In the name of God

Part 1. The Review of Linear Programming

1.2. Simplex Method

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Introduction

The Simplex Method

- The **simplex method** is an **algebraic** procedure for solving **linear programming problems**.
- Its underlying concepts are **geometric**.
- Developed by **George Dantzig** in 1947.
- It has proved to be a remarkably **efficient method** that is used routinely to solve huge problems on today's computers.

Basic Feasible Solutions

Basic Feasible Solutions

- Consider the system $\mathbf{A}\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$, where $\mathbf{A}$ is an m x n matrix and $\mathbf{b}$ is an m vector.
- Suppose that rank $(\mathbf{A}, \mathbf{b})$ = rank $(\mathbf{A})$ = m.
- Let $\mathbf{A} = [\mathbf{B}, \mathbf{N}]$ where $\mathbf{B}$ is an m x m invertible matrix and $\mathbf{N}$ is an m x (n - m) matrix.
- The solution $\mathbf{x} = \begin{bmatrix} \mathbf{x}_B \\ \mathbf{x}_N \end{bmatrix}$ to the equations $\mathbf{A}x = \mathbf{b}$, where

$$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$$

$$\mathbf{x}_N = \mathbf{0}$$

Basic Feasible Solutions

- This solution is a **basic solution**
- If $\mathbf{x_B} \geq 0$, then $\mathbf{x}$ is called a **basic feasible solution**
- **B** is the **basic matrix** (or simply the **basis**)
- **N** is the **nonbasic matrix**
- $\mathbf{x_B}$ are **basic variables**
- $\mathbf{x_N}$ are **nonbasic variables**
- If $\mathbf{x_B} > 0$, then $\mathbf{x}$ is called a **nondegenerate basic feasible solution**
- If at least one component of $\mathbf{x_B}$ is zero, then $\mathbf{x}$ is called a **degenerate basic feasible solution**

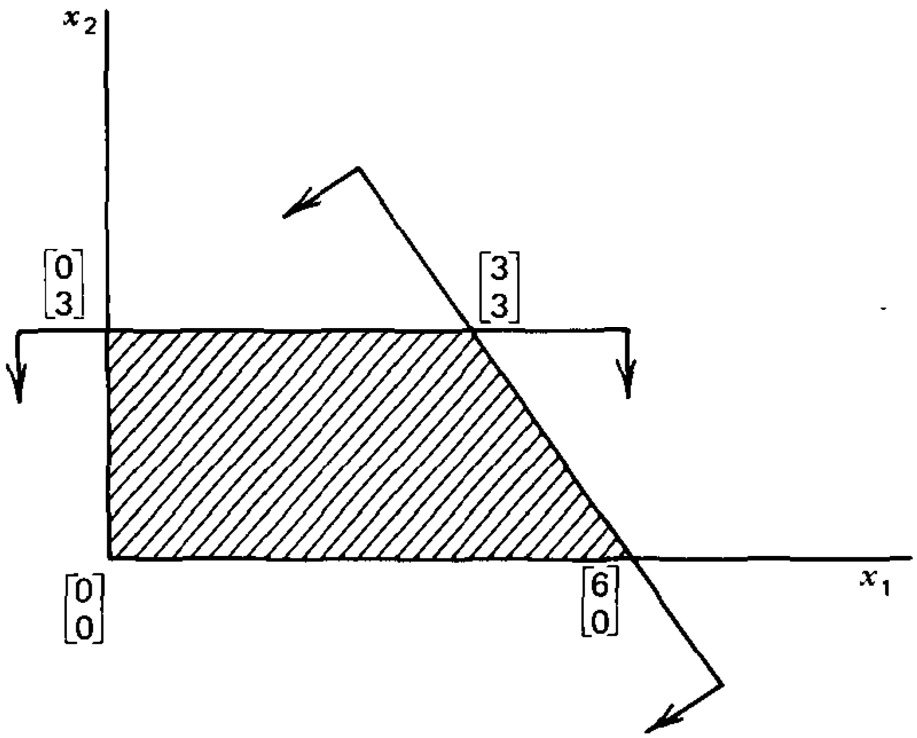
Example

- Consider the **polyhedral set** defined by the following inequalities

$$x_1 + x_2 \leqslant 6$$

$$x_2 \leqslant 3$$

$$x_1, \quad x_2 \geqslant 0$$



Example

- By introducing the slack variables x_3 and x_4, the problem is put in the following standard format:

$$\begin{aligned} x_1 + x_2 + x_3 \qquad &= 6 \\ x_2 \qquad + x_4 &= 3 \\ x_1,\ x_2,\ x_3,\ x_4 &\geqslant 0 \end{aligned}$$

- Note that the constraint matrix

$$\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4] = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Example

- Possible **Basic solutions** are:

1. $\mathbf{B} = [\mathbf{a}_1, \mathbf{a}_2] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

   $\mathbf{x}_B = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \quad \mathbf{x}_N = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

2. $\mathbf{B} = [\mathbf{a}_1, \mathbf{a}_4] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

   $\mathbf{x}_B = \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = \mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}, \quad \mathbf{x}_N = \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

3. $\mathbf{B} = [\mathbf{a}_2, \mathbf{a}_3] = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

   $\mathbf{x}_B = \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \quad \mathbf{x}_N = \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

4. $\mathbf{B} = [\mathbf{a}_2, \mathbf{a}_4] = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

