In the name of God

Part 4. Decomposition Algorithms

4.5. Column Generation for the Budget Design Problem

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The Budget Design Problem

Introduction

- In the network budget design problem (BDP) the cost of flowing a set of commodities through a network is minimized while observing budget constraints on the fixed costs of the links used.
- Applications of this problem are diverse and include problems such as deciding what transportation or communication infrastructure to build between cities or deciding on the topology of a computer network.

Introduction

- BDP can be used to construct a new network or modify an existing one.

- The links may be either directed or undirected.

Constraints

- **Balance equations**
  - In its simplest form, BDP includes **balance equations** for the flow of each commodity
- **Budget constraint**
  - A **budget constraint** on the sum of fixed costs of the links selected
- **Bundle constraints**
  - **Bundle constraints** to ensure that flow is allowed only on arcs which are built and that the maximum capacity of each arc is observed

Constraints

- Depending on the application, there may be other constraints, such as
  - **service level constraints** or
  - **requirements on the reliability** of the network.

Node-Arc Formulation

Node-Arc Formulation

- We first present a node-arc Mixed Integer Program (MIP) for BDP and show that the constraint matrix for this formulation is too large for the problems we wish to solve.

Node-Arc Formulation

$G = (N, A)$ is the graph with node set N and candidate arc set A.

K is the set of all commodities k designated by an origin-destination pair of nodes.

v^k is the volume of commodity k (in consistent units).

$orig(k)$ is the origin node for commodity k. $orig(a)$ is the origin of arc a.

$dest(k)$ is the destination node for commodity k. $dest(a)$ is the destination of arc a.

u_a is the capacity of arc a.

Node-Arc Formulation

$c_a \geq 0$ is the per unit cost of flow on arc a (assumed equal for all commodities).

e_a is the fixed cost for including arc a in the network.

B is the budget for fixed costs for the entire network.

$B(i)$ is the budget for fixed costs of arcs leaving i.

Node-Arc Formulation

x_a^k is the proportion of commodity k's volume flowing on arc a.

$$y_a = \begin{cases} 1 & \text{if arc } a \text{ is included in the network,} \\ 0 & \text{otherwise.} \end{cases}$$

Node-Arc Formulation

$$\min \quad \sum_{k \in K} \sum_{a \in A} c_a v_k x_a^k \tag{1.1a}$$

s.t.

Bundle
$$\sum_{k \in K} v_k x_a^k \le u_a y_a \qquad \forall a \in A \tag{1.1b}$$

Balance
$$\sum_{\substack{a \in A \\ orig(a)=i}} x_a^k - \sum_{\substack{a \in A \\ dest(a)=i}} x_a^k = \begin{cases} 1 & orig(k) = i \\ -1 & dest(k) = i \\ 0 & \text{otherwise} \end{cases} \qquad \forall i \in N, k \in K \tag{1.1c}$$

Budget
$$\sum_{a \in A} e_a y_a \le B \tag{1.1d}$$

Node Budget
$$\sum_{\substack{a \in A \\ orig(a)=i}} e_a y_a \le B(i) \qquad \forall i \in N \tag{1.1e}$$

$$y_a \in \{0,1\} \qquad \forall a \in A$$

$$0 \le x_a^k \le 1 \qquad \forall a \in A, k \in K$$

Objective

- The objective is to minimize the sum of the costs of delivering each commodity using the network formed by arcs for which $y_a = 1$.

Constraints

- For each arc, constraints (1.1b) prevent flow on arcs which are not built and enforce the upper bound U_a on flow for arcs which are built.
- For each node, constraints (1.1c) are balance equations for the flow of each commodity.
- The single budget constraint (1.1d) limits the sum of the fixed costs e_a for all arcs selected for the network to the total budget B.
- Constraints (1.1e) enforce the node-budget limit $B(i)$ for the sum of the fixed costs of the arcs which leave the node.

BDP & Other Problems

- BDP is related to several other problems, such as, the fixed-charge network design problem and the multi-commodity flow problem.
- The fixed charge network design problem is different in that the fixed costs for the arcs appear in the objective function (minimize the total fixed and variable costs) and not in the constraints.
- The multi-commodity flow problem is a BDP with the binary arc selection variables fixed.

