In the name of God

Part 1. The Review of Linear Programming

1.3. Starting Solution and Convergence

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Outline

- The Initial Basic Feasible Solution
- The Two-Phase Method
- The Big-M Method
- Degeneracy and Cycling
- References

The Initial Basic Feasible Solution

The Initial Basic Feasible Solution

- In order to initialize the simplex method, a basis $\mathbf{B}$ with $\bar{\mathbf{b}} = \mathbf{B}^{-1}\mathbf{b} \geq \mathbf{0}$ must be available.
- We shall show that the simplex method can always be initiated with a very simple basis, namely the identity.

Easy Case

- Suppose that the constraints are of the form:

$$\mathbf{Ax} \le \mathbf{b},\ \mathbf{x} \ge \mathbf{0}$$

  - where A is an m x n matrix and **b** is an m nonnegative vector.

- By adding the slack vector $\mathbf{x}_s$, the constraints are put in the following **standard form**:

$$\mathbf{Ax} + \mathbf{x}_s = \mathbf{b},\ \mathbf{x} \ge \mathbf{0},\ \mathbf{x}_s \ge \mathbf{0}$$

- The new m x $(m + n)$ constraint matrix $(\mathbf{A}, \mathbf{I})$ has rank m, and **a basic feasible solution** of this system is at hand, by letting $\mathbf{x}_s = \mathbf{b}$ be the basic vector, and $\mathbf{x} = \mathbf{0}$ be the nonbasic vector.

Some Bad Cases

- **First case**:
  - The constraints are of the form $\mathbf{Ax} \le \mathbf{b}$, $\mathbf{x} \ge \mathbf{0}$ but the vector $\mathbf{b}$ is **not nonnegative**.
  - After introducing the slack vector $\mathbf{x}_s$, $\mathbf{x}_s = \mathbf{b}$ violates the nonnegativity requirement.
- **Second case**:
  - the constraints are of the form $\mathbf{Ax} \ge \mathbf{b}$, $\mathbf{x} \ge \mathbf{0}$, where $\mathbf{b} \ge \mathbf{0}$.
  - After subtracting the slack vector $\mathbf{x}_s$, we get

$$\mathbf{Ax} - \mathbf{x}_s = \mathbf{b},\ \mathbf{x} \ge \mathbf{0},\ \mathbf{x}_s \ge \mathbf{0}$$

  - There is no obvious way of picking a basis $\mathbf{B}$ from the matrix $(\mathbf{A}, \mathbf{-I})$ with $\bar{\mathbf{b}} = \mathbf{B}^{-1}\mathbf{b} \ge \mathbf{0}$.

Artificial Variables

- We can use **artificial variables** to get a starting basic feasible solution.
- We change the restrictions by adding an artificial vector $\mathbf{x_a}$ leading to the system:

$$\mathbf{Ax} + \mathbf{x_a} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0},\ \mathbf{x_a} \geq \mathbf{0}.$$

- We forced an **identity matrix** corresponding to the artificial vector.
- This gives an immediate basic feasible solution of the new system, namely $\mathbf{x_a} = \mathbf{b}$ and $\mathbf{x} = \mathbf{0}$, and the simplex method can be applied.

Artificial Variables

- We have changed the problem.
- In order to get back to our original problem, we must force these artificial variables to zero, because $\mathbf{Ax} = \mathbf{b}$ if and only if $\mathbf{Ax} + \mathbf{x_a} = \mathbf{b}$ with $\mathbf{x_a} = \mathbf{0}$.
- Artificial variables are only a tool to get the simplex method started.

Slack Variables vs. Artificial Variables

- **Slack variable**
  - are introduced to put the problem in equality form
  - the slack variable can be positive, which means that the inequality holds as a strict inequality.
- **Artificial variables**
  - are introduced to facilitate the initiation of the simplex method.
  - These variables must eventually drop to zero in order to attain feasibility in the original problem.

Methods to Eliminate the Artificial Variables

- Two methods that can be used to eliminate the artificial variables:
  - The two-phase method
  - The big-M method

The Two-Phase Method

The Two-Phase Method

- **Phase I**
  - We reduce artificial variables to value zero
  - As the artificial variables drop to zero, they leave the basis, and legitimate variables enter instead.
  - If after solving the problem we have a positive artificial variable, then the original problem has no feasible solution
- **Phase II**
  - minimizes the original objective function starting with the basic feasible solution obtained at the end of the phase I.

