In the name of God

Part 4. Decomposition Algorithms

4.4. Column Generation for the Constrained Shortest Path Problem

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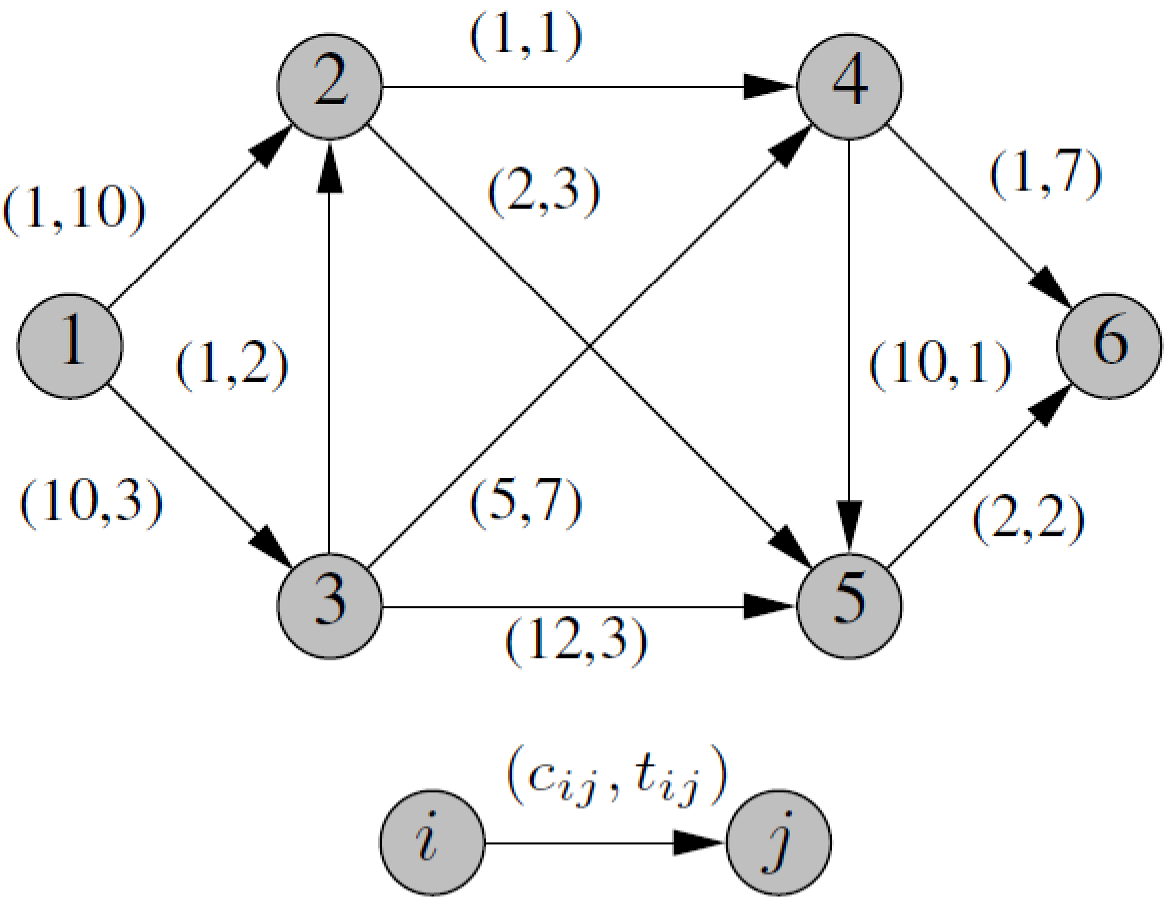
Constrained Shortest Path Problem

Constrained Shortest Path Problem

- We start by solving **a constrained shortest path problem**.



Constrained Shortest Path Problem

- c_{ij} : cost of arc (i, j)
- t_{ij} : traversal time arc (i, j)
- Our goal is to find a shortest path from node 1 to node 6 such that the total traversal time of the path does not exceed 14 time units.

Constrained Shortest Path Problem

- The integer program

$$z^\star := \min \sum_{(i,j)\in A} c_{ij}x_{ij}$$

$$\text{subject to} \quad \sum_{j:(1,j)\in A} x_{1j} = 1$$

$$\sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji} = 0 \qquad i = 2,3,4,5$$

$$\sum_{i:(i,6)\in A} x_{i6} = 1$$

$$\sum_{(i,j)\in A} t_{ij}x_{ij} \le 14$$

$$x_{ij} = 0 \text{ or } 1 \quad (i,j) \in A$$

Constrained Shortest Path Problem

- Constraints:
  - One unit of flow has to leave the source
  - Flow conservation holds at all other nodes
  - One unit of flow has to enter the sink
  - With the time resource constraint
- There are **nine** possible paths, three of which consume too much time.
- The optimal integer solution is path 13246 of cost 13 with a traversal time of 13.

