In the name of God

Part 4. Decomposition Algorithms

4.2. Benders' Decomposition Algorithm

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Benders Decomposition Algorithm

Benders Decomposition Algorithm

- The Dantzig-Wolfe decomposition procedure is equivalent for linear programming problems to two other well known partitioning / decomposition / relaxation techniques:
  - **Benders decomposition / partitioning method**
  - **Lagrangian relaxation method**

Benders Decomposition Algorithm

- Consider a linear programming problem P:

$$\begin{aligned} \text{P:} \quad & \text{Minimize} \quad \mathbf{cx} \\ & \text{Subject to} \quad \mathbf{Ax} = \mathbf{b} \\ & \qquad\qquad\quad \mathbf{x} \in X = \{\mathbf{x} : \mathbf{Dx} \geq \mathbf{d}, \mathbf{x} \geq \mathbf{0}\} \end{aligned}$$

- Let us assume that X is nonempty and bounded.
- Now let us write the dual D to problem P.

$$\begin{aligned} \text{D:} \quad & \text{Maximize} \quad \mathbf{wb} + \mathbf{vd} \\ & \text{Subject to} \quad \mathbf{wA} + \mathbf{vD} \leq \mathbf{c} \\ & \qquad\qquad\quad \mathbf{w} \quad \text{unrestricted}, \mathbf{v} \geq \mathbf{0} \end{aligned}$$

- We designate w and v as the dual variables associated with the constraints $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{Dx} \geq \mathbf{d}$ respectively.

Benders Decomposition Algorithm

- Observe that when **w** is fixed at some arbitrary value, we obtain a linear programming problem in the variables **v**.
- In particular this linear program may be specially structured or easy to solve.
- Let us proceed by **partitioning** problem D, while treating the variables **w** as **complicating** variables as follows:

$$D: \quad \underset{\mathbf{w}\ \text{unres}}{\text{Maximize}} \left\{ \begin{array}{l} \mathbf{wb} + \text{Maximum } \mathbf{vd} \\ \qquad\quad \text{Subject to } \mathbf{vD} \leq \mathbf{c} - \mathbf{wA} \\ \qquad\qquad\qquad \mathbf{v} \geq \mathbf{0} \end{array} \right\}$$

$$= \underset{\mathbf{w}\ \text{unres}}{\text{Maximum}} \left\{ \mathbf{wb} + \underset{\mathbf{x} \in X}{\text{Minimum}} (\mathbf{c} - \mathbf{wA})\mathbf{x} \right\}$$

Benders Decomposition Algorithm

- Here we have written the **dual to the inner optimization problem** at the last step
- The inner minimization problem attains an extreme point optimal solution.
- Denoting $\mathbf{x}_1$, $\mathbf{x}_2$, …, $\mathbf{x}_t$ as the vertices of X, we have that D is equivalent to the problem of

$$\text{maximizing } \{\mathbf{wb} + \text{Minimum}_{j=1,\ldots,t}(\mathbf{c} - \mathbf{wA})\mathbf{x}_j\}$$

  – over unrestricted values of **w**.
- Denoting z as the objective function in $\{\cdot\}$

Benders Decomposition Algorithm

- This may be rewritten as the following **Benders Master Problem**, **MP**:

$$
\begin{aligned}
\textbf{MP:}\quad & \text{Maximize} \quad z \\
& \text{Subject to} \quad z \le \mathbf{wb} + (\mathbf{c} - \mathbf{wA})\mathbf{x}_j \qquad \text{for } j = 1, \ldots, t \\
& \qquad\qquad\quad\; z, \mathbf{w} \quad \text{unrestricted}
\end{aligned}
$$

- This MP is inconvenient to solve directly because it typically has far **too many constraints**.
- Hence we can adopt a **relaxation strategy**, in which only a few of the constraints are explicitly maintained.

Benders Decomposition Algorithm

- For a **relaxed master program** we obtain an optimal solution $(\bar{z}, \bar{\mathbf{w}})$.
- Then $\bar{z}$ is an upper bound on the optimal value to the original problem.
- Furthermore $(\bar{z}, \bar{\mathbf{w}})$ is optimal for MP if and only if $(\bar{z}, \bar{\mathbf{w}})$ is feasible to all constraints.
- In order to check if any constraints are violated, we wish to check if:

$$\bar{z} \leq \bar{\mathbf{w}}\mathbf{b} + (\mathbf{c} - \bar{\mathbf{w}}\mathbf{A})\mathbf{x}_j$$

  – for all $j = 1, \ldots, t$,

Benders Decomposition Algorithm

- That is, if:

$$\bar{z} \le \bar{\mathbf{w}}\mathbf{b} + \text{minimum}_{j=1,\dots,t}\{(\mathbf{c} - \bar{\mathbf{w}}\mathbf{A})\mathbf{x}_j\}.$$

- This is equivalent to the linear programming **Benders' subproblem**:

$$\bar{\mathbf{w}}\mathbf{b} + \underset{\mathbf{x}\in X}{\text{minimum}}\{(\mathbf{c} - \bar{\mathbf{w}}\mathbf{A})\mathbf{x}\}.$$

