In the name of God

Part 1. The Review of Linear Programming

1.4. The Revised Simplex Method

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Instructor: Dr. Masoud Yaghini

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Introduction

The Revised Simplex Method

- **The revised simplex method** is a systematic procedure for implementing the steps of the simplex method in a smaller array, thus saving storage space.
- Let us begin by reviewing the steps of the simplex method for a **minimization problem**.

$$\text{Minimize} \quad \mathbf{cx}$$

$$\text{Subject to } \mathbf{Ax} = \mathbf{b}$$

$$\mathbf{x} \geqslant \mathbf{0}$$

- Suppose that we are given a basic feasible solution with basis **B** (and basis inverse $\mathbf{B}^{-1}$). Then:

Steps of the Simplex Method (Minimization Problem)

1. The basic feasible solution is given by $\mathbf{x_B} = \mathbf{B}^{-1}\mathbf{b} = \bar{\mathbf{b}}$ and $\mathbf{x_N} = \mathbf{0}$.
   - The objective $z = \mathbf{c_B}\mathbf{B}^{-1}\mathbf{b} = \mathbf{c_B}\,\bar{\mathbf{b}}$

2. Calculate the simplex multipliers $\mathbf{w} = \mathbf{c_B}\mathbf{B}^{-1}$.
   - For each nonbasic variable, calculate

$$z_j - c_j = \mathbf{c_B}\mathbf{B}^{-1}\mathbf{a_j} - c_j = \mathbf{w}\mathbf{a_j} - c_j.$$

   - Let $z_k - c_k$ = Maximum $z_j - c_j$.
   - If $z_k - c_k \le 0$, then stop; the current solution is **optimal**. Otherwise go to step 3.

Steps of the Simplex Method (Minimization Problem)

3. Calculate $\mathbf{y_k} = \mathbf{B^{-1}a_k}$.
   - The objective $z = \mathbf{c_B B^{-1} b} = \mathbf{c_B} = \bar{\mathbf{b}}$
     If $\mathbf{y_k} \le \mathbf{0}$, then stop; the optimal solution is **unbounded**.
   - Otherwise determine the index of the variable x_{Br} leaving the basis as follows:

$$\frac{\bar{b}_r}{y_{rk}} = \underset{1 \leq i \leq m}{\text{Minimum}} \left\{ \frac{\bar{b}_i}{y_{ik}} : y_{ik} > 0 \right\}$$

   - Update the basis $\mathbf{B}$ by replacing $\mathbf{a_{Br}}$ with $\mathbf{a_k}$, and go to step 1.

The Revised Simplex Method

- The simplex method can be executed using a smaller array.
- Suppose that we have a basic feasible solution with a known $\mathbf{B}^{-1}$.
- The **revised simplex tableau**:

BASIS INVERSE	RHS
$\mathbf{w}$	$\mathbf{c}_B\bar{\mathbf{b}}$
$\mathbf{B}^{-1}$	$\bar{\mathbf{b}}$

- where $\mathbf{w} = \mathbf{c}_B\mathbf{B}^{-1}$ and $\bar{\mathbf{b}} = \mathbf{B}^{-1}\mathbf{b}$

The Revised Simplex Method

- In step 1 of simplex method:
  - the right-hand side denotes the values of the objective function and the basic variables.
- In step 2 of simplex method:
  - In order to determine whether to stop or to introduce a new variable into the basis, we need to see is the $z_j - c_j = \mathbf{c_B B^{-1} a_j} - c_j = \mathbf{wa_j} - c_j$.
  - Since **w** is known, $z_j - c_j$ can be calculated
- In step 3 of simplex method:
  - Suppose that $z_k - c_k > 0$, then using $\mathbf{B^{-1}}$ we may compute $\mathbf{y_k = B^{-1} a_k}$
  - If $\mathbf{y_k} \leq \mathbf{0}$, then stop; the optimal solution is **unbounded**.