Constrained Shortest Path Problem

- The time resource constraint prevents us from solving our problem with a classical shortest path algorithm.
- In fact, no polynomial time algorithm is likely to exist since the resource constrained shortest path problem is NP-hard.

An Equivalent Reformulation: Arcs vs. Paths

An Equivalent Reformulation: Arcs vs. Paths

- If we ignore the time constraint, the easily tractable remainder is:

$$X = \{ \quad \sum_{j:(1,j)\in A} x_{1j} = 1$$

$$\sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji} = 0 \qquad i = 2, 3, 4, 5$$

$$\sum_{i:(i,6)\in A} x_{i6} = 1$$

$$x_{ij} = 0 \text{ or } 1 \quad (i,j) \in A$$

$$\}$$

An Equivalent Reformulation: Arcs vs. Paths

- An extreme point $\mathbf{x_p} = x_{pij}$ defined by the convex hull of X corresponds to a path $p \in P$ in the network.
- Any arc flow can be expressed as a convex combination of path flows:

$$x_{ij} = \sum_{p \in P} x_{pij} \lambda_p \quad (i,j) \in A$$

$$\sum_{p \in P} \lambda_p = 1$$

$$\lambda_p \geq 0 \quad p \in P.$$

An Equivalent Reformulation: Arcs vs. Paths

- If we substitute for x_{ij} in:

$$z^\star := \min \sum_{(i,j)\in A} c_{ij} x_{ij}$$

$$\sum_{(i,j)\in A} t_{ij} x_{ij} \le 14$$

An Equivalent Reformulation: Arcs vs. Paths

- We obtain the **master problem**:

$$z^\star = \min \sum_{p \in P} \Big(\sum_{(i,j) \in A} c_{ij} x_{pij} \Big) \lambda_p$$

$$\text{subject to} \sum_{p \in P} \Big(\sum_{(i,j) \in A} t_{ij} x_{pij} \Big) \lambda_p \leq 14$$

$$\sum_{p \in P} \lambda_p = 1$$

$$\lambda_p \geq 0 \qquad p \in P$$

An Equivalent Reformulation: Arcs vs. Paths

- The cost coefficient of λ_p is the cost of path p and its coefficient in first constraint is path p's duration.

An Equivalent Reformulation: Arcs vs. Paths

- We may recover a solution **x** to **original problem** from a master problem's solution by:

$$\sum_{p \in P} x_{pij} \lambda_p = x_{ij} \qquad (i,j) \in A$$

$$x_{ij} = 0 \text{ or } 1 \quad (i,j) \in A$$

- Always integrality must hold for the original x variables.

The Linear Relaxation of the Master Problem

The Linear Relaxation of the Master Problem

- Associate **dual variables** with constraints:

$$z^\star = \min \sum_{p \in P} \Big(\sum_{(i,j) \in A} c_{ij} x_{pij} \Big) \lambda_p$$

dual variables

$$\text{subject to} \sum_{p \in P} \Big(\sum_{(i,j) \in A} t_{ij} x_{pij} \Big) \lambda_p \leq 14 \qquad \pi_1$$

$$\sum_{p \in P} \lambda_p = 1 \qquad \pi_0$$

$$\lambda_p \geq 0 \qquad p \in P$$

The Linear Relaxation of the Master Problem

- There are a problem with **nine path variables** and **two constraints**.
- For large networks, the cardinality of P becomes prohibitive, and we cannot even explicitly state all the variables of the master problem.

The Linear Relaxation of the Master Problem

- The appealing idea of **column generation** is to work only with a sufficiently meaningful subset of variables, forming the so-called **restricted master problem (RMP)**.
- More variables are added only when needed
- Like in the simplex method we have to find in every iteration a promising variable to enter the basis.