   $\mathbf{x}_B = \begin{bmatrix} x_2 \\ x_4 \end{bmatrix} = \mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \end{bmatrix}, \quad \mathbf{x}_N = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

5. $\mathbf{B} = [\mathbf{a}_3, \mathbf{a}_4] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

   $\mathbf{x}_B = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}, \quad \mathbf{x}_N = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Example

- We have **four basic feasible solutions:**

$$\mathbf{x}_1 = \begin{bmatrix} 3 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 6 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 0 \\ 3 \\ 3 \\ 0 \end{bmatrix}, \quad \mathbf{x}_4 = \begin{bmatrix} 0 \\ 0 \\ 6 \\ 3 \end{bmatrix}$$

- These **basic feasible solutions**, in the (x_1, x_2) space—give rise to the following four points:

$$\begin{bmatrix} 3 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 6 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- These points are precisely the **extreme points** of the feasible region.

Basic Feasible Solutions

- In this example, the possible number of basic feasible solutions is bounded by the number of ways of extracting **two columns out of four columns** to form the basis.
- Therefore the number of basic feasible solutions is less or equal to:

$$\binom{4}{2} = \frac{4!}{2!2!} = 6.$$

- Out of these six possibilities, one point violates the nonnegativity of $\mathbf{B^{-1}b}$. Furthermore, a_1 and a_3, could not have been used to form a basis since $a_1 = a_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ are **linearly dependent**, and hence the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ qualify as a basis.

Basic Feasible Solutions

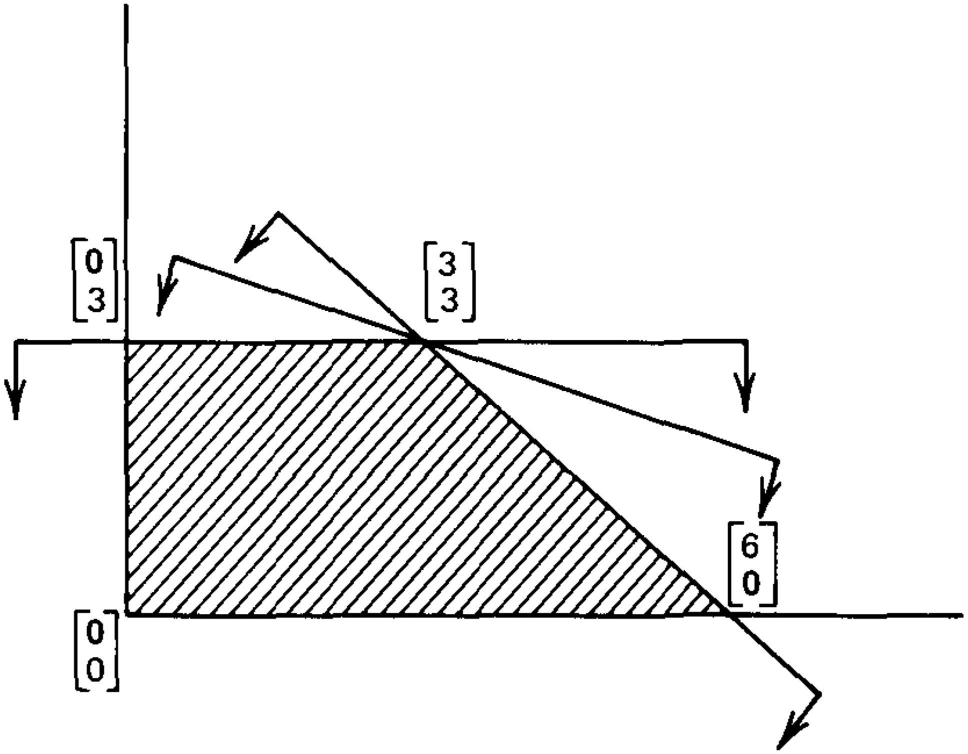
- In general, the number of basic feasible solutions is less than or equal to:

$$\binom{n}{m} = \frac{n!}{m!\,(n-m)!}$$

Example: Degenerate Basic Feasible Solutions

- Consider the following system of inequalities:

$$x_1 + x_2 \leqslant 6$$
$$x_2 \leqslant 3$$
$$x_1 + 2x_2 \leqslant 9$$
$$x_1, \quad x_2 \geqslant 0$$



- The third restriction is **redundant**.

Example: Degenerate Basic Feasible Solutions

- After adding the slack variables, we get

$$
\begin{aligned}
x_1 + x_2 + x_3 \qquad\qquad &= 6 \\
x_2 \qquad + x_4 \qquad &= 3 \\
x_1 + 2x_2 \qquad\qquad + x_5 &= 9 \\
x_1, \quad x_2, \quad x_3, \; x_4, \; x_5 &\geqslant 0
\end{aligned}
$$

- Note that

$$
\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5] = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 1 \end{bmatrix}
$$

Example: Degenerate Basic Feasible Solutions

- Let us consider the basic feasible solution with **B** = [**a**$_1$, **a**$_2$, **a**$_3$]

$$\mathbf{x}_B = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 6 \\ 3 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \\ 9 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}$$

$$\mathbf{x}_N = \begin{bmatrix} x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- Note that this basic feasible solution is degenerate since the basic variable $x_3 = 0$.