The Revised Simplex Method

- Otherwise the updated column of x_k is inserted to the right of the above tableau:

BASIS INVERSE	RHS	x_k
$\mathbf{w}$	$\mathbf{c}_B\bar{\mathbf{b}}$	$z_k - c_k$
	$\bar{b}_1$	y_{1k}
	$\bar{b}_2$	y_{2k}
$\mathbf{B}^{-1}$	$\vdots$	$\vdots$
	$\bar{b}_r$	y_{rk}
	$\vdots$	$\vdots$
	$\bar{b}_m$	y_{mk}

- The index r of step 3 can now be calculated by the usual minimum ratio test.

The Revised Simplex Method in Tableau Format

The Revised Simplex Method in Tableau Format

- **Initialization Step** (Minimization Problem)
  - Find an initial basic feasible solution with basis inverse $\mathbf{B}^{-1}$.
  - Calculate $\mathbf{w} = \mathbf{c}_B\mathbf{B}^{-1}$, $\bar{\mathbf{b}} = \mathbf{B}^{-1}\mathbf{b}$, and form the **revised simplex tableau**.

BASIS INVERSE	RHS
$\mathbf{w}$	$\mathbf{c}_B\bar{\mathbf{b}}$
$\mathbf{B}^{-1}$	$\bar{\mathbf{b}}$

The Revised Simplex Method in Tableau Format

- **Main Step**
  - For each nonbasic variable, calculate $z_j - c_j = \mathbf{wa_j} - c_j$.
  - Let $z_k - c_k = \text{Maximum } z_j - c_j$.
  - If $z_k - c_k \leq 0$, stop; the current basic feasible solution is **optimal**.
  - Otherwise, calculate $\mathbf{y_k} = \mathbf{B^{-1}a_k}$.
  - If $\mathbf{y_k} \leq \mathbf{0}$, stop; the optimal solution is unbounded.
  - Otherwise, insert the column $\left[\begin{array}{c} z_k - c_k \\ \hline \mathbf{y}_k \end{array}\right]$ to the right of the revised simplex tableau.

BASIS INVERSE	RHS		x_k
$\mathbf{w}$	$\mathbf{c}_B\bar{\mathbf{b}}$		$z_k - c_k$
$\mathbf{B}^{-1}$	$\bar{\mathbf{b}}$		$\mathbf{y}_k$

The Revised Simplex Method in Tableau Format

- Determine the index r as follows:

$$\frac{\bar{b}_r}{y_{rk}} = \underset{1 \leq i \leq m}{\text{Minimum}} \left\{ \frac{\bar{b}_i}{y_{ik}} : y_{ik} > 0 \right\}$$

- Pivot at y_{rk}.
- This updates the tableau.
- Now the column corresponding to x_k is completely eliminated from the tableau and the main step is repeated.

The Revised Simplex Method in Tableau Format

- Example:

$$
\begin{aligned}
\text{Minimize} \quad & -x_1 - 2x_2 + x_3 - x_4 - 4x_5 + 2x_6 \\
\text{Subject to} \quad & x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leqslant 6 \\
& 2x_1 - x_2 - 2x_3 + x_4 \leqslant 4 \\
& x_3 + x_4 + 2x_5 + x_6 \leqslant 4 \\
& x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5, \quad x_6 \geqslant 0
\end{aligned}
$$

- Introduce the slack variables x_7, x_8, x_9.
- The initial basis is $\mathbf{B} = [\mathbf{a}_7, \mathbf{a}_8, \mathbf{a}_9] = \mathbf{I}_3$.
- Also, $\mathbf{w} = \mathbf{c}_B\mathbf{B}^{-1} = (0, 0, 0)$ and $\bar{\mathbf{b}} = \mathbf{b}$.

The Revised Simplex Method in Tableau Format

- **Iteration 1**

	BASIS INVERSE			RHS
z	0	0	0	0
x_7	1	0	0	6
x_8	0	1	0	4
x_9	0	0	1	4

- Here w = (0, 0, 0). Noting that z_j - c_j = $\mathbf{w}\mathbf{a}_\mathbf{j}$ - c_j, we get

$$z_1 - c_1 = 1,\ z_2 - c_2 = 2,\ z_3 - c_3 = -1,$$

$$z_4 - c_4 = 1,\ z_5 - c_5 = 4,\ z_6 - c_6 = -2$$

- Therefore $k = 5$ and x_5 enters the basis:

$$\mathbf{y}_5 = \mathbf{B}^{-1}\mathbf{a}_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

The Revised Simplex Method in Tableau Format

- Insert the vector

$$\left[\frac{z_5 - c_5}{\mathbf{y}_5}\right] = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 2 \end{bmatrix}$$

- to the right of the tableau and pivot at $y_{35} = 2$.

