In the name of God

Part 1. The Review of Linear Programming

1.1. Introduction

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Outline

- The Linear Programming Problem
- Geometric Solution
- References

The Linear Programming Problem

Basic Definitions

- **Linear programming problem**
  - A problem of **minimizing** or **maximizing** a **linear function**
  - in the presence of **linear constraints** of the **inequality** and/or the **equality** type.

Basic Definitions

- **Formulation of LP problem:**
  - Identify the **decision variables**.
  - Identify the problem **constraints** and express the constraints as a series of **linear equations**.
  - Identify the **objective function** as a linear equation, and state whether the objective is **maximization** or **minimization**.

Basic Definitions

- A linear programming problem

$$\text{Minimize} \quad c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

$$\begin{aligned} \text{Subject to} \quad & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \geqslant b_1 \\ & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \geqslant b_2 \\ & \quad \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \qquad\quad \vdots \\ & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \geqslant b_m \\ & x_1, \quad x_2, \quad \ldots, \quad x_n \geqslant 0 \end{aligned}$$

Basic Definitions

- **Objective function**
  - Here $c_1x_1 + c_2x_2 + , ..., + c_nx_n$ is the objective function to be minimized and will be denoted by z.
- **Cost coefficients**
  - The coefficients $c_1, c_2, . . . , c_n$ are the cost coefficients
- **Decision variables**
  - $x_1, x_2, . . . , x_n$ are the decision variables (variables, or activity levels) to be determined.
- **Constraints**
  - The inequality $\sum_{j=1}^{n} a_{ij}x_j \geqslant b_i$ denotes the i th constraint.

Basic Definitions

- **Technological coefficients**
  - The coefficients a_{ij} for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$ are called the technological coefficients.
  - These technological coefficients form the constraint matrix **A** given below.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Basic Definitions

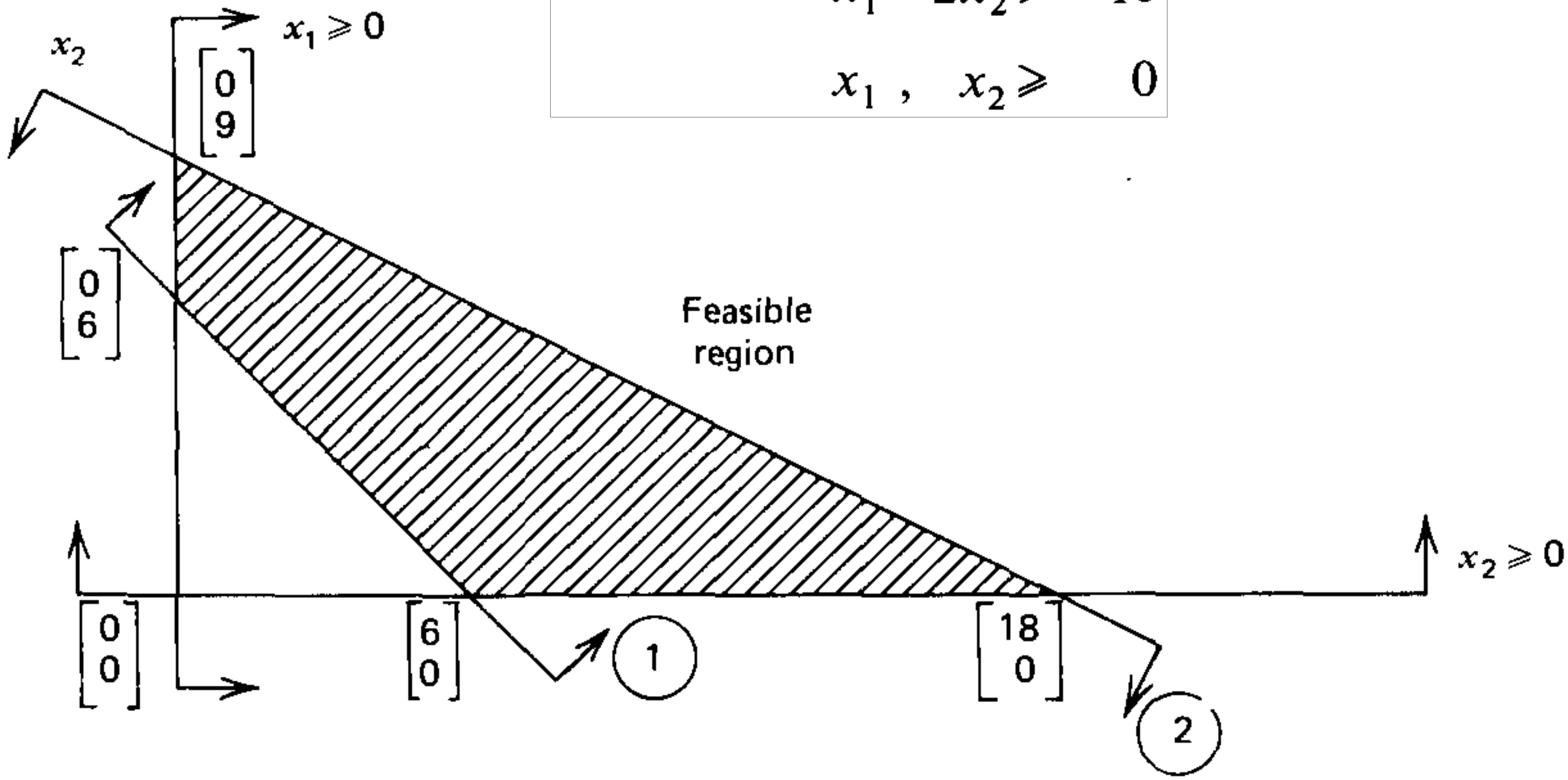
- **Right-hand-side vector**
  - The column vector whose i th component is b_i, which is referred to as the right-hand-side vector, represents the **minimal requirements** to be satisfied.
- **Nonnegativity constraints**
  - The constraints $x_1, x_2, \ldots, x_n \geq 0$ are the nonnegativity constraints.

