf{cd}_j)\,\mu_j$$

$$\text{Subject to} \quad \sum_{j=1}^{t} (\mathbf{Ax}_j)\lambda_j + \sum_{j=1}^{l} (\mathbf{Ad}_j)\,\mu_j = \mathbf{b}$$

$$\sum_{j=1}^{t} \lambda_j = 1$$

$$\lambda_j \geqslant 0 \qquad j = 1, 2, \ldots, t$$

$$\mu_j \geqslant 0 \qquad j = 1, 2, \ldots, l$$

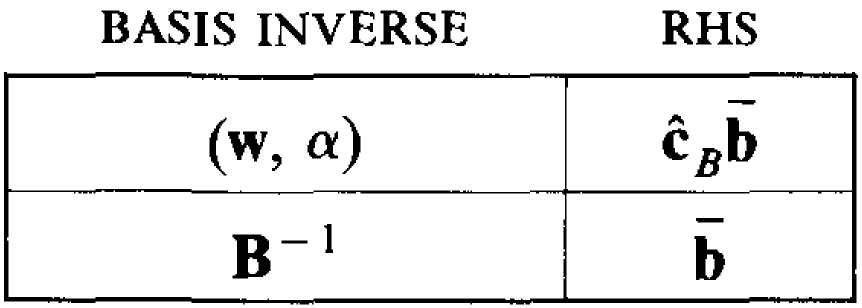
- $\lambda_1, \lambda_2, \ldots, \lambda_t$ and $\mu_1, \mu_2, \ldots, \mu_l$ are variables

The Case of Unbounded Region *X*

- Suppose that we have a basic feasible solution of the foregoing system with basis B, and let w and a be the dual
- variables corresponding to constraints G.5) and G.6) above. Further suppose

Summary of the Decomposition Algorithm

- Let the basis be **B** and form the following **master array**
- The master array includes basis inverse, the dual variables, the values of the basic variables, and the objective function



BASIS INVERSE	RHS
$(\mathbf{w}, \alpha)$	$\hat{\mathbf{c}}_B\bar{\mathbf{b}}$
$\mathbf{B}^{-1}$	$\bar{\mathbf{b}}$

  - Where $(\mathbf{w}, \alpha) = \hat{\mathbf{c}}_B \mathbf{B}^{-1}$ with $\hat{c}_j = \mathbf{c}\mathbf{x}_j$ for each basic variable λ_j
  - and $\bar{\mathbf{b}} = \mathbf{B}^{-1}\begin{pmatrix}\mathbf{b}\\ 1\end{pmatrix}$

The Case of Unbounded Region *X*

- Recall that the current solution is optimal to the overall problem if z_j - $\hat{c}_j \leq 0$ for each variable.
- In particular, the following conditions must hold at optimality:

$$\lambda_j \text{ nonbasic} \Rightarrow 0 \geqslant z_j - \hat{c}_j = (\mathbf{w}, \alpha)\begin{pmatrix} \mathbf{Ax}_j \\ 1 \end{pmatrix} - \mathbf{cx}_j = \mathbf{wAx}_j + \alpha - \mathbf{cx}_j$$

$$\mu_j \text{ nonbasic} \Rightarrow 0 \geqslant z_j - \hat{c}_j = (\mathbf{w}, \alpha)\begin{pmatrix} \mathbf{Ad}_j \\ 0 \end{pmatrix} - \mathbf{cd}_j = \mathbf{wAd}_j - \mathbf{cd}_j$$

- if theses conditions do not hold, a nonbasic variable with a positive z_k - $\hat{c}_k$ to enter the basis, is found.

The Case of Unbounded Region *X*

- We may determine whether or not optimality conditions hold by solving the following subproblem:

$$\begin{array}{ll} \text{Maximize} & (\mathbf{wA} - \mathbf{c})\mathbf{x} + \alpha \\ \text{Subject to} & \mathbf{x} \in X \end{array}$$

The Case of Unbounded Region X

- Suppose that the optimal solution of the subproblem is unbounded.
- This is only possible if an extreme direction $\mathbf{d_k}$ is found such that

$$(\mathbf{wA} - \mathbf{c})\mathbf{d_k} > 0$$

- Moreover,

$$z_k - c_k = (\mathbf{wA} - \mathbf{c})\mathbf{d_k} > 0$$

- and μ_k is eligible to enter the basis.

The Case of Unbounded Region *X*

- In this case

$$\begin{pmatrix} \mathbf{A}\mathbf{d}_k \\ 0 \end{pmatrix}$$

- is updated by premultiplying by $\mathbf{B}^{-1}$ and the resulting column

$$\begin{pmatrix} z_k - \hat{c}_k \\ \mathbf{y}_k \end{pmatrix}$$

- inserted in the foregoing array and the revised simplex method is continued.