	BASIS INVERSE			RHS	x_5
z	0	0	0	0	4
x_7	1	0	0	6	1
x_8	0	1	0	4	0
x_9	0	0	1	4	2

	BASIS INVERSE			RHS
z	0	0	-2	-8
x_7	1	0	$-\frac{1}{2}$	4
x_8	0	1	0	4
x_5	0	0	$\frac{1}{2}$	2

The Revised Simplex Method in Tableau Format

- **Iteration 2**
- Here **w** = (0, 0, -2). Noting that $z_j - c_j = \mathbf{w}\mathbf{a_j} - c_j$, we get

$$z_1 - c_1 = 1, z_2 - c_2 = 2, z_3 - c_3 = -3,$$

$$z_4 - c_4 = -1, z_6 - c_6 = -4,$$

$$z_9 - c_9 = -2.$$

- Therefore $k = 2$ and x_2 enters the basis:

$$\mathbf{y}_2 = \mathbf{B}^{-1}\mathbf{a}_2 = \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

- Insert the vector

$$\left[\frac{z_2 - c_2}{\mathbf{y}_2} \right] = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

The Revised Simplex Method in Tableau Format

- to the right of the above tableau and pivot at y_{12}.

	BASIS INVERSE			RHS		x_2
z	0	0	-2	-8		2
x_7	1	0	$-\frac{1}{2}$	4		1
x_8	0	1	0	4		-1
x_5	0	0	$\frac{1}{2}$	2		0

	BASIS INVERSE			RHS
z	-2	0	-1	-16
x_2	1	0	$-\frac{1}{2}$	4
x_8	1	1	$-\frac{1}{2}$	8
x_5	0	0	$\frac{1}{2}$	2

The Revised Simplex Method in Tableau Format

- **Iteration 3**
- Here **w** = (-2, 0, -1). Noting that z_j - c_j = $\mathbf{wa_j}$ - c_j, we get

$$z_1 - c_1 = -1,\ z_3 - c_3 = -4,\ z_4 - c_4 = -2,$$

$$z_6 - c_6 = -5,\ z_9 - c_9 = -1.$$

- Since $z_j - c_j \leq 0$ for all nonbasic variables (x_7 just left the basis and so $z_7 - c_7 < 0$), we stop; the basic feasible solution of the foregoing tableau is **optimal**.

Comparison Between the Simplex and the Revised Simplex Methods

Simplex Method vs. Revised Simplex Method

- An array that we need:
  - For the simplex method: $(m + 1) \times (n + 1)$
  - For the revised simplex: $(m + 1) \times (m + 2)$
- If n is significantly larger than m, this would result in a substantial saving in computer core storage.

Simplex Method vs. Revised Simplex Method

- The number of multiplications (division is considered a multiplication) and additions (subtraction is considered an addition) per iteration of both procedures are given in below:

METHOD		OPERATION: PIVOTING	OPERATION: $z_j - c_j$'s	OPERATION: TOTAL
Simplex	Multiplications	$(m+1)(n-m+1)$		$m(n-m)+n+1$
	Additions	$m(n-m+1)$		$m(n-m+1)$
Revised Simplex	Multiplications	$(m+1)^2$	$m(n-m)$	$m(n-m)+(m+1)^2$
	Additions	$m(m+1)$	$m(n-m)$	$m(n+1)$

Simplex Method vs. Revised Simplex Method

- From the Table we see that the number of operations required during an iteration of the simplex method is slightly **less than** those required for the revised simplex method.
- Note, however, that for most practical problems the density d (number of nonzero elements divided by total number of elements) of nonzero elements in the constraint matrix is usually small (in many cases $d \leq 0.05$).
- The revised simplex method can take advantage of this situation while calculating $z_j - c_j$.
- Note that $z_j = \mathbf{wa_j}$ and we can skip zero elements of a_j while performing the calculation $\mathbf{wa_j} = \sum^{\mathrm{m}}_{\mathrm{i}} w_i a_{ij}$

Simplex Method vs. Revised Simplex Method

- Therefore the number of operations in the revised simplex method for calculating the z_j - c_j is given by d times the entries of the, substantially reducing the total number of operations.
- While pivoting, for both the simplex and the revised simplex methods, no operations are skipped because the current tableaux usually fill quickly with nonzero entries, even if the original constraint matrix was sparse.

