In the name of God

Part 1. The Review of Linear Programming

1.5. Duality Theory

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- Introduction
- Formulation of the Dual Problem
- Primal-Dual Relationship
- Economic Interpretation of the Dual
- Dual Simplex Method
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Introduction

Introduction

- **Dual problem**
  - For every linear program there is another associated linear program, called **dual (linear programming) problem**
  - Dual problem may be used to obtain the solution to the original program.
- **Primal problem**
  - The **original linear program** called the **primal (linear programming) problem**.
  - Dual problem's variables provide extremely useful information about the **primal problem**.

Introduction

- **The dual of the dual is the primal.**
- The properties of dual program will lead to two new algorithms, for solving linear programs:
  - **The dual simplex method**
  - **The primal-dual algorithm**

Formulation of the Dual Problem

Formulation of the Dual Problem

- There are two important forms of duality:
  - **the canonical form of duality**
  - **the standard form of duality**
- These two forms are completely equivalent.

Canonical Form of Duality

- **Canonical Form of Duality**:

P: Minimize $\mathbf{cx}$

Subject to $\mathbf{Ax} \geqslant \mathbf{b}$

$\mathbf{x} \geqslant \mathbf{0}$

D: Maximize $\mathbf{wb}$

Subject to $\mathbf{wA} \leqslant \mathbf{c}$

$\mathbf{w} \geqslant \mathbf{0}$

- Where
  - P: the primal problem
  - D: the dual problem
- Note that:
  - for each **primal constraint** there is exactly one **dual variable**
  - for each **primal variable** there is exactly one **dual constraint**

Canonical Form of Duality

- **Example:**

$$
\begin{aligned}
\text{P: Minimize} \quad & 6x_1 + 8x_2 \\
\text{Subject to} \quad & 3x_1 + \ \ x_2 \geqslant 4 \\
& 5x_1 + 2x_2 \geqslant 7 \\
& x_1, \quad x_2 \geqslant 0
\end{aligned}
$$

$$
\begin{aligned}
\text{D: Maximize} \quad & 4w_1 + 7w_2 \\
\text{Subject to} \quad & 3w_1 + 5w_2 \leqslant 6 \\
& w_1 + 2w_2 \leqslant 8 \\
& w_1, \quad w_2 \geqslant 0
\end{aligned}
$$

Standard Form of Duality

- **Standard Form of Duality**

P: Minimize $\mathbf{cx}$

Subject to $\mathbf{Ax} = \mathbf{b}$

$\mathbf{x} \geqslant \mathbf{0}$

D: Maximize $\mathbf{wb}$

Subject to $\mathbf{wA} \leqslant \mathbf{c}$

$\mathbf{w}$ unrestricted

Standard Form of Duality

- **Example**

P: Minimize $6x_1 + 8x_2$

$$
\begin{aligned}
\text{Subject to } 3x_1 + x_2 - x_3 \quad &= 4 \\
5x_1 + 2x_2 \quad - x_4 &= 7 \\
x_1, \quad x_2, \quad x_3, \quad x_4 &\geqslant 0
\end{aligned}
$$

D: Maximize $4w_1 + 7w_2$

$$
\begin{aligned}
\text{Subject to } 3w_1 + 5w_2 &\leqslant 6 \\
w_1 + 2w_2 &\leqslant 8 \\
-w_1 \qquad &\leqslant 0 \\
-w_2 &\leqslant 0 \\
w_1, \quad w_2 \quad &\text{unrestricted}
\end{aligned}
$$

Mixed Forms of Duality

- In practice, many linear programs contain
  - some constraints of the ≤, ≥, and = types.
  - Also, variables may be "≥ 0," "≤ 0," or "unrestricted."
- This presents no problem since we may apply the **transformation techniques** to convert any "mixed" problem to one of the primal or dual forms.