- If $\bar{z}$ is less than or equal to the optimal objective subproblem value, then $\bar{z}$ will be *equal* to the optimal value

Benders Decomposition Algorithm

- Otherwise if $\mathbf{x}_k$ solves the subproblem, we have

$$\bar{z} > \bar{\mathbf{w}}\mathbf{b} + (\mathbf{c} - \bar{\mathbf{w}}\mathbf{A})\mathbf{x}_k$$

- and we can generate the constraint

$$z \le \mathbf{w}\mathbf{b} + (\mathbf{c} - \mathbf{w}\mathbf{A})\mathbf{x}_k$$

- and add it to the current **relaxed master program** and **reoptimize it**

Benders Decomposition Algorithm

- This process may be repeated until the solution $(\bar{\mathbf{z}}, \bar{\mathbf{w}})$ to **relaxed master problem** gives $\bar{z}$ equal to the optimal value in the subproblem:

$$\bar{\mathbf{w}}\mathbf{b} + \underset{\mathbf{x} \in X}{\text{minimum}} \{(\mathbf{c} - \bar{\mathbf{w}}\mathbf{A})\mathbf{x}\}.$$

- This must occur finitely since X has only a finite number of vertices.

Benders Decomposition Algorithm

- Note that the **Benders subproblem** is also the subproblem solved by the Dantzig-Wolfe decomposition method
- And the **Benders master program** is simply the **dual** to the **Dantzig-Wolfe master problem** with constraints.
- Here the nonnegative dual multipliers associated with the constraints are λ_j, $j = 1, 2, \ldots, t$, these variabls sum to unity by virtue of the column of the variables

Benders Decomposition Algorithm

- Therefore Benders' algorithm is called a **row generation** or **cutting plane technique** in contrast with the Dantzig-Wolfe **column generation procedure**.

Lagrangian Relaxation Algorithm

- **Lagrangian relaxation technique** is another general optimization strategy for P that finds an equivalence with the foregoing methods.
- Observe from

$$\underset{\mathbf{w}\text{ unres}}{\text{Maximum}}\left\{\mathbf{wb} + \underset{\mathbf{x}\in X}{\text{Minimum}}\,(\mathbf{c} - \mathbf{wA})\mathbf{x}\right\}$$

- if we denote the function in $\{\cdot\}$, which is a function of **w**, as $\boldsymbol{\theta}(\boldsymbol{w})$, we can equivalently state D as **Lagrangian dual** the problem:

$$\text{Maximize}\{\,\theta(\mathbf{w}) : \mathbf{w}\text{ unrestricted}\,\}$$

- Where **Lagrangian subproblem** is:

$$\theta(\mathbf{w}) = \mathbf{wb} + \text{Minimum}\{(\mathbf{c} - \mathbf{wA})\mathbf{x} : \mathbf{x} \in X\}$$

Numeric Example

Numeric Example

- Consider the following problem:

$$
\begin{aligned}
\text{Minimize} \quad & -2x_1 - x_2 - x_3 + x_4 \\
\text{Subject to} \quad & x_1 + x_3 \leqslant 2 \\
& x_1 + x_2 + 2x_4 \leqslant 3 \\
& x_1 \leqslant 2 \\
& x_1 + 2x_2 \leqslant 5 \\
& -x_3 + x_4 \leqslant 2 \\
& 2x_3 + x_4 \leqslant 6 \\
& x_1, \quad x_2, \quad x_3, \quad x_4 \geqslant 0
\end{aligned}
$$

Numeric Example

- Denote the dual variables associated with the constraints as (w_1, w_2, v_1, v_2, v_3, v_4) and write the dual as follows:

$$
\begin{array}{llllllll}
\text{Maximize} & 2w_1 & + 3w_2 & + 2v_1 & + 5v_2 & + 2v_3 & + 6v_4 & \\
\text{Subject to} & w_1 & + w_2 & + v_1 & + v_2 & & & \leq -2 \\
 & & w_2 & & + 2v_2 & & & \leq -1 \\
 & w_1 & & & & - v_3 & + 2v_4 & \leq -1 \\
 & & 2w_2 & & & + v_3 & + v_4 & \leq 1 \\
 & & & \mathbf{w} \leq \mathbf{0}, & \mathbf{v} \leq \mathbf{0} & & &
\end{array}
$$

Numeric Example

- Treating **w** as the complicating variables, we obtain the **Benders master problem** as follows:

$$
\begin{aligned}
\text{Maximize} \quad & z \\
\text{Subject to} \quad & z \le 2w_1 + 3w_2 + x_{j1}(-2 - w_1 - w_2) \\
& \quad + x_{j2}(-1 - w_2) + x_{j3}(-1 - w_1) + x_{j4}(1 - 2w_2) \\
& \qquad\qquad \text{for } j = 1, \ldots, t \\
& z \quad \text{unrestricted}, \ \mathbf{w} \le \mathbf{0}
\end{aligned}
$$