Simplex Method vs. Revised Simplex Method

- To summarize, if n is significantly larger than m, and if the density d is small, the computational effort of the revised simplex method is <u>significantly smaller</u> than that of the simplex method.
- Also, in the revised simplex method, the use of the original data for calculating the $z_j - c_j$ and the updated column $\mathbf{y_k}$ tends to <u>reduce</u> the cumulative round-off error.

Product Form of the Inverse

Product Form of the Inverse

- In another implementation of the revised simplex method, the **inverse of the basis** is stored as the product of **elementary matrices**
  - an **elementary matrix** is a square matrix that differs from the **identity** in only **one row or one column**.
- This method provides greater numerical stability by **reducing accumulated round-off errors**.

Product Form of the Inverse

- Consider a basis $\mathbf{B}$ composed of the columns $\mathbf{a}_{B1}, \mathbf{a}_{B2}, \ldots, \mathbf{a}_{Bm}$ and suppose that $\mathbf{B}^{-1}$ is known.
- Now suppose that the nonbasic column $\mathbf{a}_k$ replaces $\mathbf{a}_{Br}$ resulting in the new basis $\hat{\mathbf{B}}$.
- We wish to find $\hat{\mathbf{B}}^{-1}$ in terms of $\mathbf{B}^{-1}$.
- Noting that $\mathbf{a}_k = \mathbf{B}\mathbf{y}_k$ ($\mathbf{y}_k = \mathbf{B}^{-1}\mathbf{a}_k$) and $\mathbf{a}_{Bi} = \mathbf{B}\mathbf{e}_i$ where
  - $\mathbf{e}_i$ is a vector of zeros except for 1 at the *i* th position, we have

$$\begin{aligned}\hat{\mathbf{B}} &= (\mathbf{a}_{B_1}, \mathbf{a}_{B_2}, \ldots, \mathbf{a}_{B_{r-1}}, \mathbf{a}_k, \mathbf{a}_{B_{r+1}}, \ldots, \mathbf{a}_{B_m}) \\ &= (\mathbf{B}\mathbf{e}_1, \mathbf{B}\mathbf{e}_2, \ldots, \mathbf{B}\mathbf{e}_{r-1}, \mathbf{B}\mathbf{y}_k, \mathbf{B}\mathbf{e}_{r+1}, \ldots, \mathbf{B}\mathbf{e}_m) \\ &= \mathbf{B}\mathbf{T}\end{aligned}$$

  - where $\mathbf{T}$ is the identity with the *r* th column replaced by $\mathbf{y}_k$.

Product Form of the Inverse

- Let $\mathbf{E} = \mathbf{T}^{-1}$, since $\hat{\mathbf{B}} = \mathbf{TB}$, therefore $\hat{\mathbf{B}}^{-1} = \mathbf{T}^{-1}\,\mathbf{B}^{-1} = \mathbf{EB}^{-1}$ where the elementary matrix $\mathbf{E}$ is:

$$\begin{array}{cc}
 & \begin{array}{c} r\text{th column} \\ \downarrow \end{array} \\
\mathbf{E} = & \begin{bmatrix}
1 & 0 & \cdots & 0 & -y_{1k}/y_{rk} & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 & -y_{2k}/y_{rk} & 0 & \cdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
0 & 0 & \cdots & 0 & 1/y_{rk} & 0 & \cdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
0 & 0 & \cdots & 0 & -y_{mk}/y_{rk} & 0 & \cdots & 1
\end{bmatrix} \begin{array}{c} \\ \\ \\ \leftarrow r\text{th row} \\ \\ \\ \end{array}
\end{array}$$

- To summarize, the basis inverse at a new iteration can be obtained by premultiplying the basis inverse at the previous iteration by an elementary matrix **E**.
- The nonidentity column **g,** as the **eta vector**, and its position *r* need be stored to specify **E**.