Example: The Case of Unbounded Region X

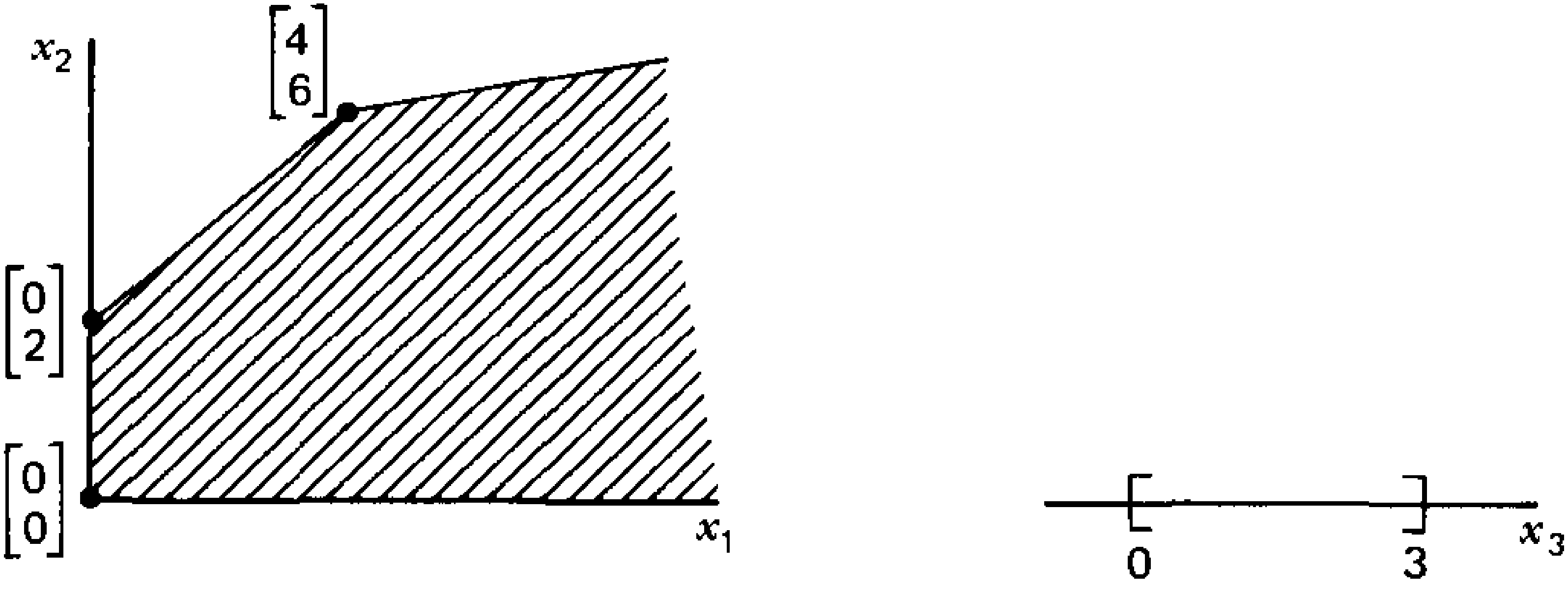
Example: The Case of Unbounded Region *X*

- **Example:**

$$
\begin{array}{llcl}
\text{Minimize} & -x_1 - 2x_2 - x_3 & & \\
\text{Subject to} & x_1 + x_2 + x_3 & \leqslant & 12 \\
& -x_1 + x_2 & \leqslant & 2 \\
& -x_1 + 2x_2 & \leqslant & 8 \\
& x_3 & \leqslant & 3 \\
& x_1, \quad x_2, \quad x_3 & \geqslant & 0
\end{array}
$$

Example: The Case of Unbounded Region *X*

- The first constraint is handled as $\mathbf{Ax} \leq \mathbf{b}$ and the rest of the constraints are treated by X.
- Note that X decomposes into the two sets of Figure.



$$-x_1 + x_2 \leqslant 2$$
$$-x_1 + 2x_2 \leqslant 8$$

$$x_3 \leqslant 3$$

Example: The Case of Unbounded Region *X*

- The problem is transformed into $\lambda_1, \lambda_2, \ldots, \lambda_t$ and $\mu_1, \mu_2, \ldots, \mu_l$, variables as follows:

$$
\begin{array}{lll}
\text{Minimize} & \displaystyle\sum_{j=1}^{t} (\mathbf{c}\mathbf{x}_j)\lambda_j + \sum_{j=1}^{l} (\mathbf{c}\mathbf{d}_j)\,\mu_j & \\
& \displaystyle\sum_{j=1}^{t} (\mathbf{A}\mathbf{x}_j)\lambda_j + \sum_{j=1}^{l} (\mathbf{A}\mathbf{d}_j)\,\mu_j \leqslant \mathbf{b} & \\
& \displaystyle\sum_{j=1}^{t} \lambda_j = 1 & \\
& \lambda_j \geqslant 0 & j = 1, 2, \ldots, t \\
& \mu_j \geqslant 0 & j = 1, 2, \ldots, l
\end{array}
$$

Example: The Case of Unbounded Region *X*

- Note that $\mathbf{x}_1 = (0, 0, 0)$ belongs to *X* and $\mathbf{Ax}_1 = 0 + 0 + 0 \leq 12$.
- Therefore the initial basis consists of λ_1 (corresponding to x,) plus the slack variable **s**.
- This leads to the following array, where $w = \alpha = 0$.

	BASIS INVERSE		RHS
z	0	0	0
s	1	0	12
λ_1	0	1	1

Iteration 1: SUBPROBLEM

SUBPROBLEM

$$\begin{array}{ll} \text{Maximize} & (\mathbf{wA} - \mathbf{c})\mathbf{x} + \alpha \\ \text{Subject to} & \mathbf{x} \in X \end{array}$$

- Since w = α = 0 and **A** = (1, 1, 1):

$$\begin{array}{lrcl} \text{Maximize} & x_1 + 2x_2 + x_3 + 0 & & \\ \text{Subject to} & -x_1 + \ \ x_2 & & \leqslant 2 \\ & -x_1 + 2x_2 & & \leqslant 8 \\ & & x_3 & \leqslant 3 \\ & x_1, \quad x_2, & x_3 & \geqslant 0 \end{array}$$

Iteration 1: SUBPROBLEM

- The problem decomposes into two problems in (x_1, x_2) and x_3. The optimal value of x_3 is 3.
- The (x_1, x_2) problem can be solved by the simplex method below, where x_4 and x_5 are the slack variables:

	z	x_1	x_2	x_4	x_5	RHS
z	1	−1	−2	0	0	0
x_4	0	−1	①	1	0	2
x_5	0	−1	2	0	1	8

	z	x_1	x_2	x_4	x_5	RHS
z	1	−3	0	2	0	4
x_2	0	−1	1	1	0	2
x_5	0	①	0	−2	1	4

Iteration 1: SUBPROBLEM

	z	x_1	x_2	x_4	x_5	RHS
z	1	0	0	-4	3	16
x_2	0	0	1	-1	1	6
x_1	0	1	0	-2	1	4

- The optimal is unbounded.
- As x_4 increases by 1 unit, x_1 increases by 2 units and x_2 increases by 1 unit
- In the (x_1, x_2) space we have found a direction leading to an unbounded solution.

$$\mathbf{d}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Iteration 1: SUBPROBLEM

- In the (x_1, x_2, x_3) space, $\mathbf{d_1}$ is given by $(2, 1, 0)^t$
- Also ($\mathbf{wA}$ - $\mathbf{c}$) $\mathbf{d_1} = 4$
  - (the negative of -4 in row 0 under x_4)
- So μ_1, is introduced in the basis.