Basic Definitions

- **Feasible point / feasible vector**
  - A set of variables $x_1, \ldots, x_n$ satisfying all the constraints is called a feasible point or a feasible vector.
- **Feasible region**
  - The set of all feasible points constitutes the feasible region or the feasible space.
- **The linear programming problem**
  - Among all feasible vectors, find that which minimizes (or maximizes) the objective function.

Example

$$\begin{aligned}
\text{Minimize} \quad & 2x_1 + 5x_2 \\
\text{Subject to} \quad & x_1 + x_2 \geqslant 6 \\
& -x_1 - 2x_2 \geqslant -18 \\
& x_1, \; x_2 \geqslant 0
\end{aligned}$$



Assumptions of Linear Programming

- **Proportionality**
  - The contribution of each activity to the value of the objective function or constraint is **proportional** to the level of the activity
  - No savings (or extra costs) are realized by using more of an activity
  - No setup cost, for starting the activity is realized.
- **Additivity**
  - Every function in a linear programming model is the **sum** of the individual contributions of the respective activities.

Assumptions of Linear Programming

- **Divisibility**
  - It is being assumed that the activities can be run at **fractional value**.
  - noninteger values for the decision variables are permitted
- **Certainty**
  - The value assigned to each parameter of a linear programming model is assumed to be a **known constant**.

Problem Manipulation

- By simple manipulations the LP problem can be transformed from one form to another equivalent form.
- These manipulations are:
  - **Inequalities and equations**
  - **Nonnegativity of the variables**
  - **Minimization and maximization problems**

Inequalities and Equations

- An **inequality** can be easily transformed into an **equation** by adding a nonnegative **slack variable**
- The constraint $\sum_{j=1}^{n} a_{ij} x_j \geqslant b_i$

  is equivalent to $\sum_{j=1}^{n} a_{ij} x_j - x_{n+i} = b_i$

  and $x_{n+i} \geq 0$
- The constraint $\sum_{j=1}^{n} a_{ij} x_j \leqslant b_i$

  is equivalent to $\sum_{j=1}^{n} a_{ij} x_j + x_{n+i} = b_i$

  and $x_{n+i} \geq 0$

Inequalities and Equations

- Also an equation of the form can be transformed into the two inequalities
- The equation $\sum_{j=1}^{n} a_{ij} x_j = b_i$

  is equivalent to $\sum_{j=1}^{n} a_{ij} x_j \leqslant b_i$

  $$\sum_{j=1}^{n} a_{ij} x_j \geqslant b_i.$$

Nonnegativity of The Variables

- For most practical problems the variables represent physical quantities and hence must be nonnegative.
- If a variable x_j is **unrestricted in sign**, then it can be replaced by $x'_j - x''_j$ where $x'_j \geq 0$ and $x''_j \geq 0$.
- If x_1, …., x_k are some k variables that all unrestricted variable, then only one additional variable x'' is needed in the equivalent transformation $x' = x'_j - x''$ for $j = 1, \ldots, k$, where $x'_j \geq 0$ and $x'' \geq 0$.

Nonnegativity of The Variables

- If $x_j \geq l_j$, then the new variable $x_j = x_j - l_j$ is automatically nonnegative.
- Also if a variable x_j is restricted such that $x \leq u_j$, where $u_j \leq 0$, then the substitution $x'_j = u_j - x_j$ produces a nonnegative variable x_j.