Example: Degenerate Basic Feasible Solutions

- Now consider the basic feasible solution with $\mathbf{B} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4]$

$$\mathbf{x}_B = \begin{bmatrix} x_1 \\ x_2 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 6 \\ 3 \\ 9 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \\ 9 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}$$

$$\mathbf{x}_N = \begin{bmatrix} x_3 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- This basic feasible solution gives rise to the same point

Example: Degenerate Basic Feasible Solutions

- It can be also checked that the basic feasible solution with basis $\mathbf{B} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_5]$ is given by

$$\mathbf{x}_B = \begin{bmatrix} x_1 \\ x_2 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} \qquad \mathbf{x}_N = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- All three of the foregoing basic feasible solutions with **different bases** are represented by the **single extreme** point $(x_1, x_2, x_3, x_4, x_5) = (3, 3, 0, 0, 0)$.
- Each of the three basic feasible solutions is degenerate since each contains a basic variable at level zero.

Key to the Simplex Method

Key to the Simplex Method

- Since the number of basic feasible solutions is bounded by $\binom{n}{m}$,
- One may think of simply listing all basic feasible solutions, and picking the one with the minimal objective value.
- This is not satisfactory, for a number of reasons

Key to the Simplex Method

- The reasons:
  - Firstly, the number of basic feasible solutions is large, even for moderate values of m and n.
  - Secondly, this approach does not tell us if the problem has an unbounded solution that may occur if the feasible region is unbounded.
  - Lastly, if the feasible region is empty, we shall discover that the feasible region is empty, only after all possible ways of extracting m columns out of n columns of the matrix **A** fail to produce a basic feasible solution, on the grounds that **B** does not have an inverse, or else $\mathbf{B}^{-1}\mathbf{b} \ngeq \mathbf{0}$

Key to the Simplex Method

- **The key to the simplex method** lies in recognizing the optimality of a given basic feasible solution (extreme point solution) based on local considerations without having to (globally) enumerate all basic feasible solutions.

Key to the Simplex Method

- Consider the following linear programming problem.

$$\begin{aligned} \text{Minimize} \quad & \mathbf{cx} \\ \text{Subject to} \quad & \mathbf{Ax} = \mathbf{b} \\ & \mathbf{x} \geqslant \mathbf{0} \end{aligned}$$

  - where **A** is an $m \times n$ matrix.

- Suppose that we have a basic feasible solution $\begin{pmatrix} \mathbf{B}^{-1}\mathbf{b} \\ \mathbf{0} \end{pmatrix}$ whose objective value z_0 is given by

$$z_0 = \mathbf{c}\begin{pmatrix} \mathbf{B}^{-1}\mathbf{b} \\ \mathbf{0} \end{pmatrix} = (\mathbf{c}_B, \mathbf{c}_N)\begin{pmatrix} \mathbf{B}^{-1}\mathbf{b} \\ \mathbf{0} \end{pmatrix} = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$$

Key to the Simplex Method

- Let $\mathbf{x_B}$ and $\mathbf{x_N}$ denote the set of **basic** and **nonbasic** variables for the given basis.
- Then feasibility requires that $\mathbf{x_B} \geq \mathbf{0}$, $\mathbf{X_N} \geq \mathbf{0}$, and that

$$\mathbf{b} = \mathbf{Ax} = \mathbf{Bx_B} + \mathbf{Nx_N}$$

- Multiplying the last equation by $\mathbf{B^{-1}}$ and rearranging the terms:

$$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N$$

$$= \mathbf{B}^{-1}\mathbf{b} - \sum_{j \in R} \mathbf{B}^{-1}\mathbf{a}_j x_j$$

$$= \bar{\mathbf{b}} - \sum_{j \in R} (\mathrm{y}_j)\mathrm{x}_j$$

  - where **R** is the current set of the indices of the **nonbasic variables**.

Key to the Simplex Method

- letting z denote the objective function at x, we get

$$
\begin{aligned}
z &= \mathbf{c}\mathbf{x} \\
&= \mathbf{c}_B\mathbf{x}_B + \mathbf{c}_N\mathbf{x}_N \\
&= \mathbf{c}_B\left(\mathbf{B}^{-1}\mathbf{b} - \sum_{j\in R}\mathbf{B}^{-1}\mathbf{a}_j x_j\right) + \sum_{j\in R} c_j x_j \\
&= z_0 - \sum_{j\in R}(z_j - c_j)x_j
\end{aligned}
$$

- where $z_j = \mathbf{c}_\mathbf{B}\mathbf{B}^{-1}\,\mathbf{a}_\mathbf{j}$ for each nonbasic variable.

Key to the Simplex Method

- The LP problem may be rewritten as:

$$\text{Minimize} \quad z = z_0 - \sum_{j \in R} (z_j - c_j) x_j$$

$$\text{Subject to} \quad \sum_{j \in R} (\mathrm{y}_j) \mathrm{x}_j + \mathbf{x_B} = \bar{\mathbf{b}}$$

$$x_j \geq 0, j \in R, \quad \text{and} \quad \mathbf{x_B} \geq 0$$

Key to the Simplex Method

- Since we are to minimize z, whenever $z_j - c_j > 0$, it would be to our advantage to increase x_j (from its current level of zero).
- If $(z_j - c_j) \leq 0$ for all $j \in R$, then
  - the current feasible solution is optimal.
  - $z \geq z_0$ for any feasible solution, and for the current (basic) feasible solution
  - $z = z_0$ since $x_j = 0$ for all $j \in R$