Iteration 1: MASTER STEP

$$z_1 - \hat{c}_1 = 4$$

$$\mathbf{Ad}_1 = (1,\ 1,\ 1)\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = 3$$

$$\mathbf{y}_1 = \mathbf{B}^{-1}\begin{pmatrix} \mathbf{Ad}_1 \\ 0 \end{pmatrix}$$

$$\mathbf{y}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

Iteration 1: MASTER STEP

- Introduce the column $\begin{pmatrix} z_1 - \hat{c}_1 \\ \mathbf{y}_1 \end{pmatrix}$
- in the master array and pivot. The μ_1 column is eliminated after pivoting.

	BASIS INVERSE		RHS
z	0	0	0
s	1	0	12
λ_1	0	1	1

μ_1
4
③
0

	BASIS INVERSE		RHS
z	$-\frac{4}{3}$	0	-16
μ_1	$\frac{1}{3}$	0	4
λ_1	0	1	1

Iteration 2

$$w = -\tfrac{4}{3} \quad \text{and} \quad \alpha = 0$$

- Since $w < 0$, s is not a candidate to enter the basis.

Iteration 2: SUBPROBLEM

SUBPROBLEM

$$\begin{array}{ll} \text{Maximize} & (\mathbf{wA} - \mathbf{c})\mathbf{x} + \alpha \\ \text{Subject to} & \mathbf{x} \in X \end{array}$$

- This reduces to the following:

$$\begin{array}{ll} \text{Maximize} & -\tfrac{1}{3}x_1 + \tfrac{2}{3}x_2 + 0 \\ \text{Subject to} & -x_1 + x_2 \leqslant 2 \\ & -x_1 + 2x_2 \leqslant 8 \\ & x_1, \quad x_2 \geqslant 0 \end{array}$$

$$\begin{array}{ll} \text{Maximize} & -\tfrac{1}{3}x_3 \\ \text{Subject to} & 0 \leqslant x_3 \leqslant 3 \end{array}$$

Iteration 2: SUBPROBLEM

- Here the value $\alpha = 0$ is added to only one of the subproblems.
- Obviously $x_3 = 0$.
- The new problem in (x_1, x_2) is solved by utilizing the corresponding tableau of the last iteration, deleting row 0, and introducing the new costs.

Iteration 2: SUBPROBLEM

	z	x_1	x_2	x_4	x_5	RHS
z	1	$\frac{1}{3}$	$-\frac{2}{3}$	0	0	0
x_2	0	0	1	-1	1	6
x_1	0	1	0	-2	1	4

Multiply row 1 by $\frac{2}{3}$ and row 2 by $-\frac{1}{3}$ and add to row 0.

	z	x_1	x_2	x_4	x_5	RHS
z	1	0	0	0	$\frac{1}{3}$	$\frac{8}{3}$
x_2	0	0	1	-1	1	6
x_1	0	1	0	-2	1	4

Iteration 2: SUBPROBLEM

- The foregoing tableau is optimal (not unique).
- The optimal objective of the subproblem is

$$z_2 - \hat{c}_2 = \frac{8}{3} > 0$$

- and so λ_2 corresponding to $\mathbf{x_2}$ is introduced.

$$\mathbf{x_2} = (x_1, x_2, x_3) = (4, 6, 0)$$

Iteration 2: MASTER STEP

$$z_2 - \hat{c}_2 = \tfrac{8}{3}$$

$$\mathbf{Ax}_2 = 10$$

$$\mathbf{y}_2 = \mathbf{B}^{-1}\begin{pmatrix} \mathbf{Ax}_2 \\ 1 \end{pmatrix} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 10 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{10}{3} \\ 1 \end{bmatrix}$$

- Introduce the following column in the master array and pivot. The λ_2 column is eliminated after pivoting.

$$\begin{pmatrix} z_2 - \hat{c}_2 \\ \mathbf{y}_2 \end{pmatrix}$$

Iteration 1: MASTER STEP

	BASIS INVERSE		RHS
z	$-\frac{4}{3}$	0	-16
μ_1	$\frac{1}{3}$	0	4
λ_1	0	1	1

λ_2
$\frac{8}{3}$
$\frac{10}{3}$
①

	BASIS INVERSE		RHS
z	$-\frac{4}{3}$	$-\frac{8}{3}$	$-\frac{56}{3}$
μ_1	$\frac{1}{3}$	$-\frac{10}{3}$	$\frac{2}{3}$
λ_2	0	1	1

Iteration 3

$$w = -\frac{4}{3}$$

- $w < 0$, So s is still not a candidate to enter the basis.
- Also the optimal solution of the last subproblem remains the same (see iteration 2).
- The objective value of $\frac{8}{3}$ was for the previous dual solution with $\alpha = 0$.
- For $\alpha = -\frac{8}{3}$ we have

$$z_3 - \hat{c}_3 = \frac{8}{3} - \frac{8}{3} = 0,$$

- which is the termination criterion, and the optimal solution is at hand.

Iteration 3

- More specifically, the optimal **x*** is given by:

$$\mathbf{x}^* = \lambda_2\mathbf{x}_2 + \mu_1\mathbf{d}_1$$

$$= 1\begin{bmatrix}4\\6\\0\end{bmatrix} + \frac{2}{3}\begin{bmatrix}2\\1\\0\end{bmatrix} = \begin{bmatrix}\frac{16}{3}\\\frac{20}{3}\\0\end{bmatrix}$$

The objective is $-\frac{56}{3}$.

Block Diagonal Structure

Block Diagonal Structure

- When the set X has a block diagonal structure, X can itself be decomposed into several sets:

  $X_1, X_2, \ldots , X_T$

  - Each involving a subset of the variables, which do not appear in any other set.