Mixed Forms of Duality

- Consider the following linear program.

$$
\begin{array}{llcccccccl}
P: & \text{Minimize} & c_1x_1 & + & c_2x_2 & + & c_3x_3 & & \\
 & \text{subject to} & A_{11}x_1 & + & A_{12}x_2 & + & A_{13}x_3 & \geq & b_1 \\
 & & A_{21}x_1 & + & A_{22}x_2 & + & A_{23}x_3 & \leq & b_2 \\
 & & A_{31}x_1 & + & A_{32}x_2 & + & A_{33}x_3 & = & b_3 \\
 & & & x_1 \geq 0, & & x_2 \leq 0, & & x_3 & \text{unrestricted.}
\end{array}
$$

- Converting this problem to canonical form by
  - multiplying the second set of inequalities by -1,
  - writing the equality constraint set equivalently as two inequalities
  - substituting $x_2 = -x_2'$, $x_3 = x_3' - x_3''$, we get

Mixed Forms of Duality

- After converting, we get

$$
\begin{array}{llllllllll}
\text{Minimize} & \mathbf{c}_1\mathbf{x}_1 & - & \mathbf{c}_2\mathbf{x}_2' & + & \mathbf{c}_3\mathbf{x}_3' & - & \mathbf{c}_3\mathbf{x}_3'' & & \\
\text{subject to} & \mathbf{A}_{11}\mathbf{x}_1 & - & \mathbf{A}_{12}\mathbf{x}_2' & + & \mathbf{A}_{13}\mathbf{x}_3' & - & \mathbf{A}_{13}\mathbf{x}_3'' & \geq & \mathbf{b}_1 \\
 & -\mathbf{A}_{21}\mathbf{x}_1 & + & \mathbf{A}_{22}\mathbf{x}_2' & - & \mathbf{A}_{23}\mathbf{x}_3' & + & \mathbf{A}_{23}\mathbf{x}_3'' & \geq & -\mathbf{b}_2 \\
 & \mathbf{A}_{31}\mathbf{x}_1 & - & \mathbf{A}_{32}\mathbf{x}_2' & + & \mathbf{A}_{33}\mathbf{x}_3' & - & \mathbf{A}_{33}\mathbf{x}_3'' & \geq & \mathbf{b}_3 \\
 & -\mathbf{A}_{31}\mathbf{x}_1 & + & \mathbf{A}_{32}\mathbf{x}_2' & - & \mathbf{A}_{33}\mathbf{x}_3' & + & \mathbf{A}_{33}\mathbf{x}_3'' & \geq & -\mathbf{b}_3 \\
 & \mathbf{x}_1 \geq \mathbf{0}, & & \mathbf{x}_2' \geq \mathbf{0}, & & \mathbf{x}_3' \geq \mathbf{0}, & & \mathbf{x}_3'' \geq \mathbf{0}. & &
\end{array}
$$

- Denoting the dual variables associated with the four constraint sets as w_1,w_2', w_3', and w_3'' respectively,

Mixed Forms of Duality

- We obtain the dual to this problem as follows:

$$
\begin{array}{lrcrcrcrcr}
\text{Maximize} & w_1 b_1 & - & w_2' b_2 & + & w_3' b_3 & - & w_3'' b_3 & & \\
\text{subject to} & w_1 A_{11} & - & w_2' A_{21} & + & w_3' A_{31} & - & w_3'' A_{31} & \leq & c_1 \\
 & -w_1 A_{12} & + & w_2' A_{22} & - & w_3' A_{32} & + & w_3'' A_{32} & \leq & -c_2 \\
 & w_1 A_{13} & - & w_2' A_{23} & + & w_3' A_{33} & - & w_3'' A_{33} & \leq & c_3 \\
 & -w_1 A_{13} & + & w_2' A_{23} & - & w_3' A_{33} & + & w_3'' A_{33} & \leq & -c_3 \\
 & \multicolumn{9}{c}{w_1 \geq 0, \quad w_2' \geq 0, \quad w_3' \geq 0, \quad w_3'' \geq 0.}
\end{array}
$$