Algebra of the Simplex Method

Algebra of the Simplex Method

- If $(z_j - c_j) \le 0$ for all $j \in R$, then $x_j = 0$, $j \in R$ and $x_B = \bar{\mathbf{b}}$ is optimal for LP.
- If $z_k - c_k > 0$, and it is the most positive of all $z_j - c_j$, it would be to our benefit to increase x_k as much as possible.
- As x_k is increased, the current basic variables must be modified according to: $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{a}_k x_k = \bar{\mathbf{b}} - \mathbf{y}_k x_k$

$$\begin{bmatrix} x_{B_1} \\ x_{B_2} \\ \vdots \\ x_{B_r} \\ \vdots \\ x_{B_m} \end{bmatrix} = \begin{bmatrix} \bar{b}_1 \\ \bar{b}_2 \\ \vdots \\ \bar{b}_r \\ \vdots \\ \bar{b}_m \end{bmatrix} - \begin{bmatrix} y_{1k} \\ y_{2k} \\ \vdots \\ y_{rk} \\ \vdots \\ y_{mk} \end{bmatrix} x_k$$

Algebra of the Simplex Method

- If $y_{ik} < 0$, then x_{Bi} increases as x_k increases and so x_{Bi} continues to be nonnegative.
- If $y_{ik} = 0$, then x_{Bi} is not changed as x_k increases and so x_{Bi} continues to be nonnegative.
- If $y_{ik} > 0$, then x_{Bi} will decrease as x_k increases.
  - In order to satisfy nonnegativity, x_k is increased until the first point at which a basic variable x_{Br} drops to zero.
- It is then clear that the first basic variable dropping to zero corresponds to the minimum of $\bar{b}_i / y_{ik}$ for $y_{ik} > 0$.

Algebra of the Simplex Method

- More precisely:

$$x_k = \frac{\bar{b}_r}{y_{rk}} = \underset{1 \leq i \leq m}{\text{Minimum}} \left\{ \frac{\bar{b}_i}{y_{ik}} \; : y_{ik} > 0 \right\}$$

- In the absence of degeneracy $\bar{b}_r > 0$, and $x_k = \bar{b}_r / y_{rk} > 0$.
- As x_k increases from level 0 to $\bar{b}_r / y_{rk}$, a new feasible solution is obtained.

Algebra of the Simplex Method

- Substituting $x_k = \bar{b}_r / y_{rk}$ in this Equation

$$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{a}_k x_k = \bar{\mathbf{b}} - \mathbf{y}_k x_k$$

- gives:

$$x_{B_i} = \bar{b}_i - \frac{y_{ik}}{y_{rk}} \bar{b}_r \qquad i = 1, 2, \ldots, m$$

$$x_k = \frac{\bar{b}_r}{y_{rk}}$$

all other x_j's are zero

Algebra of the Simplex Method

- The corresponding columns are

  $\mathbf{a}_{B1}, \mathbf{a}_{B2}, \ldots, \mathbf{a}_{r-1}, \mathbf{a}_{k}, \mathbf{a}_{r+1}, \ldots, \mathbf{a}_{m}$.

- If $\mathbf{a}_{B1}, \mathbf{a}_{B2}, \ldots, \mathbf{a}_{m}$ are linearly independent, and if $\mathbf{a}_{k}$ replaces $\mathbf{a}_{Br}$, then the new columns are linearly independent if and only if $y_{rk} \neq 0$.

Example

$$\text{Minimize } x_1 + x_2$$

$$\text{Subject to } x_1 + 2x_2 \leqslant 4$$

$$x_2 \leqslant 1$$

$$x_1, \quad x_2 \geqslant 0$$

- Introduce the slack variables x_3 and x_4 to put the problem in a standard form. This leads to the following constraint matrix **A**:

$$\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4] = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Example

- Consider the basic feasible solution corresponding to $\mathbf{B} = [\mathbf{a}_1, \mathbf{a}_2]$. In other words, x_1 and x_2 are the basic variables while x_3 and x_4 are the nonbasic variables.
- First compute

$$\mathbf{B}^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}, \qquad \mathbf{c}_B\mathbf{B}^{-1} = (1,\ 1)\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = (1, -1).$$

- Hence

$$\mathbf{y}_3 = \mathbf{B}^{-1}\mathbf{a}_3 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$\mathbf{y}_4 = \mathbf{B}^{-1}\mathbf{a}_4 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix},$$

$$\bar{\mathbf{b}} = \mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Example

- Also

$$z_0 = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b} = (1,-1)\begin{pmatrix}4\\1\end{pmatrix} = 3,$$

$$z_3 - c_3 = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{a}_3 - c_3 = (1,-1)\begin{bmatrix}1\\0\end{bmatrix} - 0 = 1,$$

$$z_4 - c_4 = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{a}_4 - c_4 = (1,-1)\begin{bmatrix}0\\1\end{bmatrix} - 0 = -1.$$

- The required representation of the problem is

$$\begin{array}{llllllll}
\text{Minimize} & 3 - x_3 & + & x_4 & & & & \\
\text{subject to} & x_3 & - & 2x_4 & + & x_1 & & = 2 \\
 & & & x_4 & & & + \; x_2 & = 1 \\
 & x_1, & & x_2, & & x_3, & x_4 & \geq 0.
\end{array}$$

Example

- Since $z_3 - c_3 > 0$, then the objective improves by increasing x_3
- The modified solution is given by

$$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{a}_3 x_3$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} x_3$$