- The vectors $\mathbf{x}$, $\mathbf{c}$, and the matrix $\mathbf{A}$ of the master constraints $\mathbf{Ax} = \mathbf{b}$, can be decomposed as:

  $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_T,$

  $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_T$

  $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_T$

Block Diagonal Structure

$$
\begin{array}{lccccccl}
\text{Minimize} & \mathbf{c}_1\mathbf{x}_1 & + & \mathbf{c}_2\mathbf{x}_2 & + \cdots + & \mathbf{c}_T\mathbf{x}_T & & \\
\text{Subject to} & \mathbf{A}_1\mathbf{x}_1 & + & \mathbf{A}_2\mathbf{x}_2 & + \cdots + & \mathbf{A}_T\mathbf{x}_T & = & \mathbf{b} \\
 & \mathbf{B}_1\mathbf{x}_1 & & & & & \leqslant & \mathbf{b}_1 \\
 & & & \mathbf{B}_2\mathbf{x}_2 & & & \leqslant & \mathbf{b}_2 \\
 & & & & \ddots & & & \vdots \\
 & & & & & \mathbf{B}_T\mathbf{x}_T & \leqslant & \mathbf{b}_T \\
 & \mathbf{x}_1, & & \mathbf{x}_2, & \ldots, & \mathbf{x}_T & \geqslant & \mathbf{0}
\end{array}
$$

where $X_i = \{\mathbf{x}_i : \mathbf{B}_i\mathbf{x}_i \leqslant \mathbf{b}_i, \mathbf{x}_i \geqslant \mathbf{0}\}$

for $i = 1, 2, \ldots, T.$

Block Diagonal Structure

- For subproblem i, $\mathbf{x_i} \in X_i$ if and only if:

$$\mathbf{x}_i = \sum_{j=1}^{t_i} \lambda_{ij}\mathbf{x}_{ij} + \sum_{j=1}^{l_i} \mu_{ij}\mathbf{d}_{ij}$$

$$\sum_{j=1}^{t_i} \lambda_{ij} = 1$$

$$\lambda_{ij} \geqslant 0 \qquad j = 1, 2, \ldots, t_i$$

$$\mu_{ij} \geqslant 0 \qquad j = 1, 2, \ldots, l_i$$

  - where the $\mathbf{x_{ij}}$, $\mathbf{d_{ij}}$ are the extreme points and the extreme directions of X_i.

Block Diagonal Structure

- Replacing each $\mathbf{x}_i$, the original problem can be reformulated as follows:

$$\text{Minimize} \quad \sum_{i=1}^{T} \sum_{j=1}^{t_i} (\mathbf{c}_i \mathbf{x}_{ij})\lambda_{ij} + \sum_{i=1}^{T} \sum_{j=1}^{l_i} (\mathbf{c}_i \mathbf{d}_{ij})\,\mu_{ij}$$

$$\text{Subject to} \quad \sum_{i=1}^{T} \sum_{j=1}^{t_i} (\mathbf{A}_i \mathbf{x}_{ij})\lambda_{ij} + \sum_{i=1}^{T} \sum_{j=1}^{l_i} (\mathbf{A}_i \mathbf{d}_{ij})\,\mu_{ij} = \mathbf{b}$$

$$\sum_{j=1}^{t_i} \lambda_{ij} = 1 \qquad i = 1, 2, \ldots, T$$

$$\lambda_{ij} \geqslant 0 \qquad j = 1, 2, \ldots, t_i \quad i = 1, 2, \ldots, T$$

$$\mu_{ij} \geqslant 0 \qquad j = 1, 2, \ldots, l_i \quad i = 1, 2, \ldots, T$$

Block Diagonal Structure

- Here we allow different convex combinations and linear combinations for each subproblem i since we have T convexity constraints.
- This adds more flexibility but at the same time increases the number of constraints

  from $m + 1$ to $m + T$

Block Diagonal Structure

- Suppose that we have a basic feasible solution of the foregoing system with an $(m + 1) \times (m + T)$ basis B.
- Each basis must contain at least one variable λ_{ij} from each block i.
- Suppose that $\mathbf{B}^{-1}$ and the following values are known:

$$\bar{\mathbf{b}} = \mathbf{B}^{-1}\begin{pmatrix} \mathbf{b} \\ \mathbf{1} \end{pmatrix}$$

$$(\mathbf{w}, \boldsymbol{\alpha}) = (w_1, \ldots, w_m, \alpha_1, \ldots, \alpha_T) = \hat{\mathbf{c}}_B\mathbf{B}$$

$$\hat{\mathbf{c}}_B : \hat{c}_{ij} = \mathbf{c}_i\mathbf{x}_{ij} \text{ for } \lambda_{ij}$$

$$\hat{c}_{ij} = \mathbf{c}_i\mathbf{d}_{ij} \text{ for } \mu_{ij}$$

Block Diagonal Structure

- The master array is:

BASIS INVERSE	RHS
$(\mathbf{w}, \boldsymbol{\alpha})$	$\hat{\mathbf{c}}_B\bar{\mathbf{b}}$
$\mathbf{B}^{-1}$	$\bar{\mathbf{b}}$

- This solution is optimal if $z_{ij} - \hat{c}_{ij} \leq 0$ for each variable (naturally $z_{ij} - \hat{c}_{ij} = 0$ for each basic variable).
- In particular the following conditions must hold at optimality:

$$\lambda_{ij} \text{ nonbasic} \Rightarrow 0 \geqslant z_{ij} - \hat{c}_{ij} = \mathbf{w}\mathbf{A}_i\mathbf{x}_{ij} + \alpha_i - \mathbf{c}_i\mathbf{x}_{ij}$$

$$\mu_{ij} \text{ nonbasic} \Rightarrow 0 \geqslant z_{ij} - \hat{c}_{ij} = \mathbf{w}\mathbf{A}_i\mathbf{d}_{ij} - \mathbf{c}_i\mathbf{d}_{ij}$$

Block Diagonal Structure

- Optimality conditions can be easily verified by solving the following subproblems.

$$\begin{array}{ll}\text{Maximize} & (\mathbf{w}\mathbf{A}_i - \mathbf{c}_i)\mathbf{x}_i + \alpha_i \\ \text{Subject to} & \mathbf{x}_i \in X_i\end{array}$$

Block Diagonal Structure

- Each subproblem *i* is solved in turn.
- If subproblem *i* yields an unbounded solution,
  - Such that $(\mathbf{wA_i} - \mathbf{c_i})\mathbf{d_{ik}} > \mathbf{0}$
  - Then an extreme direction $\mathbf{d_{ik}}$ is a candidate to enter the master basis
  - Introducing μ_{ik} will improve the objective function
- If subproblem *i* yields a bounded optimal point
  - Such that $\mathbf{wA_ix_{ik}} + \alpha_\mathbf{i} - \mathbf{c_ix_{ik}} > \mathbf{0}$
  - Then an extreme point $\mathbf{x_{ik}}$ is a candidate to enter the master basis.
  - Introducing λ_{ik} will improve the objective function