Node-Arc Formulation

- As is typical with node-arc formulations, NODE has a large number of constraints and variables.
- Let
  - |N| : denote the number of nodes
  - |K| : the number of commodities
  - |A| : the number of possible arcs

Node-Arc Formulation

- There will be |A| bundle constraints (1.1 b), one for each arc
- Since there is one mass balance constraint for each node-commodity pair, there will be:
  - |N| |K| equality constraints of form (1.1c)
  - with a non-zero coefficient for each potential arc which is adjacent to the node.
- The single (1.1 d) budget constraint
- |N| node-budget constraints (1.1 e), one for each node.
- Total of constraints

  $|A| + |N|\,|K| + 1 + |N|$

Variables

- |A||K| continuous flow variables x_a^k, one for the flow of each commodity on each arc
- |A| binary selection variables y_a

An Example

- For a network with
  - 100 nodes,
  - 1000 commodities, and
  - 5,000 arcs (half of the possible 10,000 node pairs)

An Example

- There would be:
  - 105,101 constraints,
    - bundle constraints: 5000
    - balance constraints: 100,000
    - total budget constraints: 1
    - node-budget constraints: 100
  - 5,000,000 continuous variables, and
  - 5,000 binary variables.

Result

- Given the huge size of the constraint set, storing and manipulating this formulation becomes difficult on a workstation.

Path-Based Formulation

Path-Based Formulation

Parameters

Q(k) is the set of all legal paths for commodity *k*.

PC_q^k is the path cost for flowing one unit of commodity k on path q.

δ_a^q is the incidence indicator that equals 1 if arc a is on path q and 0 otherwise.

Decision Variables

f_q^k proportion of commodity k on path q, $\forall q \in Q(k), k \in K$

PATH Formulation

$$\min \sum_{k \in K} \sum_{q \in Q(k)} v^k \, PC_q^k \, f_q^k \qquad (1.3a)$$

s.t.

$$\sum_{k \in K} \sum_{q \in Q(k)} v^k \, f_q^k \, \delta_a^q - u_a y_a \leq 0 \qquad \forall a \in A \qquad (1.3b)$$

$$\sum_{q \in Q(k)} f_q^k = 1 \qquad \forall k \in K \qquad (1.3c)$$

$$\sum_{a \in A} e_a y_a \leq B \qquad (1.3d)$$

$$\sum_{\substack{a \in A \\ orig(a)=i}} e_a y_a \leq B(i) \qquad \forall i \in N \qquad (1.3e)$$

$$f_q^k \geq 0 \qquad \forall q \in Q(k), k \in K$$

$$y_a \in \{0, 1\} \qquad \forall a \in A.$$

Path-Based Formulation

- Substituting for $\sum_{q \in Q(k)} f_q^k \delta_a^q$ for x_a^k
- Constraints (1.3b) ensure that flow is only allowed on arcs included in the network
- Constraints (1.3c) ensure all of each commodity is delivered
- (1.3d) is the overall budget constraint
- (1.3e) are the budget constraints for each node

Path-Based Formulation

- For the same size example network as before (|N|=100, |K|=1000, |A|=5000), there would be:
  - 6,101 constraints
    - bundle constraints: 5000
    - balance constraints: 1000
    - total budget constraints: 1
    - node-budget constraints: 100
  - 5,000 binary y_a selection variables
  - an exponential number of continuous path variables

Column Generation for the BDP

Column Generation for the BDP

- In PATH, the number of paths δ^q for each commodity is generally exponential in the number of nodes of the network unless the candidate arc set A is small.
- Consequently, explicitly including all f_q^k columns in PATH may result in a formulation which is too large to store or solve easily on a workstation.
- We develop a column generation procedure to solve the linear programming (LP) relaxation of PATH