Product Form of the Inverse

- Let the basis **B**, at the first iteration be the identity **I**.
- Then the basis inverse at iteration 2 is

  $$\mathbf{B_2^{-1} = E_1 B_1^{-1} = E_1 I = E_1}$$

  where $\mathbf{E_1}$ is the elementary matrix corresponding to the first iteration.
- Similarly $\mathbf{B_3^{-1} = E_2 B_2^{-1} = E_2 E_1}$, and in general

  $$\mathbf{B_t^{-1} = E_{t-1} E_{t-2} \ldots E_2 E_1}$$

- This Equation specifies the basis inverse as the product of elementary matrices, is called the **product form of the inverse**.
- Using this form, all the steps of the simplex method can be performed **without pivoting**.
- First, it will be helpful to elaborate on multiplying a vector by an elementary matrix.

Product Form of the Inverse

- **Post Multiplying**
  - Let **E** be an elementary matrix with nonidentity column **g** appearing at the r th position. Let **c** be a row vector. Then

$$
\mathbf{cE} = (c_1, c_2, \ldots, c_m)
\begin{array}{c}
\begin{array}{cccccc} & & & \text{position } r & & \\ & & & \downarrow & & \end{array} \\
\begin{bmatrix}
1 & 0 & \cdots & g_1 & \cdots & 0 \\
0 & 1 & \cdots & g_2 & \cdots & 0 \\
\vdots & \vdots & & \vdots & & \\
0 & 0 & \cdots & g_m & \cdots & 1
\end{bmatrix}
\end{array}
$$

$$
= \left(c_1, c_2, \ldots, c_{r-1}, \sum_{i=1}^{m} c_i g_i, c_{r+1}, \ldots, c_m\right)
$$

$$
= (c_1, c_2, \ldots, c_{r-1}, \mathbf{cg}, c_{r+1}, \ldots, c_m)
$$

  - **cE** is equal to **c** except that the r th component is replaced by **cg**.

Product Form of the Inverse

- **Premultiplying**
  - Let **a** be an **m** vector. Then

$$\mathbf{Ea} = \begin{bmatrix} 1 & \cdots & g_1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & g_r & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & g_m & \cdots & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_r \\ \vdots \\ a_m \end{bmatrix}$$

$$= \begin{bmatrix} a_1 + g_1 a_r \\ \vdots \\ g_r a_r \\ \vdots \\ a_m + g_m a_r \end{bmatrix} = \begin{bmatrix} a_1 \\ \vdots \\ 0 \\ \vdots \\ a_m \end{bmatrix} + a_r \begin{bmatrix} g_1 \\ \vdots \\ g_r \\ \vdots \\ g_m \end{bmatrix}$$

  - **Ea** is equal to **a** except that the *r* th component is replaced by $\mathbf{a}_r\mathbf{g}$.

Product Form of the Inverse

- **Computing the vector $\mathbf{w} = \mathbf{c_B B^{-1}}$**
  - At iteration t we wish to calculate the vector **w**. Note that

$$\mathbf{w} = \mathbf{c}_B \mathbf{B}_t^{-1} = \mathbf{c}_B \mathbf{E}_{t-1} \mathbf{E}_{t-2} \cdots \mathbf{E}_2 \mathbf{E}_1$$

  - After **w** is computed, we can calculate $z_j - c_j = \mathbf{w a_j} - c_j$ for nonbasic variables.

Product Form of the Inverse

- **Computing the updated column $\mathbf{y_k}$**
  - If x_k is to enter the basis at iteration t, then $\mathbf{y_k}$ is calculated as follows:

$$\mathbf{y}_k = \mathbf{B}_t^{-1}\mathbf{a}_k = \mathbf{E}_{t-1}\mathbf{E}_{t-2} \cdots \mathbf{E}_2\mathbf{E}_1\mathbf{a}_k$$

  - If $\mathbf{y_k} \leq 0$, we stop with the conclusion that the optimal solution is unbounded.
  - Otherwise the usual minimum ratio test determines the index r of the variable x_{Br} leaving the basis. A new elementary matrix $\mathbf{E_t}$ is generated where the nonidentity column $\mathbf{g}$ is given by:

$$\begin{bmatrix} \frac{-y_{1k}}{y_{rk}} \\ \vdots \\ \frac{1}{y_{rk}} \\ \vdots \\ \frac{-y_{mk}}{y_{rk}} \end{bmatrix}$$

  - and appears at position r.