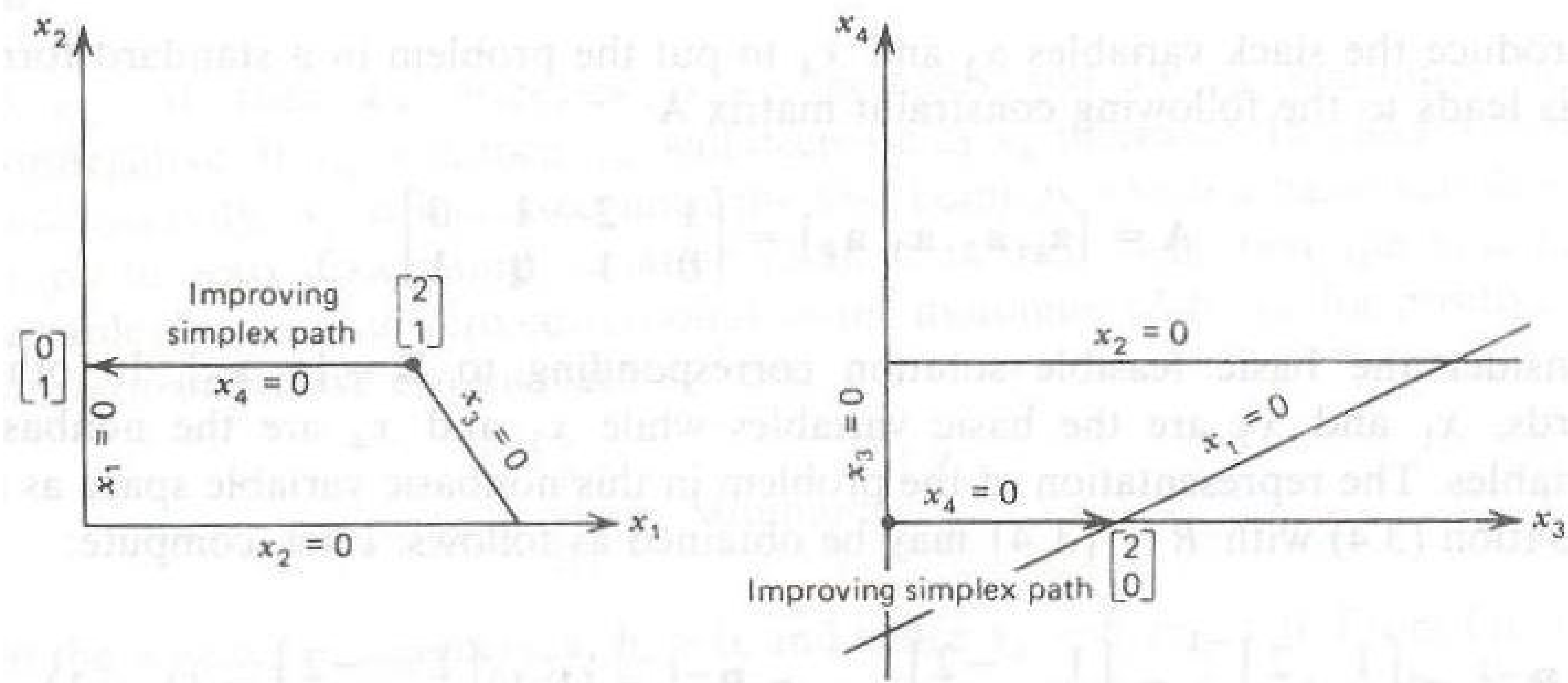
- The maximum value of x_3 is 2 (any larger value of x_3 will force x_1 to be negative). Therefore the new basic feasible solution is

$$(x_1, x_2, x_3, x_4) = (0, 1, 2, 0)$$

- Here x_3 enters the basis and x_1 leaves the basis. Note that the new point has an objective value equal to 1, which is an improvement over the previous objective value of 3. The improvement is precisely $(z_3 - c_3)\, x_3 = 2$.

Example

- The feasible region of the problem in both the original (x_1, x_2) space as well as in the current (x_3, x_4) space.



Interpretation of Entering the Basis

- Recall that

$$z = \mathbf{c}_B\bar{\mathbf{b}} - (z_k - c_k)x_k$$

  – where
$$z_k = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{a}_k = \mathbf{c}_B\mathbf{y}_k = \sum_{i=1}^{m} c_{B_i}y_{ik}$$

  – where c_{Bi} is the cost of the i th basic variable.

- If x_k is raised from zero level, while the other nonbasic variables **are kept at zero** level,
  – then the basic variables $x_{B1}, x_{B2}, \ldots, x_{Bm}$ must be modified according to

$$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{a}_k x_k = \bar{\mathbf{b}} - \mathbf{y}_k x_k$$

Interpretation of Entering the Basis

- if x_k is increased by **1 unit**:
  - if $y_{ik} > 0$, the i th basic variable will be decreased by y_{ik}
  - if $y_{ik} < 0$, the i th basic variable will be increased by y_{ik}
  - if $y_{ik} = 0$, the i th basic variable will not be changed
- z_k which is $\sum_{i=1}^{m} c_{B_i} y_{ik}$, is the cost saving that results from the modification of the basic variables, as a result of increasing x_k by 1 unit
- c_k is the cost of increasing x_k itself by 1 unit
- $z_k - c_k$ is the saving minus the cost of increasing x_k by 1 unit

Interpretation of Entering the Basis

- if z_k - $c_k > 0$,
  - it will be to our advantage to increase x_k
  - For each unit of x_k, the cost will be reduced by an amount z_k - c_k and hence it will be to our advantage to increase x_k as much as possible.
- if z_k - $c_k < 0$,
  - then by increasing x_k, the net saving is negative, and this action will result in a larger cost.
  - So this action is prohibited.
- If z_k - $c_k = 0$,
  - then increasing x_k will lead to a different solution, with the same cost.