Column Generation for the BDP

- The LP relaxation of PATH-MP with the indicated assignment of dual variables:

$$
\begin{array}{llll}
\text{(MP)} \quad min & \sum_{k\in K}\sum_{q\in Q(k)} v^k\, PC_q^k\, f_q^k & & \text{DUALS} \\
\text{s.t.} & & & \\
& \sum_{k\in K}\sum_{q\in Q(k)} v^k f_q^k \delta_a^q - u_a y_a \le 0 & \forall a \in A & (\beta_a) \\
& \sum_{q\in Q(k)} f_q^k = 1 & \forall k \in K & (\chi_k) \\
& \sum_{a\in A} e_a y_a \le B & & (\eta) \\
& \sum_{\substack{a\in A \\ orig(a)=i}} e_a y_a \le B(i) & \forall i \in N & (\theta_i) \\
& y_a \le 1 & \forall a \in A & (\mu_a) \\
& f_q^k \ge 0 & \forall q \in Q(k), k \in K & \\
& y_a \ge 0 & \forall a \in A &
\end{array}
$$

Column Generation for the BDP

- The dual of MP is:

$$\text{(DUAL)} \quad max \quad \sum_{k \in K} \chi_k + \eta B + \sum_{i \in N} B(i)\theta_i + \sum_{a \in A} \mu_a$$

$$\text{s.t.}$$

$$\sum_{a \in A} (v^k \delta_a^q)\beta_a + \chi_k - v^k PC_q^k \leq 0 \quad \forall k \in K, q \in Q(k)$$

$$-u_a\beta_a + e_a\eta + e_a\theta_{orig(a)} + \mu_a \leq 0 \quad \forall a \in A$$

$$\beta_a, \eta, \theta_i, \mu_a \leq 0$$

$$\chi_k \text{ free.}$$

- First constraints
  - correspond to the commodity path variables f_q^k
  - If some of the f_q^k variables are missing from MP, then they will correspond to missing these constraints in the dual problem.
- Second constraints correspond to the arc selection variables y_a

Column Generation for the BDP

- If there is a column f_q^k of MP not included in RMP with favourable reduced cost, it will correspond to a violated first constraint in the dual of form

Pricing Subproblem for Commodity k

- To find a violated first constraint in the dual of form, the objective of the column generation sub- problem is:

$$max \sum_{a \in A} v^k \delta_a^q \beta_a + \chi_k - v^k PC_q^k$$

- If the objective value of subproblem for a commodity *k* is greater than zero, then first constraint in the dual of form is violated and we must add a column to RMP.

Pricing Subproblem for Commodity k

- Equivalently the minimization form of subproblem

$$min \sum_{a \in A} v^k \, \delta_a^q \, (-\beta_a) - \chi_k + PC_q^k \, v^k$$

- Using the minimization form for the objective, we obtain the following column generation subproblem:

$$(SP_k) \quad min \quad \sum_{a \in A} v^k \, \delta_a^q \, (-\beta_a) - \chi_k + PC_q^k \, v^k$$

s.t.