Minimization and Maximization Problems

- Another problem manipulation is to convert a maximization problem into a minimization problem and conversely.
- Note that over any region

$$\text{Maximum} \sum_{j=1}^{n} c_j x_j = -\text{minimum} \sum_{j=1}^{n} -c_j x_j$$

- After the optimization of the new problem is completed, the objective of the old problem is -1 times the optimal objective of the new

Standard and Canonical Formats

- **Standard form**
  - All restrictions are equalities and all variables are nonnegative.
- **Canonical form**
  - For a minimization problem: all variables are nonnegative and all the constraints are of the $\geq$ type.
  - For a maximization problem: all the variables are nonnegative and all the constraints are of the $\leq$ type.
  - The canonical form is useful in exploiting duality relationships.

Standard and Canonical Formats

	MINIMIZATION PROBLEM	MAXIMIZATION PROBLEM
Standard Form	Minimize $\sum_{j=1}^{n} c_j x_j$ Subject to $\sum_{j=1}^{n} a_{ij} x_j = b_i \quad i = 1, \ldots, m$ $x_j \geqslant 0 \quad j = 1, \ldots, n$	Maximize $\sum_{j=1}^{n} c_j x_j$ Subject to $\sum_{j=1}^{n} a_{ij} x_j = b_i \quad i = 1, \ldots, m$ $x_j \geqslant 0 \quad j = 1, \ldots, n$
Canonical Form	Minimize $\sum_{j=1}^{n} c_j x_j$ Subject to $\sum_{j=1}^{n} a_{ij} x_j \geqslant b_i \quad i = 1, \ldots, m$ $x_j \geqslant 0 \quad j = 1, \ldots, n$	Maximize $\sum_{j=1}^{n} c_j x_j$ Subject to $\sum_{j=1}^{n} a_{ij} x_j \leqslant b_i \quad i = 1, \ldots, m$ $x_j \geqslant 0 \quad j = 1, \ldots, n$

Linear Programming in Matrix Notation

- Consider the following problem

$$\text{Minimize} \quad \sum_{j=1}^{n} c_j x_j$$

$$\text{Subject to} \quad \sum_{j=1}^{n} a_{ij} x_j = b_i \quad i = 1, 2, \ldots, m$$

$$x_j \geqslant 0 \quad j = 1, 2, \ldots, n$$

Linear Programming in Matrix Notation

- Denote the row vector $(c_1, c_2, \ldots, c_n)$ by **c**, and consider the following column vectors **x** and **b**, and the $m \times n$ matrix **A**.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- Then the above problem can be written as follows.

$$\begin{aligned} \text{Minimize} \quad & \mathbf{cx} \\ \text{Subject to} \quad & \mathbf{Ax} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

Linear Programming in Matrix Notation

- The problem can also be conveniently represented via the columns of **A**.
- Denoting **A** by [$\mathbf{a}_1$, $\mathbf{a}_2$, . . . , $\mathbf{a}_n$] where $\mathbf{a}_j$ is they *j* th column of **A**, the problem can be formulated as follows.

$$\text{Minimize} \quad \sum_{j=1}^{n} c_j x_j$$

$$\text{Subject to} \quad \sum_{j=1}^{n} \mathbf{a}_j x_j = \mathbf{b}$$

$$x_j \geq 0 \qquad j = 1, 2, \ldots, n$$

Geometric Solution

Geometric Solution

- **Geometric method** for solving a linear programming is only suitable for very small problems.
- It provides a great deal of insight into the linear programming problem.
- Consider the following problem.