Interpretation of Leaving the Basis

- Suppose that we decided to increase a nonbasic variable x_k with a positive z_k - c_k
- the larger the value of x_k, the smaller is the objective z.
- As x_k is increased, the basic variables are modified according to

$$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{a}_k x_k = \bar{\mathbf{b}} - \mathbf{y}_k x_k$$

- If the vector y_k has any positive component(s), then the corresponding basic variable(s) is decreased as x_k is increased.

Interpretation of Leaving the Basis

- Therefore the nonbasic variable x_k cannot be indefinitely increased, because otherwise the nonnegativity of the basic variables will be violated.
- The first basic variable x_{Br} that drops to zero is called the **blocking variable** because it blocked further increase of x_k.
- Thus x_k enters the basis and x_{Br} leaves the basis.

Termination:
Optimality and Unboundedness

Termination Optimality and Unboundedness

- Different cases for terminations of simplex methods:
  - Termination with an Optimal Solution
  - Unique and Alternative Optimal Solutions
  - Unboundedness

Termination with an Optimal Solution

- Suppose that **x*** is a basic feasible solution with basis B
- Let z* denote the objective of **x***
- Suppose z_j - $c_j \leq 0$, for all nonbasic variables
  - In this case no nonbasic variable is eligible for entering the basis.
- Fro the following equation:

$$z = z_0 - \sum_{j \in R} (z_j - c_j) x_j$$

- **x*** is an unique optimal basic feasible solution.

Unique and Alternative Optimal Solutions

- Consider the case where $z_j - c_j \leq 0$ for all nonbasic components,
- Bu $z_k - c_k = 0$ for at least one nonbasic variable x_k.
- If x_k is increased, we get (in the absence of degeneracy) points that are different from **x*** but have the same objective value.
- If x_k is increased until it is blocked by a basic variable, we get an alternative optimal basic feasible solution.
- The process of increasing x_k from level zero until it is blocked generates an **infinite number** of alternative optimal solutions.

Unboundedness

- Suppose that we have a basic feasible solution of the system $\mathbf{Ax} = \mathbf{b}$, $\mathbf{x} > 0$, with objective value z_0.
- Let us consider the case when we find a nonbasic variable x_k with $z_k - c_k > 0$ and $\mathbf{y_k} \leq 0$.
- This variable is eligible to enter the basis since increasing it will improve the objective function.
- It is to our benefit to increase x_k indefinitely, which will make z go to -∞.
- Based on the following equation:

$$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{a}_k x_k = \bar{\mathbf{b}} - \mathbf{y}_k x_k.$$

The Simplex Algorithm

The Simplex Algorithm

- Given a basic feasible solution, we can either improve it if $z_k - c_k > 0$ for some nonbasic variable x_k, or stop with an optimal point if $z_j - c_j \leq 0$ for all nonbasic variables.
- If $z_k - c_k > 0$, and the vector $\mathbf{y_k}$ contains at least one positive component, then the increase in x_k will be blocked by one of the current basic variables, which drops to zero and leaves the basis.
- On the other hand, if $z_k - c_k > 0$ and $\mathbf{y_k} \leq 0$, then x_k can be increased indefinitely, and the optimal solution is **unbounded** and has value $-\infty$.

The Simplex Algorithm

- We now give a summary of the simplex method for solving the following linear programming problem.

$$\text{Minimize } \mathbf{cx}$$

$$\text{Subject to } \mathbf{Ax} = \mathbf{b}$$

$$\mathbf{x} \geqslant \mathbf{0}$$

  – where **A** is an $m \times n$ matrix with rank m.

- **Initialization Step of Simplex Algorithm**
  – Choose a starting basic feasible solution with basis **B.**

The Simplex Algorithm: Main Step

MAIN STEP:

Step 1:

- Solve the system $\mathbf{B}\mathbf{x}_\mathbf{B} = \mathbf{b}$
- Let $\mathbf{x}_\mathbf{B} = \bar{\mathbf{b}}$, $\mathbf{x}_\mathrm{N} = \mathbf{0}$, and $z = \mathbf{c}_\mathbf{B}\mathbf{x}_\mathbf{B}$.

The Simplex Algorithm: Main Step

Step 2:

- Solve the system $\mathbf{wB} = \mathbf{c_B}$, with unique solution $\mathbf{w} = \mathbf{c_B B^{-1}}$.
  - The vector of w is **simplex multipliers**
- Calculate z_j - c_j = $\mathbf{wa_j}$ - c_j for all nonbasic variables.
  - This is known as the **pricing operation**.
- Let $z_k - c_k = \underset{j \in R}{\text{Maximum}}\ z_j - c_j$
  - where R is the current set of indices associated with the **nonbasic variables**.
- If $z_k - c_k \leq 0$, then stop with the current basic feasible solution as an **optimal solution**.
- Otherwise go to **step 3** with x_k as the entering variable.