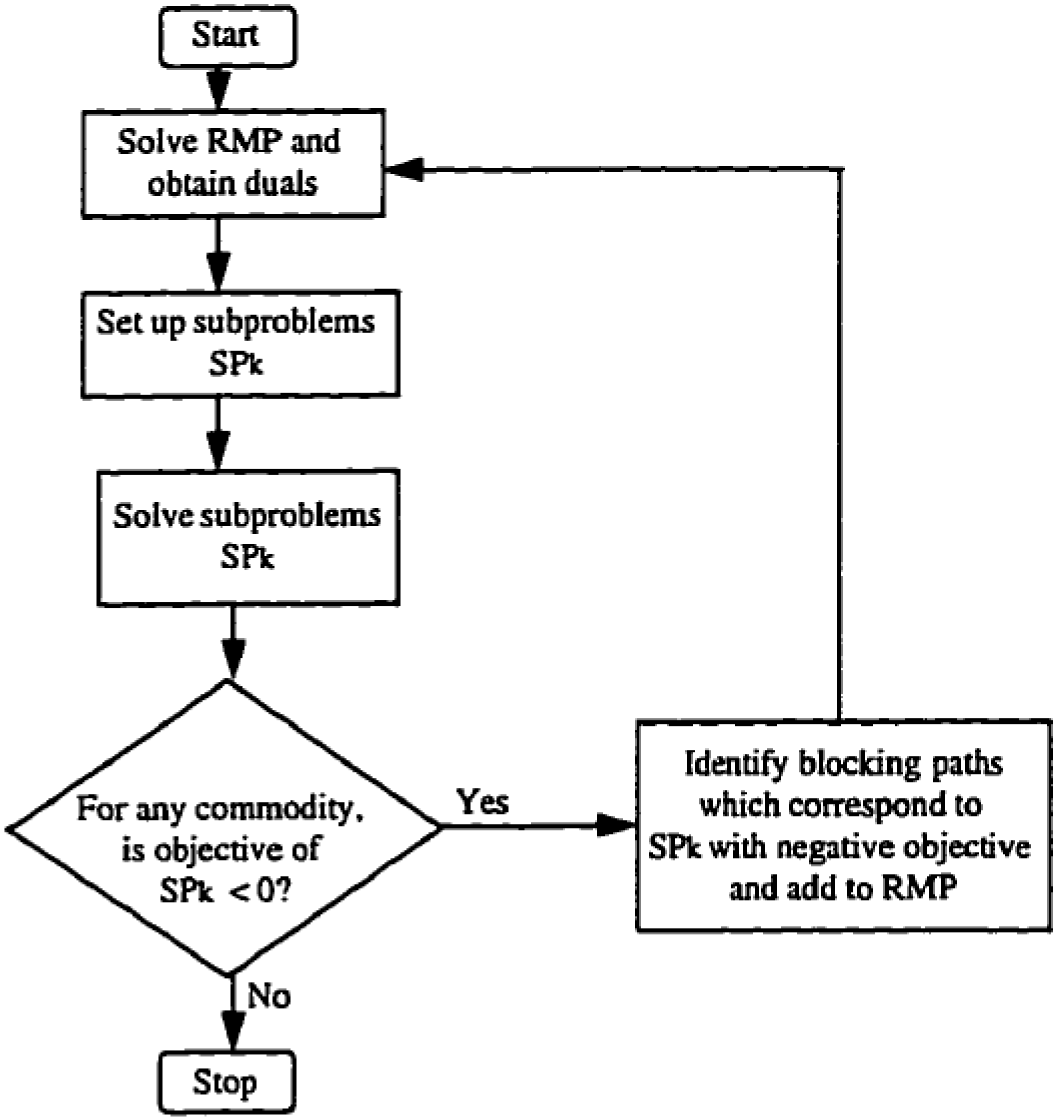
$$\text{Balance} \sum_{\substack{a \in A \\ orig(a)=i}} \delta_a^q - \sum_{\substack{a \in A \\ dest(a)=i}} \delta_a^q = \begin{cases} 1 & orig(k) = i \\ -1 & dest(k) = i \\ 0 & \text{otherwise} \end{cases} \quad \forall i \in N$$

$$0 \le \delta_a^q \le 1 \qquad \forall a \in A$$

Pricing Subproblem for Commodity k

- B_a and χ_k are fixed by the present solution to RMP.
- If the objective value of SP_k is less than zero, then the solution identifies a path with negative reduced cost for commodity *k* and we add this column to RMP.
- Any shortest path algorithm may be used to solve these subproblems.

Column Generation Algorithm



Strengthening MP with Additional Cuts

- MP is a weak LP relaxation.
- Both the node-arc and path-based formulations which we introduced above have weak LP relaxations, especially if the arc capacities u_a are large with respect to the possible flows.
- Using the notation for the node-arc formulation, the y_a values are bounded below by:

$$y_a \geq \frac{\sum_{k \in K} v^k x_a^k}{u_a}$$

  – Where x_a^k is the proportion of commodity *k* on arc *a*.

Strengthening MP with Additional Cuts

- For large u_a capacities these constraints provide little support to force the y_a to be near **one** for arcs which are used.
- Fractional y_a will allow arcs to have flow but only part of their fixed cost will be incurred in the budget constraint, so **more arcs** may have flow in the LP relaxation than in the MIP.
- Consequently, the bound provided by MP may significantly underestimate PATH and thus provide little help in working out nodes in any branch-and-bound procedure.

Strengthening MP with Additional Cuts

- To strengthen the bound of MP, we propose adding valid inequalities which we call **forcing constraints**.
- These constraints prevent the flow of an individual commodity on an arc unless the proportion of flow on the arc is less than or equal to the y_a selection variable for the arc.
- The difficulty is usually that the problem formulation becomes much larger.