Mixed Forms of Duality

- Finally. Using $\boldsymbol{w_2} = -\boldsymbol{w_2}'$, and $w_3 = w_3' - w_3''$, the foregoing problem may be equivalently stated as follows.

$$
\begin{array}{llcccccccc}
\text{D}: & \text{Maximize} & \mathbf{w}_1\mathbf{b}_1 & + & \mathbf{w}_2\mathbf{b}_2 & + & \mathbf{w}_3\mathbf{b}_3 & & \\
 & \text{subject to} & \mathbf{w}_1\mathbf{A}_{11} & + & \mathbf{w}_2\mathbf{A}_{21} & + & \mathbf{w}_3\mathbf{A}_{31} & \leq & \mathbf{c}_1 \\
 & & \mathbf{w}_1\mathbf{A}_{12} & + & \mathbf{w}_2\mathbf{A}_{22} & + & \mathbf{w}_3\mathbf{A}_{32} & \geq & \mathbf{c}_2 \\
 & & \mathbf{w}_1\mathbf{A}_{13} & + & \mathbf{w}_2\mathbf{A}_{23} & + & \mathbf{w}_3\mathbf{A}_{33} & = & \mathbf{c}_3 \\
\end{array}
$$

$$\mathbf{w}_1 \geq \mathbf{0}, \quad \mathbf{w}_2 \leq \mathbf{0}, \quad \mathbf{w}_3 \text{ unrestricted.}$$

Mixed Forms of Duality

- Relationships Between Primal and Dual Problems

	MINIMIZATION PROBLEM		MAXIMATION PROBLEM	
Variables	$\geqslant 0$	$\longleftrightarrow$	$\leqslant$	Constraints
	$\leqslant 0$	$\longleftrightarrow$	$\geqslant$	
	Unrestricted	$\longleftrightarrow$	$=$	
Constraints	$\geqslant$	$\longleftrightarrow$	$\geqslant 0$	Variables
	$\leqslant$	$\longleftrightarrow$	$\leqslant 0$	
	$=$	$\longleftrightarrow$	Unrestricted	

Mixed Forms of Duality

- **Example**, Consider the following linear program

$$\begin{array}{lrcrcrcr}
\text{Maximize} & 8x_1 & + & 3x_2 & - & 2x_3 & & \\
\text{subject to} & x_1 & - & 6x_2 & + & x_3 & \geq & 2 \\
& 5x_1 & + & 7x_2 & - & 2x_3 & = & -4 \\
\end{array}$$

$$x_1 \leq 0, \quad x_2 \geq 0, \; x_3 \; \text{unrestricted.}$$

- Applying the results of the table, we can immediately write down the dual.

$$\begin{array}{lrcrcr}
\text{Minimize} & 2w_1 & - & 4w_2 & & \\
\text{subject to} & w_1 & + & 5w_2 & \geq & 8 \\
& -6w_1 & + & 7w_2 & \leq & 3 \\
& w_1 & - & 2w_2 & = & -2 \\
\end{array}$$

$$w_1 \leq 0, \; w_2 \; \text{unrestricted.}$$

Primal-Dual Relationship

Primal-Dual Relationship

- Consider the canonical form of duality
- Let $\mathbf{x_0}$ and $\mathbf{w_0}$ are feasible solutions to the primal and dual programs
- Then $\mathbf{Ax_0} \geq \mathbf{b}$, $\mathbf{x_0} \geq \mathbf{0}$, $\mathbf{w_0}\mathrm{A} \leq \mathbf{c}$, and $\mathbf{w_0} \geq \mathbf{0}$.
- Multiplying $\mathbf{Ax_0} \geq \mathbf{b}$ on the left by $\mathbf{w_0}$ and $\mathbf{w_0 A} \leq \mathbf{c}$ on the right by $\mathbf{x_0}$, we get

$$\mathbf{cx_0} \geqslant \mathbf{w_0 A x_0} \geqslant \mathbf{w_0 b}$$

- This is known as the **weak duality property**.