The Simplex Algorithm: Main Step

Step 3:

- Solve the system $\mathbf{B}\mathbf{y}_k = \mathbf{a}_k$,with unique solution $\mathbf{y}_k = \mathbf{B}^{-1}\mathbf{a}_k$.
- If $\mathbf{y}_k \leq \mathbf{0}$, then stop with the conclusion that the optimal solution is **unbounded**
- If **NOT** $\mathbf{y}_k \leq 0$, go to **step 4**.

The Simplex Algorithm: Main Step

Step 4:

- Let x_k enters the basis and the blocking variable x_{B_r} leaves the basis,
- where the index r is determined by the following **minimum ratio test**:

$$\frac{\bar{b}_r}{y_{rk}} = \underset{1 \leq i \leq m}{\text{Minimum}} \left\{ \frac{\bar{b}_i}{y_{ik}} \ : y_{ik} > 0 \right\}$$

- Update the basis **B** where $\mathbf{a}_\mathbf{k}$ replaces $\mathbf{a}_{\mathbf{Br}}$, the index set **R** and go to **step 1**.

Modification for a Maximization Problem

- A maximization problem can be transformed into a minimization problem by multiplying the objective coefficients by - 1.
- A maximization problem can also be handled directly as follows.
- Let $z_k - c_k$ instead be the **minimum** $z_j - c_j$ for j nonbasic; the stopping criterion is that $z_k - c_k \geq 0$. Otherwise, the steps are as above.

The Simplex Method In Tableau Format

The Simplex Method In Tableau Format

- Suppose that we have a starting basic feasible solution **x** with basis **B**.
- The linear programming problem can be represented as follows:

$$
\begin{aligned}
&\text{Minimize} && z & \\
&\text{Subject to} && z - \mathbf{c}_B \mathbf{x}_B - \mathbf{c}_N \mathbf{x}_N = 0 & \quad (1)\\
& && \mathbf{B}\mathbf{x}_B + \mathbf{N}\mathbf{x}_N = \mathbf{b} & \quad (2)\\
& && \mathbf{x}_B, \quad \mathbf{x}_N \geqslant \mathbf{0} &
\end{aligned}
$$

The Simplex Method In Tableau Format

- From Equation (2) we have:

$$\mathbf{x}_B + \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N = \mathbf{B}^{-1}\mathbf{b} \quad \textbf{(3)}$$

- Multiplying (3) by $\mathbf{c_B}$ and adding to Equation (1), we get

$$z + 0\mathbf{x}_B + (\mathbf{c}_B\mathbf{B}^{-1}\mathbf{N} - \mathbf{c}_N)\mathbf{x}_N = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b} \quad \textbf{(4)}$$

- Currently $\mathbf{x_N} = \mathbf{0}$, and from Equations (3) and (4) we get $\mathbf{x_B} = \mathbf{B^{-1}b}$ and $z = \mathbf{c_B B^{-1} b}$.
- Also, from (3) and (4) we can conveniently represent the current basic feasible solution in the following tableau.

	z	$\mathbf{x}_B$	$\mathbf{x}_N$	RHS	
z	1	$\mathbf{0}$	$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{N} - \mathbf{c}_N$	$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$	Row 0
$\mathbf{x}_B$	$\mathbf{0}$	$\mathbf{I}$	$\mathbf{B}^{-1}\mathbf{N}$	$\mathbf{B}^{-1}\mathbf{b}$	Rows 1 through m

The Simplex Method In Tableau Format

- Here we think of z as a (basic) variable to be minimized.
- The objective row will be referred to as row 0 and the remaining rows are rows 1 through m.
- The **right-hand-side column (RHS)** will denote the values of the basic variables (including the objective function).
- The basic variables are identified on the far left column.
- Actually the cost row gives us $\mathbf{c_B B^{-1} N - c_N}$, which consists of the z_j - c_j for the nonbasic variables.
- So row zero will tell us if we are at the optimal solution (if each $z_j - c_j \leq 0$), and which nonbasic variable to increase otherwise.

Pivoting

- If x_k enters the basis and x_{Br} leaves the basis, then pivoting on y_{rk} can be stated as follows.

1. Divide row r by y_{rk}.
2. For $i = 1, 2, . . . ,$ m and i # r, update the i th row by adding to it $-y_{ik}$ times the new r th row.
3. Update row zero by adding to it $c_k - z_k$ times the new r th row. The two tableaux represent the situation immediately before and after pivoting.

Before and after pivoting

	z	x_{B_1}		x_{B_r}		x_{B_m}		x_j		x_k		RHS
z	1	0	$\cdots$	0	$\cdots$	0	$\cdots$	$z_j - c_j$	$\cdots$	$z_k - c_k$	$\cdots$	$\mathbf{c}_B\bar{\mathbf{b}}$
x_{B_1}	0	1	$\cdots$	0	$\cdots$	0	$\cdots$	y_{1j}	$\cdots$	y_{1k}	$\cdots$	$\bar{b}_1$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$		$\vdots$		$\vdots$
x_{B_r}	0	0	$\cdots$	1	$\cdots$	0	$\cdots$	y_{rj}	$\cdots$	$\boxed{y_{rk}}$	$\cdots$	$\bar{b}_r$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$		$\vdots$		$\vdots$
x_{B_m}	0	0	$\cdots$	0	$\cdots$	1	$\cdots$	y_{mj}	$\cdots$	y_{mk}	$\cdots$	$\bar{b}_m$