Product Form of the Inverse

- **Computing the right-hand-side $\bar{\mathbf{b}}$**
  - The new right-hand side is given by

$$\mathbf{B}_{t+1}^{-1}\mathbf{b} = \mathbf{E}_t\mathbf{B}_t^{-1}\mathbf{b}$$

  - Noting that $\mathbf{B}_t^{-1}\mathbf{b}$ is known from the last iteration.

Product Form of the Inverse

- **Updating the basis inverse**
  - The basis inverse is updated by generating $\mathbf{E_t}$ as discussed above.
  - It is worthwhile noting that the number of elementary matrices required to represent the basis inverse increases by 1 at each iteration.
  - If this number becomes large, it would be necessary to reinvert the basis and represent it as the product of m elementary matrices.
  - It is emphasized that each elementary matrix is completely described by its nonidentity column and its position.
  - Therefore an elementary matrix $\mathbf{E}$ could be stored as nonidentity column $\mathbf{g}$ and r is its position.

Product Form of the Inverse

- **Example**

$$\text{Minimize} \quad -x_1 - 2x_2 + x_3$$

$$\begin{aligned} \text{Subject to} \quad x_1 + x_2 + x_3 &\leqslant 4 \\ -x_1 + 2x_2 - 2x_3 &\leqslant 6 \\ 2x_1 + x_2 \qquad &\leqslant 5 \\ x_1, \quad x_2, \quad x_3 &\geqslant 0 \end{aligned}$$

- Introduce the slack variables x_4, x_5, and x_6. The original basis consists of x_4, x_5, and x_6.

Product Form of the Inverse

- **Iteration 1**

$$\bar{\mathbf{b}} = \begin{bmatrix} 4 \\ 6 \\ 5 \end{bmatrix}$$

$$\mathbf{x}_B = \begin{bmatrix} x_{B_1} \\ x_{B_2} \\ x_{B_3} \end{bmatrix} = \begin{bmatrix} x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 5 \end{bmatrix} \qquad \mathbf{x}_N = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$z = 0$$

$$\mathbf{w} = \mathbf{c}_B = (0, 0, 0)$$

  - Note that $z_j - c_j = \mathbf{w}\mathbf{a}_j - c_j$ Therefore

$$z_1 - c_1 = 1,\ z_2 - c_2 = 2,\ z_3 - c_3 = -1$$

  - Thus, $k = 2$, and x_2 enters the basis.

$$\mathbf{y}_2 = \mathbf{a}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Product Form of the Inverse

- Here x_{B_r} leaves the basis where r is determined by

$$\text{Minimum}\left\{ \frac{\bar{b}_1}{y_{12}}, \frac{\bar{b}_2}{y_{22}}, \frac{\bar{b}_3}{y_{32}} \right\} = \text{Minimum}\left\{ \frac{4}{1}, \frac{6}{2}, \frac{5}{1} \right\} = 3$$

- Therefore $r = 2$; that is, $x_{B2} = x_5$ leaves the basis and x_2 enters the basis. The nonidentity column of $\mathbf{E}_1$ is given by

$$\mathbf{g} = \begin{bmatrix} -\dfrac{y_{12}}{y_{22}} \\ \dfrac{1}{y_{22}} \\ -\dfrac{y_{32}}{y_{22}} \end{bmatrix} = \begin{bmatrix} -\dfrac{1}{2} \\ \dfrac{1}{2} \\ -\dfrac{1}{2} \end{bmatrix}$$

  – and $\mathbf{E}_1$ is represented by $\begin{bmatrix} \mathbf{g} \\ 2 \end{bmatrix}$