Block Diagonal Structure

- When all subproblems have z_{ij} - $\hat{c}_{ij} \leq 0$
  - no subproblem yields a candidate to enter the master basis
  - then the optimal solution to the original problem is obtained.
- Otherwise, we must select one from among the various candidates to enter the master basis, We may use
  - the rule of the most positive z_{ik} - $\hat{c}_{ik}$
  - the rule of first positive z_{ik} - $\hat{c}_{ik}$
    - we may stop solving the subproblems after the first candidate comes available

Calculation of Lower Bounds

Calculation of Lower Bounds

- This is a natural extension of the case for one bounded subproblem presented earlier.
- Let $\mathbf{x_1}$, $\mathbf{x_2}$, ..., $\mathbf{x_T}$, represent a feasible solution of the overall problem
- so that $\mathbf{x_i} \in X_i$ for each i and $\Sigma_{\mathrm{i}}\mathbf{A_i x_i} = \mathbf{b}$.

Calculation of Lower Bounds

- We have

$$(\mathbf{wA}_i - \mathbf{c}_i)\mathbf{x}_i + \alpha_i \leqslant (z_{ik} - \hat{c}_{ik})$$

or

$$\mathbf{c}_i\mathbf{x}_i \geqslant \mathbf{wA}_i\mathbf{x}_i + \alpha_i - (z_{ik} - \hat{c}_{ik})$$

- Summing on *i* we get

$$\sum_i \mathbf{c}_i\mathbf{x}_i \geqslant \mathbf{w}\sum_i \mathbf{A}_i\mathbf{x}_i + \sum_i \alpha_i - \sum_i (z_{ik} - \hat{c}_{ik}).$$

- But $\Sigma_i\mathbf{c}_i\mathbf{x}_i = \mathbf{cx}$ and $\Sigma_i\mathbf{A}_i\mathbf{x}_i = \mathbf{b}$, Thus we get

$$\mathbf{cx} \geqslant \mathbf{wb} + \boldsymbol{\alpha}\mathbf{1} - \sum_i (z_{ik} - \hat{c}_{ik})$$

- or

$$\mathbf{cx} \geqslant \hat{\mathbf{c}}_B\bar{\mathbf{b}} - \sum_i (z_{ik} - \hat{c}_{ik}).$$

Numeric Example

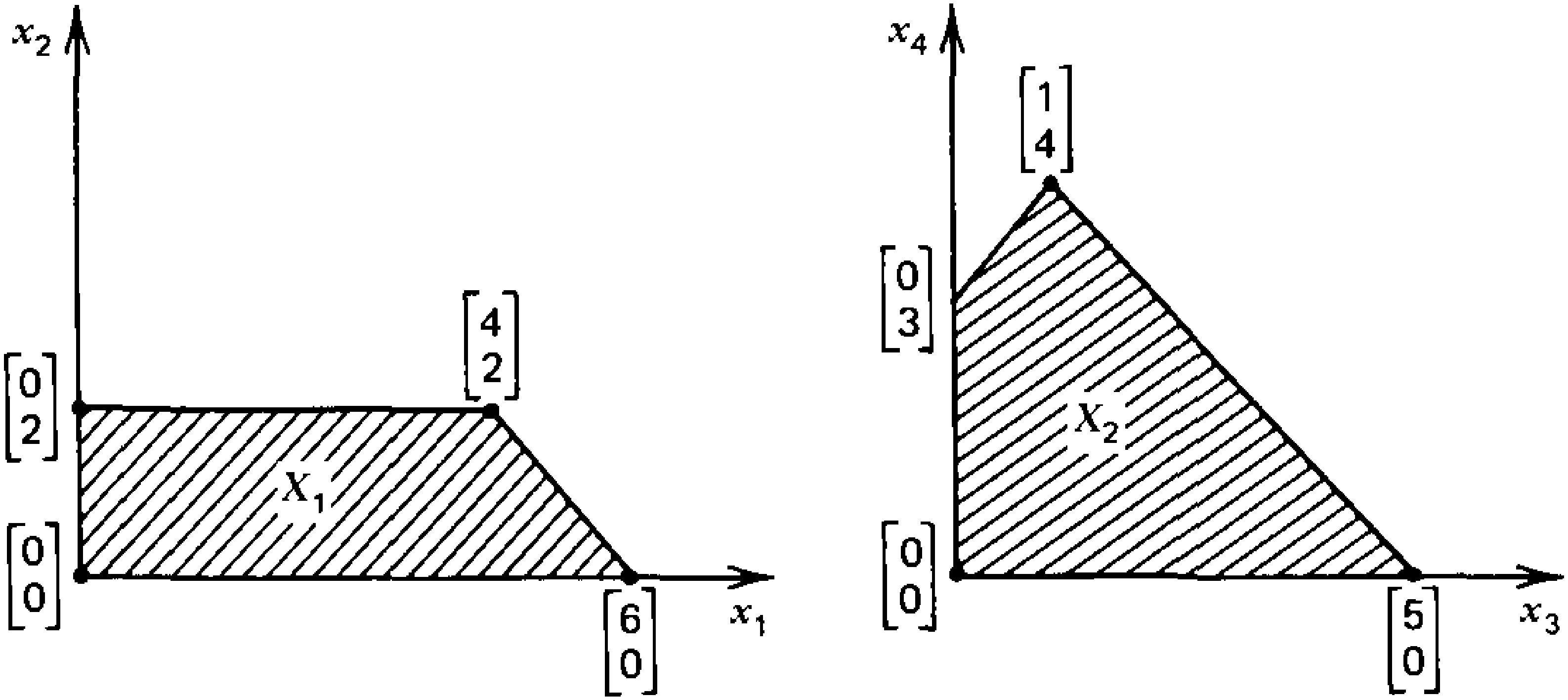
Numeric Example

$$\text{Minimize} \quad -2x_1 - x_2 - 3x_3 - x_4$$

$$\begin{array}{lrcl}
\text{Subject to} & x_1 + x_2 + x_3 + x_4 & \leqslant & 6 \\
 & x_2 + 2x_3 + x_4 & \leqslant & 4 \\
 & x_1 + x_2 & \leqslant & 6 \\
 & x_2 & \leqslant & 2 \\
 & -x_3 + x_4 & \leqslant & 3 \\
 & x_3 + x_4 & \leqslant & 5 \\
 & x_1, \; x_2, \; x_3, \; x_4 & \geqslant & 0
\end{array}$$