	z	x_{B_1}		x_{B_r}		x_{B_m}				x_k		RHS
z	1	0	$\cdots$	$\dfrac{c_k - z_k}{y_{rk}}$	$\cdots$	0	$\cdots$	$(z_j - c_j) - \dfrac{y_{rj}}{y_{rk}}(z_k - c_k)$	$\cdots$	0	$\cdots$	$\mathbf{c}_B\bar{\mathbf{b}} - (z_k - c_k)\dfrac{\bar{b}_r}{y_{rk}}$
x_{B_1}	0	1	$\cdots$	$\dfrac{-y_{1k}}{y_{rk}}$	$\cdots$	0	$\cdots$	$y_{1j} - \dfrac{y_{rj}}{y_{rk}} y_{1k}$	$\cdots$	0	$\cdots$	$\bar{b}_1 - \dfrac{y_{1k}}{y_{rk}} \bar{b}_r$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$		$\vdots$		$\vdots$
x_k	0	0	$\cdots$	$\dfrac{1}{y_{rk}}$	$\cdots$	0	$\cdots$	$\dfrac{y_{rj}}{y_{rk}}$	$\cdots$	1	$\cdots$	$\dfrac{\bar{b}_r}{y_{rk}}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$		$\vdots$		$\vdots$
x_{B_m}	0	0	$\cdots$	$\dfrac{-y_{mk}}{y_{rk}}$	$\cdots$	1	$\cdots$	$y_{mj} - \dfrac{y_{rj}}{y_{rk}} y_{mk}$	$\cdots$	0	$\cdots$	$\bar{b}_m - \dfrac{y_{mk}}{y_{1k}} \bar{b}_r$

The Simplex Method In Tableau Format

- Let us examine the implications of the pivoting operation.

1. The variable x_k entered the basis and x_{Br} left the basis. This is illustrated on the left-hand side of the tableau by replacing x_{Br} with x_k. For the purpose of the following iteration, the new x_{Br} is now x_k.
2. The right-hand side of the tableau gives the current values of the basic variables. The nonbasic variables are kept zero.
3. Pivoting results in a new tableau that gives the new $\mathbf{B}^{-1}\mathbf{N}$ under the nonbasic variables, an updated set of $z_j - c_j$ for the new nonbasic variables, and the values of the new basic variables and objective function.

The Simplex Method In Tableau Format

- Example

$$\text{Minimize} \quad x_1 + x_2 - 4x_3$$

$$\begin{aligned} \text{Subject to} \quad & x_1 + x_2 + 2x_3 \leqslant 9 \\ & x_1 + x_2 - x_3 \leqslant 2 \\ & -x_1 + x_2 + x_3 \leqslant 4 \\ & x_1, \ x_2, \ x_3 \geqslant 0 \end{aligned}$$

- Introduce the nonnegative slack variables x_4, x_5, and x_6.

$$\text{Minimize} \quad x_1 + x_2 - 4x_3 + 0x_4 + 0x_5 + 0x_6$$

$$\begin{aligned} \text{Subject to} \quad & x_1 + x_2 + 2x_3 + x_4 = 9 \\ & x_1 + x_2 - x_3 + x_5 = 2 \\ & -x_1 + x_2 + x_3 + x_6 = 4 \\ & x_1, \ x_2, \ x_3, \ x_4, \ x_5, \ x_6 \geqslant 0 \end{aligned}$$

The Simplex Method In Tableau Format

- Since $\mathbf{b} \geq \mathbf{0}$, then we can choose our initial basis as $\mathbf{B} = [\mathbf{a}_4, \mathbf{a}_5, \mathbf{a}_6] = I_3$, and we indeed have $\mathbf{B}^{-1}\mathbf{b} \geq 0$. This gives the following initial tableau.

Iteration 1

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	−1	−1	4	0	0	0	0
x_4	0	1	1	2	1	0	0	9
x_5	0	1	1	−1	0	1	0	2
x_6	0	−1	1	(1)	0	0	1	4

Iteration 2

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	3	−5	0	0	0	−4	−16
x_4	0	(3)	−1	0	1	0	−2	1
x_5	0	0	2	0	0	1	1	6
x_3	0	−1	1	1	0	0	1	4

The Simplex Method In Tableau Format

Iteration 3

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	0	-4	0	-1	0	-2	-17
x_1	0	1	$-\frac{1}{3}$	0	$\frac{1}{3}$	0	$-\frac{2}{3}$	$\frac{1}{3}$
x_5	0	0	2	0	0	1	1	6
x_3	0	0	$\frac{2}{3}$	1	$\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{13}{3}$

- This is the optimal tableau since $z_j - c_j \leq 0$ for all nonbasic variables. The optimal solution is given by

$$x_1 = \frac{1}{3},\ x_2 = 0,\ x_3 = \frac{13}{3}$$

$$z = -17$$

The Simplex Method In Tableau Format

- Note that the current optimal basis consists of the columns **a1**, **a5**, and **a3** namely

$$\mathbf{B} = [\mathbf{a}_1, \mathbf{a}_5, \mathbf{a}_3] = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\mathbf{B}^{-1} = \begin{bmatrix} \frac{1}{3} & 0 & \frac{-2}{3} \\ 0 & 1 & 1 \\ \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix}$$

References

References

- M.S. Bazaraa, J.J. Jarvis, H.D. Sherali, **Linear Programming and Network Flows**, Wiley, 1990. (Chapter 3)

The End