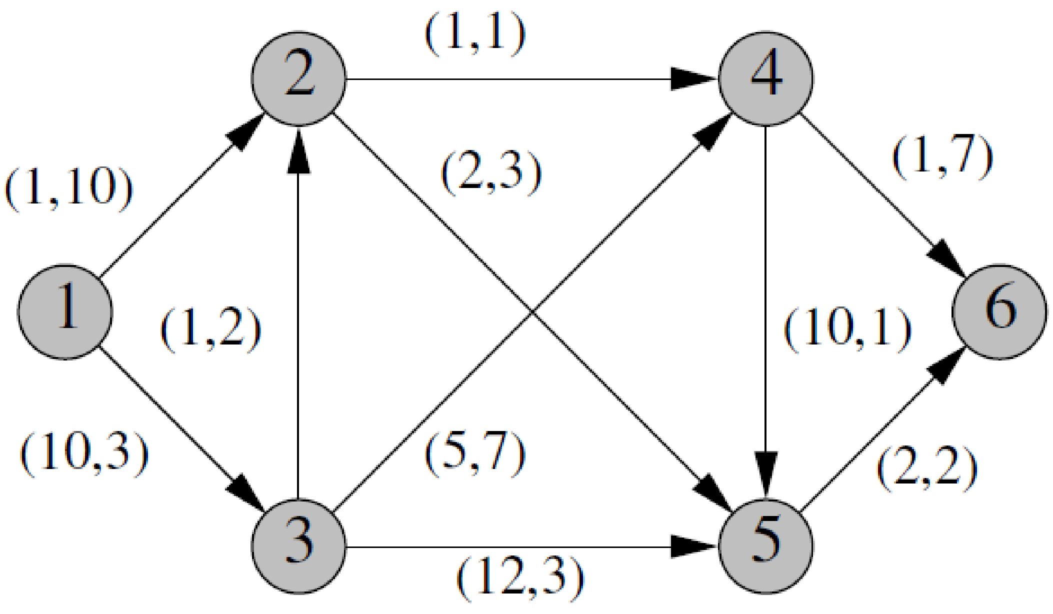
The Linear Relaxation of the Master Problem

- In column generation an iteration consists:
  - (a) optimizing the restricted master problem in order to determine the current optimal objective function value $\bar{z}$ and dual multipliers **π**, and
  - (b) finding, if there still is one, a variable *p* with **negative reduced cost**

$$\bar{c}_p = \sum_{(i,j)\in A} c_{ij} x_{pij} - \pi_1 \left(\sum_{(i,j)\in A} t_{ij} x_{pij} \right) - \pi_0 \; < 0$$

The Linear Relaxation of the Master Problem

- The search for a minimum reduced cost variable amounts to optimizing a **subproblem**, precisely in our case: a shortest path problem in the network of :



The Linear Relaxation of the Master Problem

- **The subproblem**:

$$\bar{c}^\star = \min \sum_{(i,j)\in A} (c_{ij} - \pi_1 t_{ij}) x_{ij} - \pi_0$$

$$\text{subject to} \quad \sum_{j:(1,j)\in A} x_{1j} = 1$$

$$\sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji} = 0 \qquad i = 2,3,4,5$$

$$\sum_{i:(i,6)\in A} x_{i6} = 1$$

$$x_{ij} = 0 \text{ or } 1 \quad (i,j) \in A$$

The Linear Relaxation of the Master Problem

- Clearly, if $\bar{c}^* \geq 0$ there is no improving variable and the linear relaxation of the master problem is optimal
- Otherwise, the variable found is added to the **RMP** and we repeat.
- We now give full numerical details of the solution of our particular instance.

The Linear Relaxation of the Master Problem

- $BBn.i$: iteration number i at node number n ($n = 0$ represents the root node).
- p : the index of path
- c_p : the cost of path p
- t_p : the duration of path p
- $\bar{z}$: the objective function value of restricted master problem
- $\boldsymbol{\pi}$: dual variables
- $\bar{c}^*$: the objective function value of subproblem

The Linear Relaxation of the Master Problem

- The linear programming relaxation of the master problem:

Iteration	Master Solution	$\bar{z}$	π_0	π_1	$\bar{c}^\star$	p	c_p	t_p
BB0.1	$y_0 = 1$	100.0	100.00	0.00	-97.0	1246	3	18
BB0.2	$y_0 = 0.22, \lambda_{1246} = 0.78$	24.6	100.00	-5.39	-32.9	1356	24	8
BB0.3	$\lambda_{1246} = 0.6, \lambda_{1356} = 0.4$	11.4	40.80	-2.10	-4.8	13256	15	10
BB0.4	$\lambda_{1246} = \lambda_{13256} = 0.5$	9.0	30.00	-1.50	-2.5	1256	5	15
BB0.5	$\lambda_{13256} = 0.2, \lambda_{1256} = 0.8$	7.0	35.00	-2.00	0			
Arc flows: $x_{12} = 0.8, x_{13} = x_{32} = 0.2, x_{25} = x_{56} = 1$								

The Linear Relaxation of the Master Problem

- **Iteration 1**
  - Since we have no feasible initial solution at iteration BB0.1, we adopt a big-M approach and introduce an artificial variable y_0 with a large cost, say 100, for the convexity constraint.
  - We do not have any path variables yet and the RMP contains two constraints and the artificial variable.
  - This problem is solved by inspection: $y_0 = 1$; $\bar{z} = 100$, and the dual variables are $\pi_0 = 100$ and $\pi_1 = 0$.
  - The subproblem returns path 1246 at reduced cost $\bar{c}^* = -97$, cost 3 and duration 18.