Strengthening MP with Additional Cuts

- These forcing constraints for NODE are:

$$x_a^k - y_a \leq 0 \qquad \forall a \in A, k \in K$$

- and for PATH are:

$$\sum_{q \in Q(k)} f_q^k \, \delta_a^q - y_a \leq 0 \qquad \forall a \in A, k \in K$$

Strengthening MP with Additional Cuts

- Adding all possible valid inequalities for PATH to one test instance with six nodes and thirty commodities
- It reduced the number of nodes evaluated in the branch-and-bound tree from 1161 to 33.
- So, these cuts seem to be strong and may be high dimensional faces of the convex hull of feasible solutions.
- We add the forcing constraints to MP and let ϕ_a^k denote the corresponding dual variable for commodity k and arc a

Strengthening MP with Additional Cuts

- The LP relaxation of PATH-MP with force constraints:

$$(MP) \quad \min \quad \sum_{k \in K} \sum_{q \in Q(k)} v^k PC_q^k f_q^k \qquad \text{DUALS}$$

s.t.

$$\sum_{k \in K} \sum_{q \in Q(k)} v^k f_q^k \delta_a^q - u_a y_a \leq 0 \qquad \forall a \in A \qquad (\beta_a)$$

$$\sum_{q \in Q(k)} f_q^k = 1 \qquad \forall k \in K \qquad (\chi_k)$$

$$\sum_{a \in A} e_a y_a \leq B \qquad (\eta)$$

$$\sum_{\substack{a \in A \\ orig(a)=i}} e_a y_a \leq B(i) \qquad \forall i \in N \qquad (\theta_i)$$

$$\sum_{q \in Q(k)} f_q^k \delta_a^q - y_a \leq 0 \qquad \forall a \in A, k \in K \qquad (\phi_a^k)$$

$$y_a \leq 1 \qquad \forall a \in A \qquad (\mu_a)$$

$$f_q^k \geq 0 \qquad \forall q \in Q(k), k \in K$$

$$y_a \geq 0 \qquad \forall a \in A$$

Strengthening MP with Additional Cuts

- Then constraint of DUAL will become:

$$\sum_{a \in A} \{\delta_a^q (v^k \beta_a + \phi_a^k)\} + \chi_k - v^k PC_q^k \le 0 \quad \forall k \in K, q \in Q(k)$$

- Correspondingly, the objective of SP_k will become:

$$min \quad \sum_{a \in A} \{\delta_a^q (-v^k \beta_a - \phi_a^k)\} - \chi_k + PC_q^k v^k$$

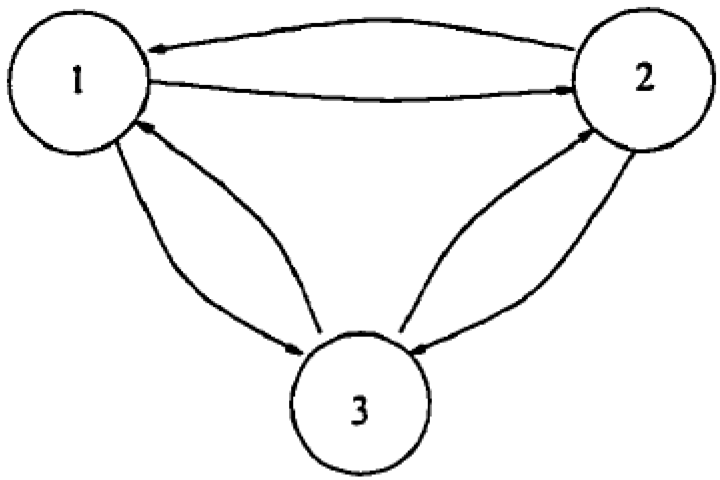
Branch-and-Price Algorithm

- Column generation procedure to solve the LP-relaxation of PATH in a branch-and-bound search for an integer solution to PATH, is called **branch-and-price algorithm**.
- Branching on the (fractional) y_a can be accommodated in SP_k by deleting arcs a for which y_a is fixed at zero.