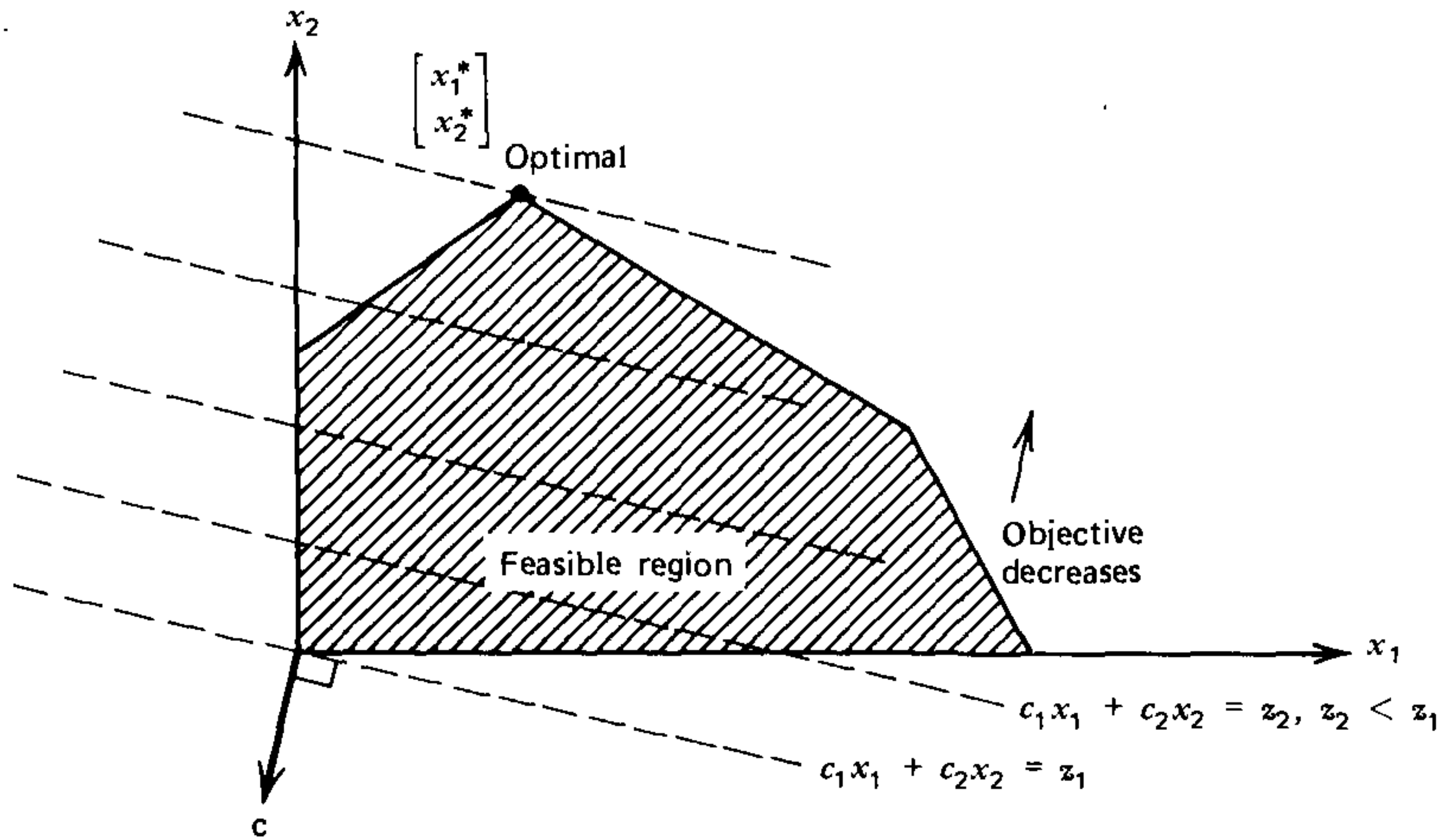
$$\begin{aligned} \text{Minimize} \quad & \mathbf{cx} \\ \text{Subject to} \quad & \mathbf{Ax} \geqslant \mathbf{b} \\ & \mathbf{x} \geqslant \mathbf{0} \end{aligned}$$

Geometric Solution

- The feasible region consists of all vectors **x** satisfying **Ax** ≥ **b** and **x** ≥ **0**.
- We want to find a point with minimal **cx** value.
- The points with the same objective z satisfy the equation **cx** = z, that is, $\sum_{j=1}^{n} c_j x_j = z$
- Since z is to be minimized, then the line (in a two-dimensional space) $\sum_{j=1}^{n} c_j x_j = z$ must be moved parallel to itself in the direction that minimizes the objective most.
- This direction is **-c**, and hence the plane is moved in the direction **-c** as much as possible.