Product Form of the Inverse

- **Iteration 2**
  - Update $\bar{\mathbf{b}}$

$$\bar{\mathbf{b}} = \mathbf{E}_1 \begin{bmatrix} 4 \\ 6 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix} + 6 \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

$$\mathbf{x}_B = \begin{bmatrix} x_{B_1} \\ x_{B_2} \\ x_{B_3} \end{bmatrix} = \begin{bmatrix} x_4 \\ x_2 \\ x_6 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \qquad \mathbf{x}_N = \begin{bmatrix} x_1 \\ x_5 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$z = 0 - \bar{b}_2(z_2 - c_2) = -6$$

$$\mathbf{w} = \mathbf{c}_B\mathbf{E}_1 = (0, -2, 0)\mathbf{E}_1.$$

  - Then $\mathbf{w} = (0, -1, 0)$. Note that $z_j - c_j = \mathbf{w}\mathbf{a_j} - c_j$,Therefore

$$z_1 - c_1 = 2,\ z_3 - c_3 = 1$$

  - Thus, $k = 1$ and x_1, enters the basis.

Product Form of the Inverse

- Noting

$$\mathbf{y}_1 = \mathbf{E}_1\mathbf{a}_1 = \mathbf{E}_1\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ \frac{5}{2} \end{bmatrix}$$

- Then x_{Br} leaves the basis where r is determined by

$$\text{Minimum}\left\{ \frac{\bar{b}_1}{y_{11}}, \frac{\bar{b}_3}{y_{31}} \right\} = \text{Minimum}\left\{ \frac{1}{\frac{3}{2}}, \frac{2}{\frac{5}{2}} \right\} = \frac{2}{3}$$

- Therefore $r = 1$; that is, $x_{B1} = x_4$ leaves and x_1, enters the basis. The nonidentity column of $\mathbf{E}_2$ is represented by $\begin{bmatrix} \mathbf{g} \\ 1 \end{bmatrix}$

$$\mathbf{g} = \begin{bmatrix} \frac{1}{y_{11}} \\ -\frac{y_{21}}{y_{11}} \\ -\frac{y_{31}}{y_{11}} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ -\frac{5}{3} \end{bmatrix}$$

Product Form of the Inverse

- **Iteration 3**
  - Update $\bar{\mathbf{b}}$

$$\bar{\mathbf{b}} = \mathbf{E}_2 \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ -\frac{5}{3} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{10}{3} \\ \frac{1}{3} \end{bmatrix}$$

$$\mathbf{x}_B = \begin{bmatrix} x_{B_1} \\ x_{B_2} \\ x_{B_3} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_6 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{10}{3} \\ \frac{1}{3} \end{bmatrix} \qquad \mathbf{x}_N = \begin{bmatrix} x_4 \\ x_5 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$z = -6 - \bar{b}_1(z_1 - c_1) = -\frac{22}{3}$$

$$\mathbf{w} = \mathbf{c}_B \mathbf{E}_2 \mathbf{E}_1 = (-1, -2, 0)\mathbf{E}_2\mathbf{E}_1.$$

Product Form of the Inverse

– Calculate **w**

$$\mathbf{c}_B \mathbf{E}_2 = \left(-\tfrac{4}{3}, -2, 0\right)$$

$$\mathbf{w} = (\mathbf{c}_B \mathbf{E}_2)\mathbf{E}_1 = \left(-\tfrac{4}{3}, -\tfrac{1}{3}, 0\right)$$

– Note that $z_j - c_j = \mathbf{wa}_j - c_j$ Therefore

$$z_3 - c_3 = -\tfrac{5}{3}, \; z_5 - c_5 = -\tfrac{1}{3}$$

– Since $z_j - c_j \leq 0$ for all nonbasic variables, then the optimal solution is at hand. The objective value is -22/3 and

$$(x_1, x_2, x_3, x_4, x_5, x_6) = \left(\tfrac{2}{3}, \tfrac{10}{3}, 0, 0, 0, \tfrac{1}{3}\right)$$

References

References

- M.S. Bazaraa, J.J. Jarvis, H.D. Sherali, **Linear Programming and Network Flows**, Wiley, 1990. (Chapter 5)

The End