The Two-Phase Method

- **Phase I**:
  - Solve the following linear program starting with the basic feasible solution $\mathbf{x} = \mathbf{0}$ and $\mathbf{x_a} = \mathbf{b}$.

$$\begin{aligned} \text{Minimize} \quad & \mathbf{1}\mathbf{x}_a \\ \text{Subject to} \quad & \mathbf{A}\mathbf{x} + \mathbf{x}_a = \mathbf{b} \\ & \mathbf{x},\ \mathbf{x}_a \geqslant \mathbf{0} \end{aligned}$$

  - If at optimality $\mathbf{x_a} \neq \mathbf{0}$, then stop; the original problem has no feasible solutions.
  - Otherwise let the basic and nonbasic legitimate variables be $\mathbf{x_B}$ and $\mathbf{x_N}$.
    - We are assuming that all artificial variables left the basis.
  - **Go to phase II**.

The Two-Phase Method

- **Phase II**:
  - Solve the following linear program starting with the basic feasible solution $\mathbf{x_B} = \mathbf{B^{-1}b}$ and $\mathbf{x_N} = \mathbf{0}$.

$$
\begin{array}{ll}
\text{Minimize} & \mathbf{c}_B\mathbf{x}_B + \mathbf{c}_N\mathbf{x}_N \\
\text{Subject to} & \mathbf{x}_B + \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N = \mathbf{B}^{-1}\mathbf{b} \\
 & \mathbf{x}_B,\ \mathbf{x}_N \geqslant \mathbf{0}
\end{array}
$$

  - The optimal solution of the original problem is given by the optimal solution of this problem.

The Big-M Method

The Big-M Method

- The two-phase method is one way to get rid of the artificials.
- However, during phase I of the two-phase method the original cost coefficients are essentially ignored.
- Phase I of the two-phase method seeks any basic feasible solution, not necessarily a good one.
- Another possibility for eliminating the artificial variables is the **big-M method**.

The Big-M Method

- The original problem P and the big-M problem P(M) are stated below, where the vector $\mathbf{b} \geq \mathbf{0}$

$$
\begin{array}{llll}
\textit{Problem } P: & \text{Minimize} & \mathbf{c}\,\mathbf{x} & \\
& \text{Subject to} & \mathbf{A}\mathbf{x} = \mathbf{b} & \\
& & \quad \mathbf{x} \geqslant \mathbf{0} & \\
\textit{Problem } P(M): & \text{Minimize} & \mathbf{c}\,\mathbf{x} + M\,\mathbf{1}\mathbf{x}_a & \\
& \text{Subject to} & \mathbf{A}\mathbf{x} + \mathbf{x}_a = \mathbf{b} & \\
& & \quad \mathbf{x}, \quad \mathbf{x}_a \geqslant \mathbf{0} &
\end{array}
$$

- The starting basic feasible solution is given by

$$\mathbf{x}_{\mathbf{a}} = \mathbf{b} \text{ and } \mathbf{x} = \mathbf{0}$$

The Big-M Method

- The term $\mathbf{M\,1x_a}$ can be interpreted as a penalty to be paid by any solution with $\mathbf{x_a} \neq \mathbf{0}$.
- The simplex method will try to get the artificial variables out of the basis, and then continue to find the optimal solution of the original problem.

The Big-M Method

- Analysis of the Big-M Method
  - Case A: P(M) has an finite optimal solution
    - Subcase A1: ($\mathbf{x}^*$, $\mathbf{0}$) is an optimal solution of P(M) and $\mathbf{x_a} = \mathbf{0}$
    - Subcase A2: ($\mathbf{x}^*$, $\mathbf{x}^*_\mathbf{a}$) is an optimal solution of P(M) and $\mathbf{x}^*_\mathbf{a} \neq \mathbf{0}$
  - Case B: P(M) has an unbounded optimal solution value, that is, $z \rightarrow -\infty$.
    - Subcase B1: $z_k - c_k = \text{Maximum}(z_i - c_i) > 0$, $\mathbf{y_k} \leq \mathbf{0}$, and $\mathbf{x_a} = \mathbf{0}$
    - Subcase B2: $z_k - c_k = \text{Maximum}(z_i - c_i) > 0$, $\mathbf{y_k} \leq \mathbf{0}$, and $\mathbf{x_a} \neq \mathbf{0}$