Geometric Solution

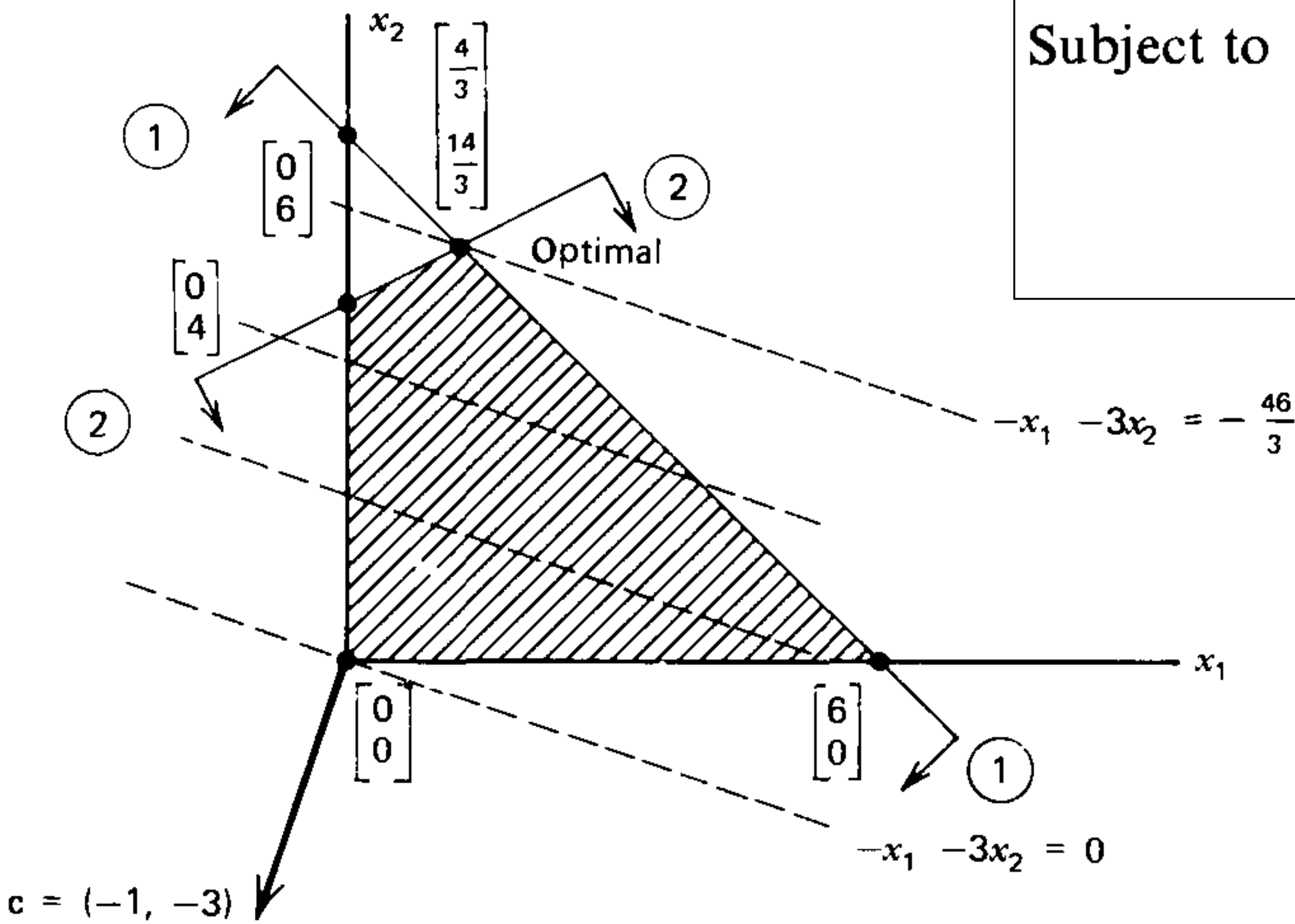
- This process is illustrated in this Figure.



Geometric Solution

- The **optimal point** **x*** is reached, the line $c_1x_1 + c_2x_2 = z^*$, where $z^* = c_1x^*_1 + c_2x^*_2$, cannot be moved farther in the direction $-\mathbf{c} = (-c1, -c2)$ because this will lead to only points outside the feasible region.
- We therefore conclude that x^* is indeed the **optimal solution**.
- The optimal point x^* is one of the five **corner points** that are called **extreme points**.
- If a linear program has a finite optimal solution, then it has an **optimal corner** (or **extreme**) **solution**.

Example



Minimize	$-x_1 - 3x_2$
Subject to	$x_1 + x_2 \leqslant 6$
	$-x_1 + 2x_2 \leqslant 8$
	$x_1, \quad x_2 \geqslant 0$