Numeric Example

- The first two constraints are handled by $\mathbf{Ax} \leq \mathbf{b}$, and the rest of the constraints are treated by X.
- Note that X decomposes into two sets as shown:



Numeric Example

- The problem is transformed into the following:

$$\text{Minimize} \quad \sum_{j=1}^{t_1} (\mathbf{c}_1\mathbf{x}_{1j})\lambda_{1j} + \sum_{j=1}^{t_2} (\mathbf{c}_2\mathbf{x}_{2j})\,\lambda_{2j}$$

$$\text{Subject to} \quad \sum_{j=1}^{t_1} (\mathbf{A}_1\mathbf{x}_{1j})\lambda_{1j} + \sum_{j=1}^{t_2} (\mathbf{A}_2\mathbf{x}_{2j})\lambda_{2j} \leqslant \mathbf{b}$$

$$\sum_{j=1}^{t_1} \lambda_{1j} = 1$$

$$\sum_{j=1}^{t_2} \lambda_{2j} = 1$$

$$\lambda_{1j} \geqslant 0 \qquad j = 1, 2, \ldots, t_1$$

$$\lambda_{2j} \geqslant 0 \qquad j = 1, 2, \ldots, t_2$$

Numeric Example

- Where

$$\mathbf{c}_1 = (-2, -1),\ \mathbf{c}_2 = (-3, -1),\ \mathbf{A}_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},\ \text{and } \mathbf{A}_2 = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}.$$

- That

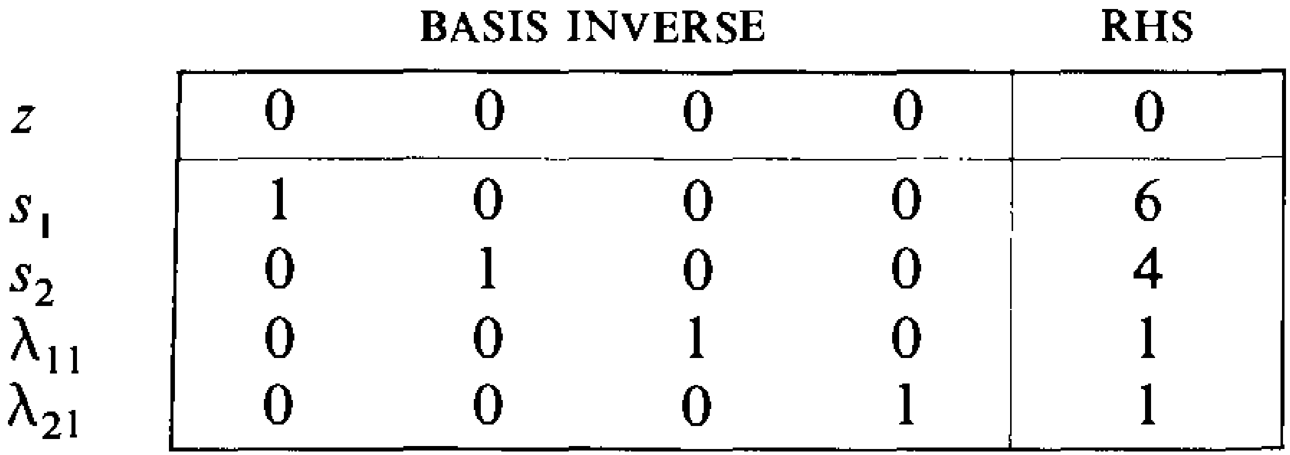
$$\mathbf{x}_{11} = (x_1, x_2) = (0, 0)$$
$$\mathbf{x}_{21} = (x_3, x_4) = (0, 0)$$

  - belong to X_1 and X_2 and satisfy the master constraints.

- Therefore we have a basic feasible solution of the overall system where the basis consists of s_1, s_2, λ_{11}, and λ_{21}

Numeric Example

- This leads to the following master array.

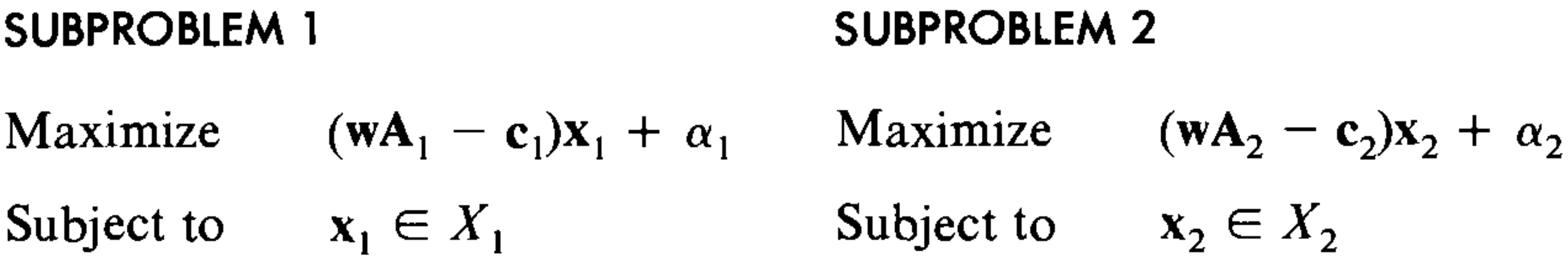


	BASIS INVERSE				RHS
z	0	0	0	0	0
s_1	1	0	0	0	6
s_2	0	1	0	0	4
λ_{11}	0	0	1	0	1
λ_{21}	0	0	0	1	1

- The first four entries of row 0 give w_1, w_2, α_1, and α_2 respectively.
- Under these entries $\mathbf{B}^{-1}$ is stored.