Example

Example

- The node and arc set are:



Example

- The nodes are $N = \{1, 2, 3\}$.
- All node-pairs are potential arcs; i.e., $A = \{(1, 2), (1,3), (2,1), (2,3), (3, 1), (3, 2)\}$.
- The flow cost and fixed cost of each arc is one, $c_a = 1$, $e_a = 1$.
- The arc capacities are $u_{12} = 2$, $u_{13} = 3$, $u_{21} = 1$, $u_{23} = 2$, $u_{31} = 0$, $u_{32} = 1$.
- The commodities (each with unit volume) are $\{(1, 2), (1,3), (2,3)\}$
- Node budget capacities $B(i)$ are all one.
- The overall budget $B = 3$.

Example

- Since all of the flow costs are one, the objective is equivalent to minimizing the sum of the number of arcs used to deliver each commodity.
- Since $u_{31} = 0$, y_{31} may be deleted from the formulation.
- The LP relaxation gives an integer solution for this example, but, in general, branching will be required to reach an integer solution.

Example

- We initially populate RMP with paths for each commodity that use the single arc from the commodity's origin to its destination.
- In order to use column generation, a feasible solution must exist among the columns in RMP, so we introduce the artificial variables a_t with high objective cost to the budget constraints.
- Let f_{i-j}^{ij} represent the path (i-j) for commodity with origin i and destination j .

Example

- The initial RMP is:

$$
\begin{array}{lccccccccccc}
\min & f^{12}_{1-2} & +f^{13}_{1-3} & +f^{23}_{2-3} & & & & & & +M\sum_{i=0}^{3} a_i & & \\
\text{s.t.} & & & & & & & & & & & \\
\text{bundle}_{12} & f^{12}_{1-2} & & & -2y_{12} & & & & & & \leq 0 \\
\text{bundle}_{13} & & f^{13}_{1-3} & & & -3y_{13} & & & & & \leq 0 \\
\text{bundle}_{21} & & & & & & -y_{21} & & & & \leq 0 \\
\text{bundle}_{23} & & & f^{23}_{2-3} & & & & -2y_{23} & & & \leq 0 \\
\text{bundle}_{32} & & & & & & & & -y_{32} & & \leq 0 \\
\text{convex}^{12} & f^{12}_{1-2} & & & & & & & & & = 1 \\
\text{convex}^{13} & & f^{13}_{1-3} & & & & & & & & = 1 \\
\text{convex}^{23} & & & f^{23}_{2-3} & & & & & & & = 1 \\
\text{budget}_{1} & & & & y_{12} & +y_{13} & & & & -a_1 & \leq 1 \\
\text{budget}_{2} & & & & & & y_{21} & +y_{23} & & -a_2 & \leq 1 \\
\text{budget}_{3} & & & & & & & & y_{32} & -a_3 & \leq 1 \\
\text{budget} & & & & y_{12} & +y_{13} & +y_{21} & +y_{23} & +y_{32} & -a_0 & \leq 3 \\
\text{forcing}^{12}_{12} & f^{12}_{1-2} & & & -y_{12} & & & & & & \leq 0 \\
\text{forcing}^{13}_{13} & & f^{13}_{1-3} & & & -y_{13} & & & & & \leq 0 \\
\text{forcing}^{23}_{23} & & & f^{23}_{2-3} & & & & -y_{23} & & & \leq 0 \\
 & & & & & & & & 0 & \leq y_a & \leq 1 \\
 & & & & & & & & & f^k_q & \geq 0.
\end{array}
$$

Example

- Let M = 10,000,0000.
- Solving RMP gives the following solution with an objective value of 500,003.

$$y_{12} = y_{13} = y_{23} = 1.0$$

$$f^{12}_{1-2} = f^{13}_{1-3} = f^{23}_{2-3} = 1$$

$$a_1 = 0.005$$

- With these nonzero duals:

$$\chi_{12} = \chi_{13} = 500001$$

$$\chi_{23} = 1$$

$$\phi^{12}_{12} = -500000$$

Example

- Solving SP_k for commodity (1,2) identifies the shortest path (1-3-2).
- The objective value for SP_k is:

$$(-\beta_{13}) + c_{13}v^{12} - \phi_{13}^{12} + (-\beta_{32}) + c_{32}v^{12} - \phi_{32}^{12} - \chi_{12}$$

$$= 1 + 1 - 500001 = -499999$$

- Since the objective is less than zero, the path (1-3-2) prices out for commodity (1,2) and will be added to RMP.