Example

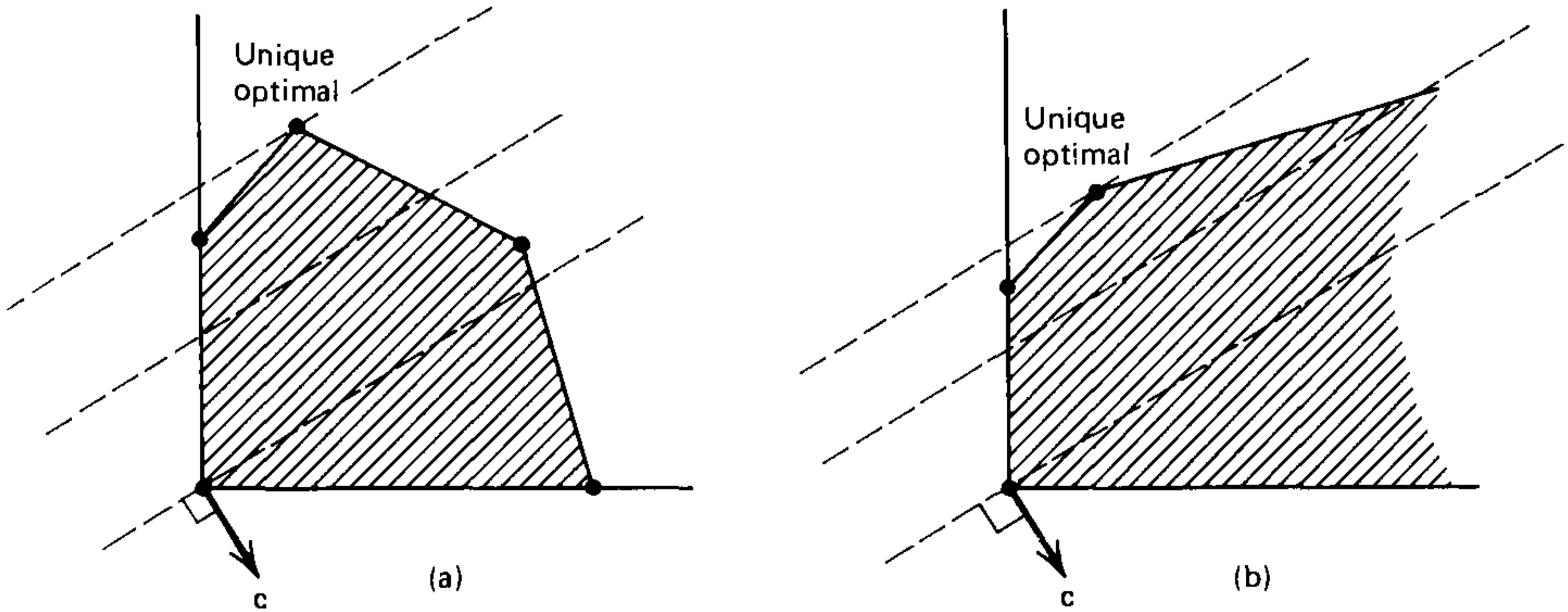
- The equations $-x1 - 3x2 = z$ are called the **objective contours** and are represented by dotted lines in the Figure.
- In particular the contour $-x1 - 3x2 = z = 0$ passes through the origin.
- The contours are moved in the direction **-c =** (1, 3) as much as possible until the optimal point (4/3, 14/3) is reached.

Geometric Solution

- In the example we had a unique optimal solution.
- Other cases may occur depending upon the problem structure.
- All possible cases that may arise are summarized below (for a minimization problem):
  - **Unique Finite Optimal Solution.**
  - **Alternative Finite Optimal Solutions**
  - **Unbounded Optimal Solution**
  - **Empty Feasible Region**