The Linear Relaxation of the Master Problem

- **Iteration 2**
  - In iteration BB0.2, the RMP contains two variables: y_0 and λ_{1246}.
  - An optimal solution with $\bar{z} = 24.6$ is $y_0 = 0.22$ and $\lambda_{1246} = 0.78$, which is still infeasible.
  - The dual variables assume values $\pi_0 = 100$ and $\pi_1 = -5.39$.
  - Solving the subproblem gives the feasible path 1356 of reduced cost -32.9, cost 24, and duration 8.

The Linear Relaxation of the Master Problem

- In total, four path variables are generated during the column generation process.
- In iteration BB0.5, we use 0.2 times the feasible path 13256 and 0.8 times the infeasible path 1256.
- The optimal objective function value is 7, with $\pi_0 = 35$ and $\pi_1 = -0.2$.
- The arc flow values provided at the bottom of Table are identical to those found when solving the LP relaxation of the original problem.

The Linear Relaxation of the Master Problem

- In order to obtain integer solutions to our original problem, we have to embed column generation within a **branch-and-bound** framework.

Column Generation with Branch-and-Bound

Column Generation with Branch-and-Bound

- A subsequent branch-and-bound process is used to compute an optimal integer solution.
- To proceed, we have to start the reformulation and column generation again in each node.

Column Generation with Branch-and-Bound

- **First strategy**: a single arc flow variable of branching on fractional variables
  - branching on $x_{12} = 0.8$.
  - For $x_{12} = 0$, the impact on the RMP is that we have to remove path variables 1246 and 1256, that is, those paths which contain arc (1, 2).
  - In the subproblem, this arc is removed from the network.
  - On branch $x_{12} = 1$, arcs (1, 3) and (3, 2) cannot be used.
  - Generated paths which contain these arcs are discarded from the RMP, and both arcs are removed from the subproblem.

Column Generation with Branch-and-Bound

- **Second strategy:** involving more than a single arc flow variables
  - We branch on $x_{13} + x_{32} = 0.4$.
  - On branch $x_{13} + x_{32} = 0$, we simultaneously treat two flow variables; impacts on the RMP and the subproblem.
  - On branch $x_{13} + x_{32} = 1$, this constraint is first added to the original formulation.
  - We exploit again the path substructure X, go through the reformulation process, and obtain a new RMP to work with.
  - Details of the search tree are summarized in Table.

Column Generation with Branch-and-Bound

- The branch-and-bound decisions

Iteration	Master Solution	$\bar{z}$	π_0	π_1	π_2	$\bar{c}^\star$	p	c_p	t_p
BB1: BB0 and $x_{13} + x_{32} = 0$									
BB1.1	$y_0 = 0.067, \lambda_{1256} = 0.933$	11.3	100	-6.33	–	0			
BB1.2	Big-M increased to 1000								
	$y_0 = 0.067, \lambda_{1256} = 0.933$	71.3	1000	-66.33	–	-57.3	12456	14	14
BB1.3	$\lambda_{12456} = 1$	**14**	1000	-70.43	–	0			
BB2: BB0 and $x_{13} + x_{32} \geq 1$									
BB2.1	$\lambda_{1246} = \lambda_{13256} = 0.5$	9	15	-0.67	3.33	0			
Arc flows: $x_{12} = x_{13} = x_{24} = x_{25} = x_{32} = x_{46} = x_{56} = 0.5$									
BB3: BB2 and $x_{12} = 0$									
BB3.1	$\lambda_{13256} = 1$	**15**	15	0	0	-2	13246	13	13
BB3.2	$\lambda_{13246} = 1$	**13**	13	0	0	0			
BB4: BB2 and $x_{12} = 1$									
BB4.1	$y_0 = 0.067, \lambda_{1256} = 0.933$	111.3	100	-6.33	100	0			
Infeasible arc flows									

References

References

- J. Desrosiers and M. E. Lubbecke, **A Primer in Column Generation**, G. Desaulniers, J. Desrosiers, and M. M. Solomon, eds., Springer-Verlag, New York, 2005, ch. 1, pp. 1–32.

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