Primal-Dual Relationship

- The objective function value for any feasible solution to the minimization problem is always greater than or equal to the objective function value for any feasible solution to the maximization problem.
- The objective value of any feasible solution of the minimization problem gives an **upper bound** on the optimal objective of the maximization problem.
- Similarly, the objective value of any feasible solution of the maximization problem is a **lower bound** on the optimal objective of the minimization problem.

The Fundamental Theorem of Duality

- With regard to the primal and dual linear programming problems, exactly one of the following statements is true:
  1. If $\mathbf{cx_0} = \mathbf{w_0 b}$, then $\mathbf{x_0}$ and $\mathbf{w_0}$ are optimal solutions to their respective problems.
  2. One problem has unbounded objective value, in which case the other problem must be infeasible.
  3. Both problems are infeasible.
- From this theorem we see that duality is not completely symmetric.

The Fundamental Theorem of Duality

- The best we can say is that

P	OPTIMAL	$\Leftrightarrow$	D	OPTIMAL
P	UNBOUNDED	$\Rightarrow$	D	INFEASIBLE
D	UNBOUNDED	$\Rightarrow$	P	INFEASIBLE
P	INFEASIBLE	$\Rightarrow$	D	UNBOUNDED OR INFEASIBLE
D	INFEASIBLE	$\Rightarrow$	P	UNBOUNDED OR INFEASIBLE

Economic Interpretation of the Dual

Economic Interpretation of the Primal

- Consider the following linear program and its dual.

$$\begin{array}{llll} \text{P: Minimize} & \mathbf{cx} & \text{D: Maximize} & \mathbf{wb} \\ \text{Subject to} & \mathbf{Ax} \geqslant \mathbf{b} & \text{Subject to} & \mathbf{wA} \leqslant \mathbf{c} \\ & \mathbf{x} \geqslant \mathbf{0} & & \mathbf{w} \geqslant \mathbf{0} \end{array}$$

- If **B** is the optimal basis for the primal problem and $\mathbf{c_B}$ is the basic cost vector, then we know that

$$z^* = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b} - \sum_{j\in J}(z_j - c_j)x_j = \mathbf{w}^*\mathbf{b} - \sum_{j\in J}(z_j - c_j)x_j .$$

- Where *J* the index set of for nonbasic variables include slack and structural variables.

$$\frac{\partial z^*}{\partial \mathbf{b}} = \mathbf{c}_B\mathbf{B}^{-1} = \mathbf{w}^*$$

Economic Interpretation of the Primal

- The $\mathbf{w_i}$* is the rate of change of the optimal objective value with a unit increase in the i th right-hand-side value.
- Since $\mathbf{w_i}^* \geq 0$, z^* will increase or stay constant as b_i increases.
- Economically, we may think of **w*** as a vector of **shadow prices** for the right-hand-side vector.
- If the i th constraint represents a demand for production of at least b_i units of the i th product and **cx** represents the total cost of production, then w_i^* is the **incremental cost** of producing one more unit of the i th product.
- Put another way, **w*** is the **fair price** we would pay to have an extra unit of the i th product.

Economic Interpretation of the Primal

- Suppose that you engage a firm to produce specified outputs or goods. Let,
  - $b_1, b_2, \ldots, b_m$: amounts of m outputs.
  - n: the number of activities at varying levels to produce the outputs.
  - x_j: the level of each activity j
  - c_j: the unit cost of each activity j
  - a_{ij}: the amount of product i generated by one unit of activity j
  - b_i: the required amount of output I
  - $\sum_{j=1}^{n} a_{ij} x_j$: the units of output i that are produced and must be greater than or equal to the required amount b_i

Economic Interpretation of the Primal

- You agree to pay the total cost of production.
- You would like to have control over the firm's operations so that you can specify the mix and levels of activities that the firm will engage in so as to minimize the total production cost.
- Therefore you wish to solve the following problem, which is precisely the primal problem.