The Big-M Method

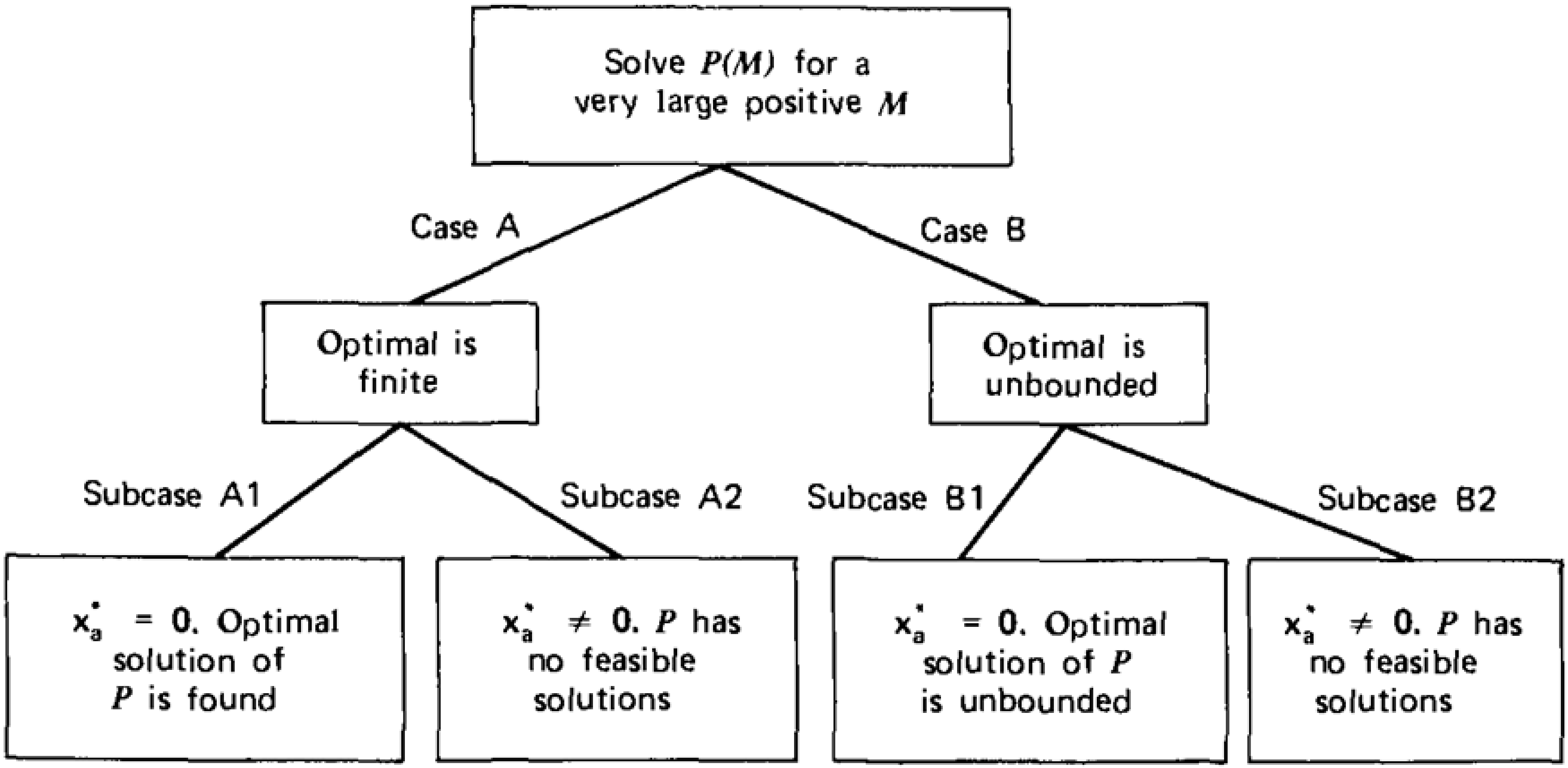
- **Case A:** P(M) has an finite optimal solution
- **Subcase A1:** ($\mathbf{x}^*$, $\mathbf{0}$) is an optimal solution of P(M) and $\mathbf{x_a} = \mathbf{0}$
  - In this case $\mathbf{x}^*$ is an optimal solution to problem P.
- **Subcase A2:** ($\mathbf{x}^*$, $\mathbf{x}^*_\mathbf{a}$) is an optimal solution of P(M) and $\mathbf{x}^*_\mathbf{a} \neq \mathbf{0}$
  - There is no feasible solution of P.

The Big-M Method

- **Case B:** P(M) has an unbounded optimal solution value
  - $z_k - c_k = \text{Maximum}(z_i - c_i) > 0$, $\mathbf{y_k} \leq \mathbf{0}$, that is, z $\rightarrow$ -∞.
- **Subcase B1:** $z_k - c_k = \text{Maximum}(z_i - c_i) > 0$, $\mathbf{y_k} \leq \mathbf{0}$, and $\mathbf{x_a} = \mathbf{0}$
  - The problem P has an unbounded optimal solution.
- **Subcase B2:** $z_k - c_k = \text{Maximum}(z_i - c_i) > 0$, $\mathbf{y_k} \leq \mathbf{0}$, and $\mathbf{x_a} \neq \mathbf{0}$
  - There could be no feasible solution of the original problem.