Iteration 1: SUBPROBLEM

- Solve the following two subproblems:



SUBPROBLEM 1

Maximize $(\mathbf{w}\mathbf{A}_1 - \mathbf{c}_1)\mathbf{x}_1 + \alpha_1$

Subject to $\mathbf{x}_1 \in X_1$

SUBPROBLEM 2

Maximize $(\mathbf{w}\mathbf{A}_2 - \mathbf{c}_2)\mathbf{x}_2 + \alpha_2$

Subject to $\mathbf{x}_2 \in X_2$

- Since $\mathbf{w} = (0, 0)$ and $\alpha = (0, 0)$, these reduce to:

  maximizing $2x_1 + x_2 + 0$

  maximizing $3x_3 + x_4 + 0$

Iteration 1: SUBPROBLEM

- The optimal solutions are respectively

$$\mathbf{x_{12}} = (x_1, x_2) = = (6, 0) \text{ with objective } 12$$
$$\mathbf{x_{22}} = (x_3, x_4) = (5, 0) \text{ with objective } 15$$

- Then

$$(\mathbf{wA_1} - \mathbf{c_1})\mathbf{x_{12}} + \alpha_1 = 12$$
$$(\mathbf{wA_2} - \mathbf{c_2})\mathbf{x_{22}} + \alpha_2 = 15$$

- Therefore λ_{12} and λ_{22} are both candidates to enter.
- Select λ_{22} since $z_{22} - \hat{c}_{22} = 15$ is the most positive.

Iteration 1: MASTER PROBLEM

- $z_{22} - \hat{c}_{22} = 15$
- Form the column

$$\begin{bmatrix} \mathbf{A}_2\mathbf{x}_{22} \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{A}_2\mathbf{x}_{22} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix},$$

$$\begin{bmatrix} \mathbf{A}_2\mathbf{x}_{22} \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 0 \\ 1 \end{bmatrix}$$

Iteration 1: MASTER PROBLEM

	BASIS INVERSE				RHS	λ_{22}
z	0	0	0	0	0	15
s_1	1	0	0	0	6	5
s_2	0	1	0	0	4	⑩
λ_{11}	0	0	1	0	1	0
λ_{21}	0	0	0	1	1	1

	BASIS INVERSE				RHS
z	0	$-\frac{3}{2}$	0	0	-6
s_1	1	$-\frac{1}{2}$	0	0	4
λ_{22}	0	$\frac{1}{10}$	0	0	$\frac{2}{5}$
λ_{11}	0	0	1	0	1
λ_{21}	0	$-\frac{1}{10}$	0	1	$\frac{3}{5}$

Note that $w_1 = 0$, $w_2 = -\frac{3}{2}$, $\alpha_1 = \alpha_2 = 0$.

Iteration 2: SUBPROBLEM

- Check the z_j - c_j values for the nonbasic slack variables
  - Since s_2 just left the basis, it will not be a candidate to immediately reenter.
- Solve the following two subproblems:

SUBPROBLEM 1		**SUBPROBLEM 2**	
Maximize	$(\mathbf{wA}_1 - \mathbf{c}_1)\mathbf{x}_1 + \alpha_1$	Maximize	$(\mathbf{wA}_2 - \mathbf{c}_2)\mathbf{x}_2 + \alpha_2$
Subject to	$\mathbf{x}_1 \in X_1$	Subject to	$\mathbf{x}_2 \in X_2$

- These problems reduce to the following:

Maximize	$2x_1 - \frac{1}{2}x_2 + 0$	Maximize	$0x_3 - \frac{1}{2}x_4 + 0$
Subject to	$(x_1, x_2) \in X_1$	Subject to	$(x_3, x_4) \in X_2$

Iteration 2: SUBPROBLEM

- The optimal solutions are:

$$\mathbf{x}_{13} = (x_1, x_2) = (6, 0) \text{ with objective}$$

$$z_{13} - \hat{c}_{13} = (\mathbf{w}\mathbf{A}_1 - \mathbf{c}_1)\mathbf{x}_{13} + \alpha_1 = 12$$

$$\mathbf{x}_{23} = (x_3, x_4) = (5, 0) \text{ with objective}$$

$$z_{23} - \hat{c}_{23} = (\mathbf{w}\mathbf{A}_2 - \mathbf{c}_2)\mathbf{x}_{23} + \alpha_2 = 0$$

- Thus there is no candidate from subproblem 2 at this time and λ_{13} is a candidate to enter the master basis.

Iteration 2: MASTER PROBLEM

$$z_{13} - \hat{c}_{13} = 12$$

$$\begin{pmatrix} \mathbf{A}_1\mathbf{x}_{13} \\ 1 \\ 0 \end{pmatrix}$$

$$\mathbf{A}_1\mathbf{x}_{13} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix} \quad \begin{pmatrix} \mathbf{A}_1\mathbf{x}_{13} \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\mathbf{y}_{13} = \mathbf{B}^{-1}\begin{pmatrix} \mathbf{A}_1\mathbf{x}_{13} \\ 1 \\ 0 \end{pmatrix} = \begin{bmatrix} 6 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Iteration 2: MASTER PROBLEM

	BASIS INVERSE				RHS	λ_{13}
z	0	$-\frac{3}{2}$	0	0	-6	12
s_1	1	$-\frac{1}{2}$	0	0	4	⑥
λ_{22}	0	$\frac{1}{10}$	0	0	$\frac{2}{5}$	0
λ_{11}	0	0	1	0	1	1
λ_{21}	0	$-\frac{1}{10}$	0	1	$\frac{3}{5}$	0

	BASIS INVERSE				RHS
z	-2	$-\frac{1}{2}$	0	0	-14
λ_{13}	$\frac{1}{6}$	$-\frac{1}{12}$	0	0	$\frac{2}{3}$
λ_{22}	0	$\frac{1}{10}$	0	0	$\frac{2}{5}$
λ_{11}	$-\frac{1}{6}$	$\frac{1}{12}$	1	0	$\frac{1}{3}$
λ_{21}	0	$-\frac{1}{10}$	0	1	$\frac{3}{5}$

Note that $w_1 = -2$, $w_2 = -\frac{1}{2}$, $\alpha_1 = \alpha_2 = 0$.