$$\text{Minimize} \quad \sum_{j=1}^{n} c_j x_j$$

$$\text{Subject to} \quad \sum_{j=1}^{n} a_{ij} x_j \geqslant b_i \qquad i = 1, 2, \ldots, m$$

$$x_j \geqslant 0 \qquad j = 1, 2, \ldots, n$$

Economic Interpretation of the Dual

- Instead of trying to control the operation of the firm to obtain the most desirable mix of activities, suppose that you agree to pay the firm **fair unit prices** for each of outputs. Let,
  - w_i: the **unit price** output $i = 1, 2, \ldots, m$
  - $\sum_{i=1}^{m} a_{ij} w_i$: the unit price of activity j
- Therefore you ask the firm that the implicit price of activity j, does not exceed the actual price c_j.
- Therefore the firm must observe the constraints

$$\sum_{i=1}^{m} a_{ij} w_i \leqslant c_j \quad \text{for } j = 1, 2, \ldots, n.$$

- Within these constraints the firm would like to choose a set of prices that maximize his return

$$\sum_{i=1}^{m} w_i b_i .$$

Economic Interpretation of the Dual

- This leads to the following dual problem of the firm.

$$\text{Maximize} \quad \sum_{i=1}^{m} w_i b_i$$

$$\text{Subject to} \quad \sum_{i=1}^{m} a_{ij} w_i \leqslant c_j \qquad j = 1, 2, \ldots, n$$

$$w_i \geqslant 0 \qquad i = 1, \ldots, m$$

- The main duality theorem states that there is an equilibrium set of activities and set of prices where the minimal production cost is equal to the maximal return.

The Dual Simplex Method

The Dual Simplex Method

- The **dual simplex method** solves the dual problem directly on the **primal simplex tableau**.
- At each iteration we move from a basic feasible solution of the dual problem to an improved basic feasible solution until optimality of the dual (and also the primal) is reached,
- Or else until we conclude that the **dual is unbounded** and that the **primal is infeasible**.

The Dual Simplex Method

- In certain instances it is difficult to find a starting basic solution that is feasible (that is, all $\bar{b}_i \geq 0$) to a linear program without adding artificial variables.
- In these instances it is often possible to find a starting basic, but not necessarily feasible, solution that is dual feasible (that is, all $z_j - c_j \leq 0$ for a minimization problem).
- In such cases it is useful to use **dual simplex method** that would produce a series of simplex tableaux that maintain dual feasibility and complementary slackness and strive toward primal feasibility.

The Dual Simplex Method

- Consider the following tableau representing a basic solution at some iteration.

	z	x_1	$\cdots$	x_j	$\cdots$	x_k	$\cdots$	x_n	RHS
z	1	$z_1 - c_1$	$\cdots$	$z_j - c_j$	$\cdots$	$z_k - c_k$	$\cdots$	$z_n - c_n$	$\mathbf{c}_B\bar{\mathbf{b}}$
x_{B_1}	0	y_{11}	$\cdots$	y_{1j}	$\cdots$	y_{1k}	$\cdots$	y_{1n}	$\bar{b}_1$
x_{B_2}	0	y_{21}	$\cdots$	y_{2j}	$\cdots$	y_{2k}	$\cdots$	y_{2n}	$\bar{b}_2$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
x_{B_r}	0	y_{r1}	$\cdots$	y_{rj}	$\cdots$	$\textcircled{y_{rk}}$	$\cdots$	y_{rn}	$\bar{b}_r$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$	$\vdots$
x_{B_m}	0	y_{m1}	$\cdots$	y_{mj}	$\cdots$	y_{mk}	$\cdots$	y_{mn}	$\bar{b}_m$

The Dual Simplex Method

- Suppose that the tableau is dual feasible (that is, $z_j - c_j \leq 0$ for a minimization problem).
- Then, if the tableau is also primal feasible (that is, all $\bar{b}_i \geq 0$) then we have the optimal solution.
- Otherwise, consider some $\bar{b}_r < 0$.
- By selecting row r as a pivot row and some column k such that $y_{rk} < 0$ as a pivot column we can make the new right-hand side $\bar{b}_r' > 0$.
- Through a series of such pivots we hope to make all $\bar{b}_i \geq 0$ while maintaining all $z_j - c_j \leq 0$ and thus achieve optimality.