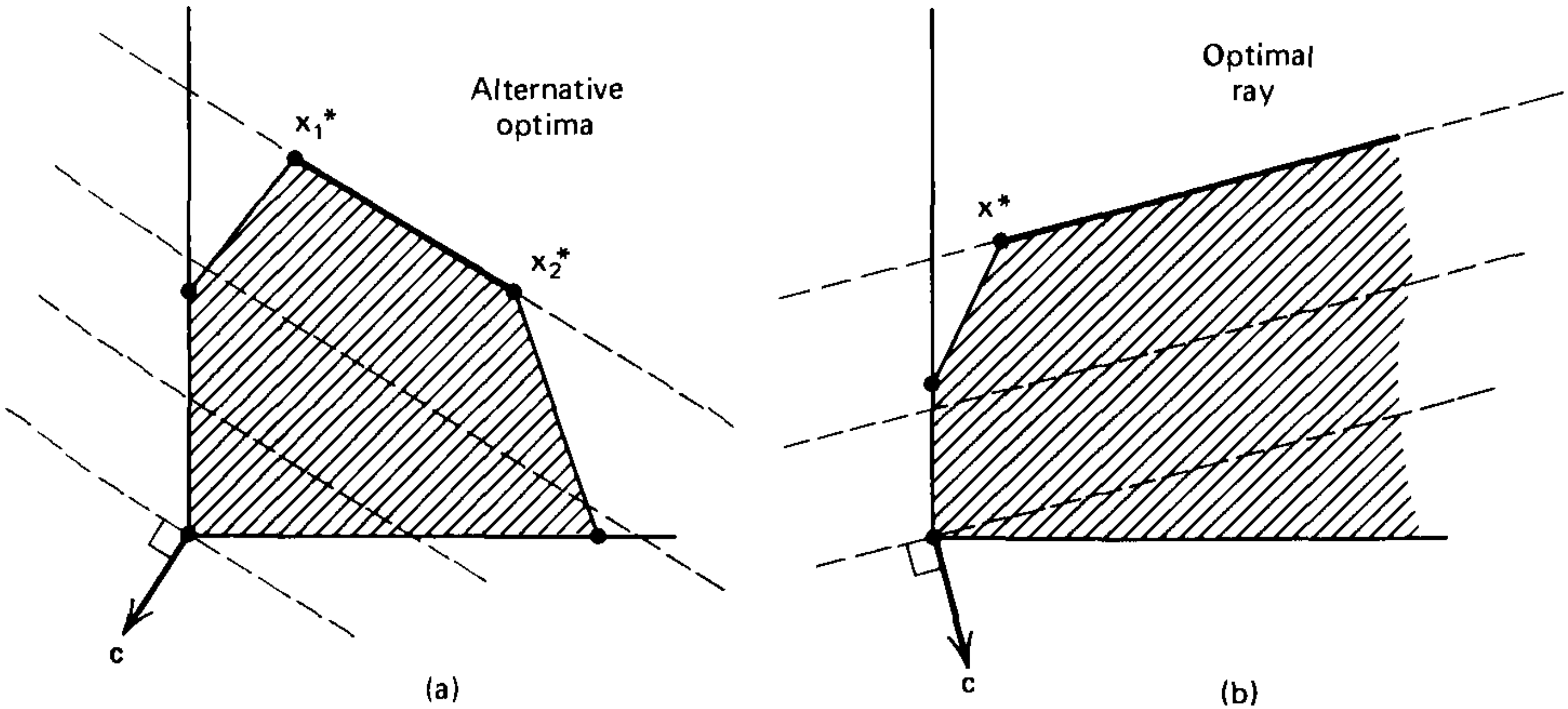
Unique Finite Optimal Solution

- If the optimal finite solution is unique, then it occurs at an extreme point. (a) Bounded region, (b) Unbounded Region.