Iteration 3: SUBPROBLEM

- Check the z_j - c_j values for the nonbasic slack variables
  - Since $w_1 < 0$ and $w_2 < 0$, neither s_1 nor s_2 are candidates to enter the basis.
- Solve the following two subproblems.

SUBPROBLEM 1		SUBPROBLEM 2	
Maximize	$(\mathbf{wA}_1 - \mathbf{c}_1)\mathbf{x}_1 + \alpha_1$	Maximize	$(\mathbf{wA}_2 - \mathbf{c}_2)\mathbf{x}_2 + \alpha_2$
Subject to	$\mathbf{x}_1 \in X_1$	Subject to	$\mathbf{x}_2 \in X_2$

- These problems reduce to the following.

Maximize	$0x_1 - \frac{3}{2}x_2 + 0$	Maximize	$0x_3 - \frac{3}{2}x_4 + 0$
Subject to	$(x_1, x_2) \in X_1$	Subject to	$(x_3, x_4) \in X_2$

Iteration 3: SUBPROBLEM

- The optimal solutions are:

$$\mathbf{x}_{14} = (x_1, x_2) = (0, 0) \text{ with objective } 0$$

$$\mathbf{x}_{24} = (x_3, x_4) = (0, 0) \text{ with objective } 0$$

$$(\mathbf{wA}_1 - \mathbf{c}_1)\mathbf{x}_{14} + \alpha_1 = 0$$

$$(\mathbf{wA}_2 - \mathbf{c}_2)\mathbf{x}_{24} + \alpha_2 = 0$$

- The optimal solution is reached.

Iteration 3: SUBPROBLEM

- From the master problem the optimal point **x*** is given by:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda_{11}\mathbf{x}_{11} + \lambda_{13}\mathbf{x}_{13}$$

$$= \tfrac{1}{3}\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \tfrac{2}{3}\begin{pmatrix} 6 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \lambda_{21}\mathbf{x}_{21} + \lambda_{22}\mathbf{x}_{22}$$

$$= \tfrac{3}{5}\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \tfrac{2}{5}\begin{pmatrix} 5 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

- Therefore **x*** = (x_1, x_2, x_3, x_4) = (4, 0, 2, 0) with objective - 14.

Economic Interpretation

Economic Interpretation

- Consider the case of a large system that is composed of smaller subsystems 1, 2, ...,T
- Each subsystem i has its own objective
- The objective function of the overall system is the sum of the objective functions of the subsystems.
- Each subsystem has its constraints designated by the set X_i, (which is assumed to be bounded for the purpose of simplification)
- All the subsystems share a few common resources, and hence the consumption of these resources by all the subsystems must not exceed the availability given by the vector **b**.

Economic Interpretation

- Recall the following economic interpretation of the dual variables (Lagrangian multipliers).
- Here w_i is the rate of change of the objective as a function of b_i
- If b_i is replaced by $b_i + \Delta$, then the objective is modified by adding $w_i\,\Delta$.
- Hence $-w_i$ can be thought of as the price of consuming one unit of the *i*th common resource.
- Similarly, $-\alpha$, can be thought of as the price of consuming a portion of the *i*th convexity constraint.

Economic Interpretation

- The decomposition algorithm can be interpreted as follows:
  - With the current proposals of the subsystems, the superordinate (total system) obtains the optimal weights of these proposals and announces a set of prices for using the common resources.
  - These prices are passed down to the subsystems, which modify their proposals according to these new prices.

Economic Interpretation

- A typical subsystem *i* solves the following subproblem:

$$\begin{array}{ll} \text{Maximize} & (\mathbf{wA}_i - \mathbf{c}_i)\mathbf{x}_i + \alpha_i \\ \text{Subject to} & \mathbf{x}_i \in X_i \end{array}$$

- or equivalently:

$$\begin{array}{ll} \text{Minimize} & (\mathbf{c}_i - \mathbf{wA}_i)\mathbf{x}_i - \alpha_i \\ \text{Subject to} & \mathbf{x}_i \in X_i \end{array}$$

Economic Interpretation

- The original objective of subsystem *i* is $\mathbf{c_i x_i}$.
- The term -$\mathbf{wA_i x_i}$ reflects the indirect price of using the common resources.
- $\mathbf{A_i x_i}$ is the amount of the common resources consumed by the $\mathbf{x_i}$ proposal.
- Since the price of using these resources is -$\mathbf{w}$, then the indirect cost of using them is -$\mathbf{wA_i x_i}$,
- And the total cost is $(\mathbf{c_i} - \mathbf{wA_i})\mathbf{x_i}$
- The term -**wAxi** makes proposals that use much of the common resources unattractive from a cost point of view.

Economic Interpretation

- Subsystem *i* announces an optimal proposal $\mathbf{x}_{\mathbf{ik}}$
- If this proposal is to be considered, then the weight of the older proposals $\mathbf{x}_{\mathbf{ij}}$'s must decrease in order to "make room" for this proposal;
- That is, $\Sigma_j\lambda_{ij}$ must decrease from its present level of 1.
- The resulting saving is precisely α_i.
- If the cost of introducing the proposal $\mathbf{x}_{\mathbf{ik}}$ is less than the saving realized; that is, if

  $$(\mathbf{c}_\mathbf{i} - \mathbf{w}\mathbf{A}_\mathbf{i})x_{\mathbf{ik}} - \alpha_i < 0, \text{ or}$$

  $$(\mathbf{w}\mathbf{A}_\mathbf{i} - \mathbf{c}_\mathbf{i})\mathbf{x}_{\mathbf{ik}} + \alpha_i > 0,$$

  - then the superordinate would consider this new proposal.

Economic Interpretation

- After all the subsystems introduce their new proposals, the superordinate calculates the optimum mix of these proposals and passes down new prices.
- The process is repeated until none of the subsystems has a new attractive proposal; that is, when

$$(\mathbf{c_i} - \mathbf{wA_i})x_{\mathbf{ik}} - \alpha_{\mathrm{i}} \geq 0 \text{ for each i}$$

References

References

- M.S. Bazaraa, J.J. Jarvis, H.D. Sherali, **Linear Programming and Network Flows**, Wiley, 1990. (Chapter 7)

The End