Example

- Solving SP_k for commodity (1,3) identifies the shortest path (1-2-3) for which the objective value of SP_k is:

$$(-\beta_{12}) + c_{12}v^{13} - \phi_{12}^{13} + (-\beta_{23}) + c_{23}v^{13} - \phi_{23}^{13} - \chi_{13}$$

$$= 1 + 1 - 500001 = -499999$$

- and the path (1-2-3) prices out for commodity (1,3) and will be added to RMP.
- The objective for SP_k for commodity (2,3) is zero, so no columns are added for this commodity.

Example

- Adding the two paths which priced out and the corresponding forcing constraints to RMP gives the following formulation:

$$
\begin{array}{llllllllllllll}
min & f^{12}_{1-2} & +f^{13}_{1-3} & +f^{23}_{2-3} & +f^{12}_{1-3-2} & +f^{13}_{1-2-3} & & & & & & +M\sum_{i=0}^{3} a_i & \\
s.t. & & & & & & & & & & & & \\
bundle_{12} & f^{12}_{1-2} & & & & +f^{13}_{1-2-3} & -2y_{12} & & & & & & \leq 0 \\
bundle_{13} & & f^{13}_{1-3} & & +f^{12}_{1-3-2} & & & -3y_{13} & & & & & \leq 0 \\
bundle_{21} & & & & & & & & -y_{21} & & & & \leq 0 \\
bundle_{23} & & & f^{23}_{2-3} & & +f^{13}_{1-2-3} & & & & -2y_{23} & & & \leq 0 \\
bundle_{32} & & & & & & & & & & -y_{32} & & \leq 0 \\
convex^{12} & f^{12}_{1-2} & & & +f^{12}_{1-3-2} & & & & & & & & = 1 \\
convex^{13} & & f^{13}_{1-3} & & & +f^{13}_{1-2-3} & & & & & & & = 1 \\
convex^{23} & & & f^{23}_{2-3} & & & & & & & & & = 1 \\
budget_1 & & & & & & y_{12} & +y_{13} & & & & -a_1 & \leq 1 \\
budget_2 & & & & & & & & y_{21} & +y_{23} & & -a_2 & \leq 1 \\
budget_3 & & & & & & & & & & y_{32} & -a_3 & \leq 1 \\
budget & & & & & & y_{12} & +y_{13} & +y_{21} & +y_{23} & +y_{32} & -a_0 & \leq 3 \\
forcing^{12}_{12} & f^{12}_{1-2} & & & & & -y_{12} & & & & & & \leq 0 \\
forcing^{12}_{32} & & & & f^{12}_{1-3-2} & & & & & & -y_{32} & & \leq 0 \\
forcing^{13}_{12} & & & & & f^{13}_{1-2-3} & & -y_{13} & & & & & \leq 0 \\
forcing^{13}_{13} & & f^{13}_{1-3} & & & & & -y_{13} & & & & & \leq 0 \\
forcing^{23}_{23} & & & f^{23}_{2-3} & & & & & & -y_{23} & & & \leq 0 \\
 & & & & & & & & & & 0 & \leq y_a & \leq 1 \\
 & & & & & & & & & & & f^k_q & \geq 0.
\end{array}
$$

Column Generation Algorithm

- Solving RMP with these new columns gives an objective value of 4.
- Solving each SP_k with the new dual values does not identified any paths with favourable reduced cost.
- So the solution to RMP is also optimal for MP.
- In general, branching will be necessary before identifying an integral LP solution; however, in this particular case, all of the y_a are integer.
- Since the solution to MP is integral, it is also an optimal for solution for PATH.

Column Generation Algorithm

- The optimal solution is:

$$y_{12} = y_{23} = 1.0$$

$$f^{12}_{1-2} = f^{23}_{2-3} = f^{13}_{1-2-3} = 1.0$$

References

References

- H. N. Newton. **Network Design under Budget Constraints with Application to the Railroad Blocking Problem**. Ph.D. dissertation, Auburn University, USA, 1996.

The End