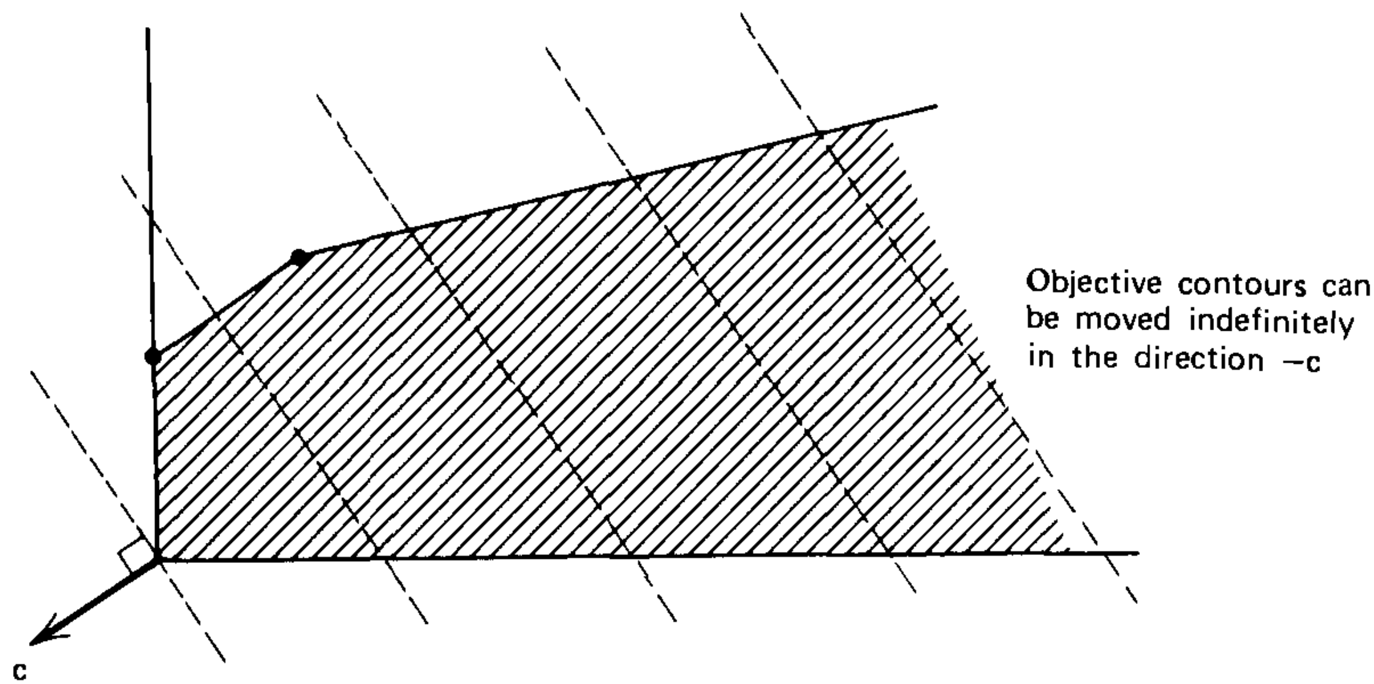
Alternative Finite Optimal Solutions

- (a) the feasible region is bounded. The two corner points x_1^* and x_2^* are optimal, and also any point on the line segment joining them.
- (b) the feasible region is unbounded but the optimal objective is finite. Any point on the ray with vertex x^* in Figure b is optimal.



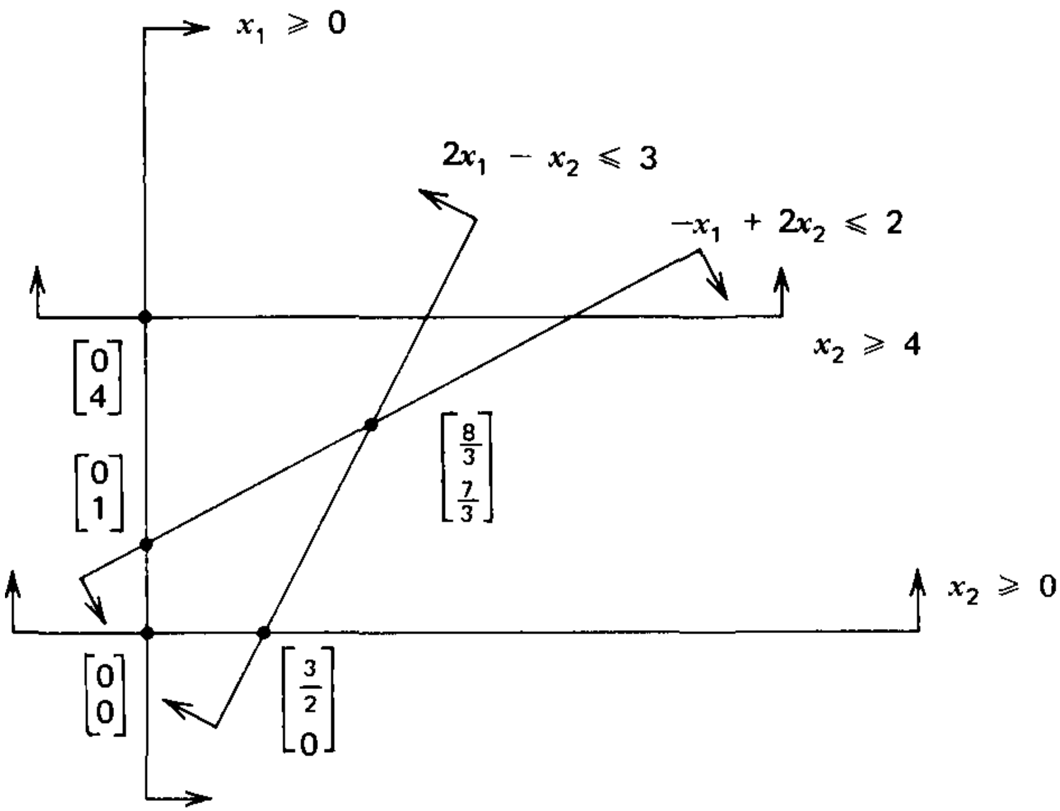
Unbounded Optimal Solution

- The feasible region and the optimal solution are unbounded. For a minimization problem the plane **cx** = z can be moved in the direction **–c** indefinitely while always intersecting with the feasible region. In this case the optimal objective is unbounded with value -∞.



Empty Feasible Region

- In this case the system of equations and/or inequalities defining the feasible region is inconsistent. Consider the following problem.



$$\begin{aligned} \text{Minimize} \quad & -2x_1 + 3x_2 \\ \text{Subject to} \quad & -x_1 + 2x_2 \leqslant 2 \\ & 2x_1 - x_2 \leqslant 3 \\ & x_2 \geqslant 4 \\ & x_1, \quad x_2 \geqslant 0 \end{aligned}$$

References

References

- M.S. Bazaraa, J.J. Jarvis, H.D. Sherali, **Linear Programming and Network Flows**, Wiley, 1990. (Chapter 1)

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