Summary of the Dual Simplex Method

- **INITIALIZATION STEP (for Minimization Problem)**
  - Find a basis **B** of the primal such that

$$z_j - c_j = \mathbf{c_B B^{-1} a_j} - c_j \leq 0$$

  - for all j

- **MAIN STEP**

1. If $\bar{\mathbf{b}} = \mathbf{B^{-1}b} \geq \mathbf{0}$, stop; the current solution is optimal. Otherwise select the pivot row r with $\bar{b}_r < 0$, say

$$\bar{b}_r = \text{Minimum } \{ \bar{b}_i \}.$$

The Dual Simplex Method

2. If $y_{rj} \geq 0$ for all j, stop; the **dual is unbounded** and the **primal is infeasible**.
   Otherwise select the pivot column k by the following minimum ratio test:

$$\frac{z_k - c_k}{y_{rk}} = \underset{j}{\text{Minimum}} \left\{ \frac{z_j - c_j}{y_{rj}} : y_{rj} < 0 \right\}$$

3. Pivot at y_{rk} and return to step 1.

The Dual Simplex Method

- Example:

$$\text{Minimize } 2x_1 + 3x_2 + 4x_3$$

$$\begin{aligned} \text{Subject to } \quad x_1 + 2x_2 + x_3 &\geq 3 \\ 2x_1 - x_2 + 3x_3 &\geq 4 \\ x_1, \quad x_2, \quad x_3 &\geq 0 \end{aligned}$$

- A starting basic solution that is dual feasible can be obtained by utilizing the slack variables x_4 and x_5.
- This results from the fact that the cost vector is nonnegative.
- Applying the dual simplex method, we obtain the following series of tableaux.

The Dual Simplex Method

	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1	-2	-3	-4	0	0	0
x_4	0	-1	-2	-1	1	0	-3
x_5	0	-2	1	-3	0	1	-4

	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1	0	-4	-1	0	-1	4
x_4	0	0	$-\frac{5}{2}$	$\frac{1}{2}$	1	$-\frac{1}{2}$	-1
x_1	0	1	$-\frac{1}{2}$	$\frac{3}{2}$	0	$-\frac{1}{2}$	2

	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1	0	0	$-\frac{9}{5}$	$-\frac{8}{5}$	$-\frac{1}{5}$	$\frac{28}{5}$
x_2	0	0	1	$-\frac{1}{5}$	$-\frac{2}{5}$	$\frac{1}{5}$	$\frac{2}{5}$
x_1	0	1	0	$\frac{7}{5}$	$-\frac{1}{5}$	$-\frac{2}{5}$	$\frac{11}{5}$

The Dual Simplex Method

- Since $\bar{\mathbf{b}} > 0$ and $z_j - c_j \leq 0$ for all j, the optimal primal and dual solutions are at hand.

$$(x_1^*, x_2^*, x_3^*, x_4^*, x_5^*) = \left(\frac{11}{5}, \frac{2}{5}, 0, 0, 0\right)$$

$$(w_1^*, w_2^*) = \left(\frac{8}{5}, \frac{1}{5}\right)$$

- Note that w_1^* and w_2^* are respectively the negatives of the $z_j - c_j$ entries under the slack variables x_4 and x_5.
- Also note that in each subsequent tableau the value of the objective function is increasing, as it should, for the dual (maximization) problem.

References

References

- M.S. Bazaraa, J.J. Jarvis, H.D. Sherali, **Linear Programming and Network Flows**, Wiley, 1990. (Chapter 6)

The End