The Big-M Method

- Analysis of the big-M method



Degeneracy and Cycling

Degeneracy and Cycling

- Suppose that
  - we have a basic feasible solution with basis **B**.
  - we have a nonbasic variable x_k with z_k - $c_k > 0$ (for a minimization problem).
- Therefore x_k enters the basis and x_{Br} leaves the basis, where the index r is determined as follows:

$$\frac{\bar{b}_r}{y_{rk}} = \underset{1 \leq i \leq m}{\text{Minimum}} \left\{ \frac{\bar{b}_i}{y_{ik}} : y_{ik} > 0 \right\}$$

  - where $\bar{\mathbf{b}} = \mathbf{B}^{-1}\mathbf{b}$ and $\mathbf{y}_k = \mathbf{B}^{-1}\mathbf{a}_k$.
  - Column $\mathbf{a}_k$ enters the basis and $\mathbf{a}_{Br}$ leaves the basis.

Degeneracy and Cycling

- The basic feasible solutions before and after pivoting are given by the following.

Before pivoting	*After pivoting*
$\mathbf{x}_B = \bar{\mathbf{b}}$	$\mathbf{x}_B = \bar{\mathbf{b}} - \dfrac{\bar{b}_r}{y_{rk}} \mathbf{y}_k$
$x_k = 0$	$x_k = \dfrac{\bar{b}_r}{y_{rk}}$
all other x_j's $= 0$	all other x_j's $= 0$

- The difference in the objective function before and after pivoting is given by $(\bar{b}_r / y_{rk})(z_k - c_k)$

Degeneracy and Cycling

- In the presence of degeneracy $\bar{b}_r = 0$.
- In this case the objective function remains constant.
- It is evident that we have the same extreme point before and after pivoting, represented by different bases
- As the process is repeated, it is conceivable that another degenerate pivot is taken
- It is therefore possible, though highly unlikely, that we may stay at a nonoptimal extreme point, and pivot through a sequence of bases $\mathbf{B_1}, \mathbf{B_2}, \ldots, \mathbf{B_t}$, where $\mathbf{B_t} = \mathbf{B_1}$.

Degeneracy and Cycling

- **Cycling problem**
  - If the same sequence of pivots is used over and over again, we shall **cycle** forever among the bases $\mathbf{B_1}$, $\mathbf{B_2}$, . . . , $\mathbf{B_t} = \mathbf{B_1}$, without reaching the optimal solution.

Degeneracy and Cycling

- Two rules that prevent cycling:
  - Lexicographic rule for selecting existing variables
  - Bland’s rule for selecting entering and leaving variables

Lexicographic rule

- Suppose that
  - we have a basic feasible solution with basis **B**.
  - we have a nonbasic variable x_k with $0 < z_k - c_k$ = Maximum $z_i - c_i$ (for a minimization problem).
- The index r of the variable x_{Br} leaving the basis is determined as follows.

$$I_0 = \left\{ r : \frac{\bar{b}_r}{y_{rk}} = \underset{1 \leqslant i \leqslant m}{\text{Minimum}} \left\{ \frac{\bar{b}_i}{y_{ik}} : y_{ik} > 0 \right\} \right\}$$

- If I_0 is a singleton, namely $I_0 = \{r\}$, then x_{Br} leaves the basis.

Lexicographic rule

- Otherwise form I_1 as follows:

$$I_1 = \left\{ r : \frac{y_{r1}}{y_{rk}} = \underset{i \in I_0}{\text{Minimum}} \left\{ \frac{y_{i1}}{y_{ik}} \right\} \right\}$$

- If I_1 is singleton, namely $I_1 = \{r\}$, then x_{Br} leaves the basis. Otherwise form I_2.
- In general I_j is formed from I_{j-1} as follows:

$$I_j = \left\{ r : \frac{y_{rj}}{y_{rk}} = \underset{i \in I_{j-1}}{\text{Minimum}} \left\{ \frac{y_{ij}}{y_{ik}} \right\} \right\}$$

Bland's rule

- This rule has been suggested by Robert Bland.
- In this rule, the variables are first ordered in some sequence. say, x_l, x_2, ..., x_n.
- Then of all nonbasic variables having = $z_j - c_j > 0$, the one that has the smallest index is selected to enter the basis.
- Similarly, of all the candidates to leave the basis (i.e. which tie in the usual minimum ratio test), the one that has the smallest index is chosen as the exiting variable.

References

References

- M.S. Bazaraa, J.J. Jarvis, H.D. Sherali, **Linear Programming and Network Flows**, Wiley, 1990. (Chapter 4)

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