- where $\mathbf{x_j} = (\mathbf{x_{j1}}, \mathbf{x_{j2}}, \mathbf{x_{j3}}, \mathbf{x_{j4}})$, $j = 1, \ldots, t$ are the vertices of X

Numeric Example

- The corresponding **Lagrangian dual problem** is:

$$\text{Maximize}\{\theta(\mathbf{w}) : \mathbf{w} \leq \mathbf{0}\}$$

- where, for a given $\mathbf{w} = (w_1, w_2)$, we have:

$$\theta(\mathbf{w}) = 2w_1 + 3w_2 + \underset{\mathbf{x} \in X}{\text{Minimum}} \{(-2 - w_1 - w_2)x_1 + (-1 - w_2)x_2 + (-1 - w_1)x_3 + (1 - 2w_2)x_4\}$$

- The evaluation of $\theta(\mathbf{w})$, given $\mathbf{w}$, is precisely the **Benders subproblem**.

Numeric Example

- Suppose that as in Dantzig-Wolfe decomposition algorithm we begin with the vertex $\mathbf{x_1} = (0,0,0,0)$ of X.
- Hence the **relaxed Benders master problem** is of the form:

$$\begin{array}{ll} \text{Maximize} & z \\ \text{Subject to} & z \leq 2w_1 + 3w_2 \\ & \mathbf{w} \leq \mathbf{0} \end{array}$$

Numeric Example

- The optimal solution is $\bar{z} = 0$, $\bar{\mathbf{w}} = (0,0)$.
- Hence using only the tangential support $z \leq 2w_1 + 3w_2$ for $\theta(.)$,
- We obtain $\bar{\mathbf{w}} = (0,0)$ as the maximizing solution.
- Solving the Benders' subproblem with $\bar{\mathbf{w}} = (0.0)$, that is, computing $\theta(\bar{\mathbf{w}})$,
- we get

$$\theta(\bar{\mathbf{w}}) = -\frac{17}{2}, \quad \text{at } \mathbf{x}_2 = (2, \tfrac{3}{2}, 3, 0)$$

Numeric Example

- Since $\theta(\bar{\mathbf{w}}) < \bar{z}$, we generate a second **Benders cut** or tangential support using $\mathbf{x_2}$:

$$z \le 2w_1 + 3w_2 + 2(-2 - w_1 - w_2) + \tfrac{3}{2}(-1 - w_2) + 3(-1 - w_1)$$
$$= -\tfrac{17}{2} - 3w_1 - \tfrac{1}{2}w_2$$

- This leads to the **Benders' master program**:

$$\begin{aligned} \text{Maximize} \quad & z \\ \text{Subject to} \quad & z \le 2w_1 + 3w_2 \\ & z \le -\tfrac{17}{2} - 3w_1 - \tfrac{1}{2}w_2 \\ & \mathbf{w} \le \mathbf{0} \end{aligned}$$

Numeric Example

- The optimal solution is

$$\bar{z} = -\tfrac{17}{5}, \text{ and } \bar{\mathbf{w}} = (-\tfrac{17}{10}, 0).$$

- The slack variables in both constraints are nonbasic, and hence both constraints are maintained.
- The respective dual variables are 3/5 and 2/5
- Solving the **Benders subproblem**, we compute

$$\theta(\bar{\mathbf{w}}) = -\tfrac{59}{10} \text{ at } \mathbf{x}_3 = (0, \tfrac{5}{2}, 0, 0).$$

- Since $\bar{z} > \theta(\bar{\mathbf{w}})$, we generate a third Benders constraint or tangential support as:

$$z \leq 2w_1 + 3w_2 + \tfrac{5}{2}(-1 - w_2)$$

$$= -\tfrac{5}{2} + 2w_1 + \tfrac{1}{2}w_2.$$

Numeric Example

- Appending this to the master problem, we obtain:

$$\begin{aligned}
\text{Maximize} \quad & z \\
\text{Subject to} \quad & z \le 2w_1 + 3w_2 \\
& z \le -\tfrac{17}{2} - 3w_1 - \tfrac{1}{2}w_2 \\
& z \le -\tfrac{5}{2} + 2w_1 + \tfrac{1}{2}w_2 \\
& \mathbf{w} \le \mathbf{0}
\end{aligned}$$

- The optimal solution to this problem is:

$$\bar{z} = -\frac{49}{10}, \text{ and } \bar{\mathbf{w}} = \left(-\frac{6}{5}, 0\right)$$

- with the slack in the first constraint being basic.

Numeric Example

- The optimal dual multipliers are:

$$\lambda_1 = 0,\ \lambda_2 = \tfrac{2}{5},\ \text{and}\ \lambda_3 = \tfrac{3}{5}.$$

- Hence the first constraint may be deleted from the current outer-linearization.
- A deleted constraint can possibly be regenerated later.

References

References

- M.S. Bazaraa, J.J. Jarvis, H.D. Sherali, **Linear Programming and Network Flows**, Wiley, 1990. (Chapter 7)

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